ENS de Lyon **Master 1 – Algebraic topology**
TD8 **Spring 2024**

The functors Tor **and** Ext

We use the following definitions: if M, N are two $\mathbb{Z}$-modules and $0 \to \mathbb{Z}^{(J)} \xrightarrow{i} \mathbb{Z}^{(I)} \to M \to 0$ is a free resolution, one defines

$$\operatorname{Tor}(M,N) := \ker(i_N : N^{(I)} \to N^{(J)})$$
$$\operatorname{Ext}(M,N) := \operatorname{coker}(i_N^*) = N^J / i_N^* N^I,$$

where $i_N : N^{(I)} \to N^{(J)}$ and $i^*N : N^I \to N^J$ are associated to the presentation. One can show that $\operatorname{Tor}(M,N)$ and $\operatorname{Ext}(M,N)$ are unique up to canonical isomorphism.

If M is a $\mathbb{Z}$module, one denotes $T(M)$ its torsion submodule.

1. Show that $T(M) = \ker(M \to M \otimes_{\mathbb{Z}} \mathbb{Q})$.

2. Show that $\operatorname{Ext}(M, \mathbb{Q}) = 0$.

3. (i) Let
$$0 \to N' \to N \to N'' \to 0$$
be an exact sequence of $\mathbb{Z}$-module and M be a $\mathbb{Z}$-module. Prove (if it is not already known) that we have an exact sequence
$$0 \to \operatorname{Tor}(M,N') \to \operatorname{Tor}(M,N) \to \operatorname{Tor}(M,N'') \to \operatorname{Hom}(M,N'') \to M \otimes N' \to M \otimes N \to M \otimes N'' \to 0.$$

(ii) Deduce that $\operatorname{Tor}(M,N) = \operatorname{Tor}(N,M)$ (canonical isomorphism).

(iii) Prove that $\operatorname{Tor}(M, \mathbb{Q}/\mathbb{Z}) = T(M)$ (canonical isomorphism).

4. (i) Let
$$0 \to N' \to N \to N'' \to 0$$
be an exact sequence of $\mathbb{Z}$-modules and M be a $\mathbb{Z}$-module. Prove that we have an exact sequence
$$0 \to \operatorname{Hom}(M,N') \to \operatorname{Hom}(M,N) \to \operatorname{Hom}(M,N'') \to \operatorname{Ext}(M,N') \to \operatorname{Ext}(M,N) \to \operatorname{Ext}(M,N'') \to 0.$$

(ii) Show that $\operatorname{Ext}(M, \mathbb{Z}) = \operatorname{Hom}(T(M), \mathbb{Q}/\mathbb{Z})$.

(iii) If M is finitely generated, show that $\mathrm{Ext}(\mathrm{Ext}(M, \mathbb{Q}/\mathbb{Z})), \mathbb{Q}/Z) = T(M)$ (canonical isomorphism).

5. Let $0 \to N \to E \to M \to 0$ be a short exact sequence of $\mathbb{Z}$-modules. It is called an *extension* of M by N

(i) If $\mathbb{Z}^{(J)} \to \mathbb{Z}^{(I)} \to M$ is a free resolution of M, viewed as an extension of M by $\mathbb{Z}^{(J)}$, show that there is a chain map from it to the extension $N \to E \to M$ which is unique up to chain homotopy. **More explicitly, show that there exists a commutative diagram**

$$\begin{array}{ccccc} \mathbb{Z}^{(J)} & \xrightarrow{i} & \mathbb{Z}^{(I)} & \xrightarrow{p} & M \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \mathrm{Id}_M} \\ N & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & M \end{array}$$

and that φ_1 is determined up to an element $u \circ i$ where $u \in \mathrm{Hom}(\mathbb{Z}^{(I)}, N)$. Deduce that this determines an element $e(N \to E \to M) \in \mathrm{Ext}(M, N)$.

(ii) Define the notion of isomorphism of extensions of M by N.

(iii) Define the sum of two extensions of M by N and the multiplication of an extension by a scalar. Show that the isomorphism classes of extensions of M by N form a $\mathbb{Z}$-module $\mathcal{E}(M, N)$. **This question is rather hard**

(iv) Show that e induces an isomorphism $\mathcal{E}(M, N) \to \mathrm{Ext}(M, N)$.

6. A $\mathbb{Z}$-module I is said to be *injective* when the functor $\mathrm{Hom}(-, I)$ is exact.

(i) Show that I is injective if and only if for any injective morphism $N \to M$ and any morphism $f : N \to I$, there is a morphism $M \to I$ which extends f.

(ii) Show that if I is injective, the multiplication by any non-zero integer is a surjective map $I \to I$ (we say that I is *divisible*).

(iii) Using Zorn's lemma, show that a divisible $\mathbb{Z}$-module is injective.

(iv) Deduce that the $\mathbb{Z}$-modules $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective.

(v) An *injective resolution* of N is an exact sequence

$$0 \to N \to I_0 \to I_1 \to \cdots$$

such that every I_i is injective. We admit the following (hard) theorem which is due to Eilenberg and Cartan: if we denote by C^* the complex

$$\mathrm{Hom}(M, I_0) \to \mathrm{Hom}(M, I_1) \to \cdots$$

then,

$$H^i(C^*) = \begin{cases} \mathrm{Hom}(M,N) & \text{if } i = 0 \\ \mathrm{Ext}(M,N) & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that, for any M, the homology groups of the complex

$$0 \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(M,I_0) \to \mathrm{Hom}(M,I_1) \to \cdots$$

is independent from the resolution.

(vi) Deduce that if I is injective, we have $\mathrm{Ext}(M,I) = 0$ for all M.