ENS de Lyon **Master 1 – Algebraic topology**
TD8 **Spring 2024**

Corrigé de l'exercice montrant que les extensions de M par N sont classifiées par $\mathrm{Ext}(M,N)$

1. Remark. This exercise remains valid if $\mathbb{Z}$ is replaced by any principal ideal domain.

(i) We want to find a commutative diagram

$$\begin{CD}
\mathbb{Z}^{(J)} @>{i}>> \mathbb{Z}^{(I)} @>{p}>> M \\
@V{\varphi_1}VV @V{\varphi_0}VV @V{\mathrm{Id}_M}VV \\
N @>{\alpha}>> {}' @>{\beta}>> M
\end{CD}$$

i.e. we want to solve

$$\beta \circ \varphi_0 = p$$
$$\alpha \circ \varphi_1 = \varphi_0 \circ i.$$

We can find φ_0 since β is onto and $\mathbb{Z}^{(I)}$ is free. Given φ_0, since $\beta \circ \varphi_0 \circ i = p \circ i = 0$, the image of $\varphi_0 \circ i$ is contained in $\ker \beta = \mathrm{im}\ \alpha$. Since α is injective, there is a unique φ_1 such that $\varphi_0 \circ i = \alpha \circ \varphi_1$.

Moreover, φ_0 is defined modulo a morphism $v : \mathbb{Z}^{(I)} \to \ker \beta = \mathrm{im}\ \alpha$. Since α is injective, $v = \alpha \circ u$ with $u \in \mathrm{Hom}(\mathbb{Z}^{(I)}, N)$. Thus φ_1 is defined modulo $u \circ i$, thus

$$e(N \to E \to M) := [\varphi_1] \in \mathrm{Hom}(\mathbb{Z}^{(J)}, N)/(\mathrm{Hom}(\mathbb{Z}^{(J)}, N) \circ i) = \mathrm{coker}(i_N^*) = \mathrm{Ext}(M,N)$$

is well-defined.

(ii) Clearly, an isomorphism between two extensions $N \to E \to M$ and $N \to E' \to M$ is a commutative diagram

$$\begin{CD}
N @>{\alpha}>> E @>{\beta}>> M \\
@V{\varphi_1}V{\approx}V @V{\varphi_0}V{\approx}V @V{\mathrm{Id}_M}VV \\
N @>{\alpha'}>> E' @>{\beta'}>> M
\end{CD}$$

(iii) The sum $(N \to E \to M) \oplus (N \to E' \to M)$ is $N \to \widetilde{E} \to M$ with

$$\begin{aligned}
\widetilde{E} &= \frac{\{(x,x') \in E \oplus E' \mid \beta(x) = \beta'(x')}{(i \times (-i'))(N)} \\
\widetilde{\alpha}(n) &= [(\alpha(n),0)] = [(0,\alpha'(n')] \\
\widetilde{\beta}([(x,x')]) &= \beta(x) = \beta'(x').
\end{aligned}$$

If $a \in \mathbb{Z}$, one defines $a(N \to E \to M) = (N \to \widetilde{E} \to M)$ with

$$\begin{aligned}
\widetilde{E} &= \frac{N \oplus E}{(a \times (-\alpha))(N)} \\
\widetilde{i}(n) &= [an, 0)] = [(0, \alpha(n))] \\
\widetilde{\beta}([(n,x)]) &= \beta(x).
\end{aligned}$$

One checks that we do obtain short exact sequences, that the sum is commutative and associative up to isomorphism, and that $a(b(N \to E \to M))$ is isomorphic to $(ab)(N \to E \to M)$.

Finally, the trivial extension $N \to N \oplus M \to M$ is neutral modulo isomorphism, and the inverse (or opposite) of $(N \to E \to M)$ up to isomorphism is $(-1)(N \to E \to M)$, or more simply $N \to E \to M$ with α replaced by $-\alpha$ and β unchanged. Thus we obtain a structure of $\mathbb{Z}$-module on $\mathcal{E}(M,N)$.

Remark. For the ring $\mathbb{Z}$, it is not necessary to define the multiplication by a scalar, since it follows from the addition and the inverse.

(iv) In the notations of (i), we want to prove that 1) $[\varphi_1]$ can be any element of $\mathrm{Ext}(M,N)$, and 2) ($[\varphi_1] = 0 \Rightarrow E$ is trivial)

1) Let $\varphi \in \mathrm{Hom}(\mathbb{Z}^{(J)}, N)$. We define

$$\begin{aligned}
E &= \frac{N \oplus \mathbb{Z}^{(I)}}{(\varphi \times (-i))(N)} \\
\alpha(n) &= [(\varphi_1(n), 0)] = [(0, i(n))] \\
\beta([n,x]) &= p(x) \\
\varphi_1 &= \varphi : \mathbb{Z}^{(J)} \to N \\
\varphi_0(x) &= [x].
\end{aligned}$$

Then we have a commutative diagram in which the second line is a short exact sequence:

$$\begin{array}{ccccc}
\mathbb{Z}^{(J)} & \xrightarrow{\;i\;} & \mathbb{Z}^{(I)} & \xrightarrow{\;p\;} & M \\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \mathrm{Id}_M} \\
N & \xrightarrow{\;\alpha\;} & E & \xrightarrow{\;\beta\;} & M.
\end{array}$$

2) If $[\varphi_1] = 0$, there exists $u : \mathbb{Z}^{(I)} \to N$ such that $\varphi_1 = u \circ i$. If $m \in E$, $m = p(x)$ for some $x \in \mathbb{Z}^{(I)}$, and $(\varphi_0 - \alpha \circ u)(x)$ depends only on m: if $p(x) = 0$, then $x = i(y)$ thus

$$(\varphi_0 - \alpha \circ u)(x) = (\varphi_0 - \alpha \circ u) \circ i(y) = \alpha \circ (\varphi_1 - u \circ i)(y) = 0.$$

Thus $m \mapsto (\varphi_0 - \alpha \circ u)(x)$ gives a section of $N \to E \to M$, and thus extension is trivial.