1. Let M be an abelian group. Let n and g be non-negative integers. Compute the (co)homology groups of the following spaces with coefficients in M.

(a) $\mathbb{CP}^n$.
(b) $\mathbb{RP}^n$.
(c) The orientable surface Σ_g of genus g.
(d) The non-orientable surface Σ'_g of genus g.

You can use the following facts without proving them.

(a) If X is a CW complex,the complex $C_*^{\mathrm{CW}}(X) \otimes_{\mathbb{Z}} M$ computes the homology groups of X with coefficients in M.
(b) If X is a CW complex, the dual complex of the complex $C_*^{\mathrm{CW}}(X)$ computes the cohomology groups of X with coefficients in M.

2. Let X be a finite connected graph with v vertices and e edges.

(a) Show that $H_1(X;\mathbb{Z})$ is a free Abelian group of rank $g = e - v + 1$. Its automorphism group is therefore $GL_g(\mathbb{Z})$.

(b) Let G be a finite group of homeomorphisms of X. Assuming that all vertices of X have degree > 1, show that the map

$$\Psi : \begin{cases} G \to \mathrm{GL}_g(\mathbb{Z}) \\ f \mapsto H_1(f) \end{cases}$$

is one to one. You may start with the case where X is a wedge sum of circles.

(c) Show that this result still holds when $\mathbb{Z}$ is replaced by $\mathbb{Z}/m\mathbb{Z}$ with $m > 2$. What happens if $m = 2$?

3. Let X be a topological space. Deduce from the short exact sequence of complexes

$$0 \to C_*(X) \overset{\times n}{\to} C_*(X) \to C_*(X;\mathbb{Z}/n\mathbb{Z}) \to 0$$

a short exact sequence of abelian groups

$$0 \to H_i(X)/n \to H_i(X,\mathbb{Z}/n\mathbb{Z}) \to n-\mathrm{torsion}(H_{i-1}(X)) \to 0.$$

4. Let X be a finite connected CW-complex and $\pi : \widehat{X} \to X$ a Galois covering with group $\mathbb{Z}$.

(a) Show that $C_*^{CW}(\widehat{X})$ and $H_*(\widehat{X})$ (resp. $C_*^{CW}(\widehat{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $H_*(\widehat{X},\mathbb{Q})$) have a natural structure of module over $\mathbb{Z}[t,t^{-1}]$ (resp. $\mathbb{Q}[t,t^{-1}]$).

(b) Show that $\mathbb{Q}[t,t^{-1}]$ is a principal ideal domain (PID).

(c) Show that $\dim_{\mathbb{Q}}(H_*(\widehat{X},\mathbb{Q}))$ is finite if and only if $H_*(\widehat{X},\mathbb{Q})$ is torsion over $\mathbb{Q}[t,t^{-1}]$.

(d) Study the example $X = (S^1 \times [0,1])/_{(z,1)\sim(z^2,0)} = (S^1 \times \mathbb{R})_{(z,t+1)\sim(z^2,t)}$, $\widehat{X} = S^1 \times \mathbb{R}$. Show that (iii) does not hold if we replace $\mathbb{Z}$ by $\mathbb{Q}$ and $\dim_{\mathbb{Q}}$ by $\mathrm{rank}_{\mathbb{Z}}$.