SOLUTIONS OF THE EXAM – April 24, 2025

Exercise 1. Are the following statements true or false? (No justification required).

(a) In the Euclidean space $(\mathbb{R}^4, g_0)$, there exists an embedding $\iota : \mathbb{T}^2 \hookrightarrow \mathbb{R}^4$ such that $(\mathbb{T}^2, \iota^* g_0)$ is flat (i.e. the Riemann tensor of $\iota^* g_0$ vanish identically).

(b) Every Riemannian submanifold of a flat Riemannian manifold is flat.

(c) If (M, g) is a complete Riemannian manifold, and $x, y \in M$ are two sufficiently close points, there exists a unique geodesic $\gamma : [0, 1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

(d) Let (M, g) be a Riemannian manifold such that, for some $x \in M$ and for some local coordinates around x, the Christoffel symbols of the Levi-Civita connection vanish at x. Then the Riemann tensor of (M, g) vanishes at x.

(e) If $\gamma : [0, 1] \hookrightarrow \mathbb{R}^n$ is a smooth embedded curve, then there exists a Riemannian metric g on $\mathbb{R}^n$ for which γ is a geodesic.

Solution.

(a) True. The product of two flat Riemannian manifolds is flat, and every one-dimensional Riemannian manifold is flat. Hence, $(\mathbb{T}^2, \iota^* g_0)$ is flat for any embedding of the form

$$\iota = (\iota_1, \iota_2) : \mathbb{T}^2 = S^1 \times S^1 \hookrightarrow \mathbb{R}^4,$$

where $\iota_i : S^1 \hookrightarrow \mathbb{R}^2$ are embeddings.

(b) False. For every $n \geq 2$, the unit sphere $S^n \subset \mathbb{R}^{n+1}$ has positive Gaussian curvature, while $\mathbb{R}^{n+1}$ is flat. Moreover, by Nash's isometric embedding theorem, any Riemannian manifold can be isometrically embedded as a submanifold of some Euclidean space.

(c) False. On the round sphere, geodesics are great circles. Any two points (not antipodal) are joined by two distinct geodesic segments.

(d) False. In any Riemannian manifold (M, g) one can choose geodesic normal coordinates at x, in which the Christoffel symbols vanish at x, regardless of the value of the Riemann tensor there.

(e) True. Since γ is embedded, it admits a compact tubular neighbourhood $U \subset \mathbb{R}^n$ and a diffeomorphism onto its image $\phi : U \to \phi(U) \subset \mathbb{R}^n$ such that $\phi \circ \gamma(t) = (t, 0)$. Then $\phi \circ \gamma$ is a geodesic for the Euclidean metric g, so γ is a geodesic for the pullback metric $\phi^* g$ on U. Finally, we can extend $\phi^* g$ to the whole $\mathbb{R}^n$.

Exercise 2. Let (M, g) be a Riemannian manifold with Riemann tensor R, and let $\lambda > 0$ be a constant. What is the relation between the sectional curvature of (M, g) and the one of $(M, \lambda g)$?

Solution. The manifolds (M, g) and $(M, \lambda g)$ share the same Levi–Civita connection, and hence the same $(1,3)$–tensor R. If $v, w \in T_x M$ are orthonormal with respect to g, then

$$K_{\lambda g}(\mathrm{span}\{v, w\}) = \frac{\lambda\, g\big(R(v,w)w, v\big)}{\|v\|^2_{\lambda g}\|w\|^2_{\lambda g} - \lambda^2 g(v,w)^2} = \frac{1}{\lambda}\, g\big(R(v,w)w, v\big) = \frac{1}{\lambda}\, K_g(\mathrm{span}\{v, w\}).$$

Exercise 3. Let (M, g) be a Riemannian manifold, and let $\gamma : [0, L] \to M$ be a geodesic such that $\|\dot{\gamma}\|_g \equiv 1$ and $d_g(\gamma(0), \gamma(L)) = L$. Is it possible that, for some $\ell \in (0, L)$, there exists a geodesic $\zeta : [0, \ell] \to M$ such that $\|\dot{\zeta}\|_g \equiv 1$, $\zeta(0) = \gamma(0)$, $\zeta(\ell) = \gamma(\ell)$, and $\zeta \neq \gamma|_{[0,\ell]}$?

Solution. No. If such a ζ existed, then at $t = \ell$ we would have $\dot{\zeta}(\ell) \neq \dot{\gamma}(\ell)$, since two distinct geodesics cannot have the same tangent vector. Then the piecewise smooth path

$$\sigma(t) = \begin{cases} \zeta(t), & t \in [0, \ell], \\ \gamma(t), & t \in [\ell, L], \end{cases}$$

would join $\sigma(0)$ to $\sigma(L)$ with length $L = d_g(\gamma(0), \gamma(L))$. This is a contradiction, since σ is not a geodesic, whereas any length-minimizing segment must be a geodesic.

Exercise 4. Let (M, g) be a closed Riemannian manifold, where as usual closed means "compact without boundary". Fix a point $x \in M$, and define a function $f : M \setminus \{x\} \to (0, \infty)$ by $f(y) = d_g(x, y)$, where $d_g : M \times M \to [0, \infty)$ is the Riemannian distance. Can f be a C^1 function?

Solution. We show that f cannot be C^1. Since M is compact, f attains a maximum at some point $y \in M \setminus \{x\}$, with $L := f(y)$. Assume by contradiction that f is C^1. In particular, y must be a critical point of f, meaning that $df(y) = 0$. However, let $\gamma : [0, L] \to M$ be a geodesic parametrized with unit speed $\|\dot\gamma\|_g \equiv 1$ joining $x = \gamma(0)$ and $y = \gamma(L)$. Notice that $f(\gamma(t)) = t$, and therefore

$$\|df(y)\|_g \geq df(y)\,\dot\gamma(L) = \frac{d}{dt}\Big|_{t=L} f(\gamma(t)) = 1,$$

which contradicts $df(y) = 0$.

Exercise 5. Let (M,g) be a Riemannian manifold, $N \subset M$ an embedded submanifold, and $x \in M \setminus N$. Let Ω be the space of smooth curves $\gamma : [0,1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) \in N$. Characterize the critical points of the energy functional

$$E : \Omega \to [0,\infty), \qquad E(\gamma) = \frac{1}{2}\int_0^1 g(\dot\gamma(t),\dot\gamma(t))\,dt.$$

Namely, characterize the smooth curves $\gamma \in \Omega$ such that $\frac{d}{ds}\big|_{s=0} E(\Gamma_s) = 0$ for all smooth homotopies

$$\Gamma : (-\epsilon,\epsilon) \times [0,1] \to M, \qquad \Gamma(s,t) = \Gamma_s(t),$$

such that $\Gamma_0 = \gamma$ and $\Gamma_s \in \Omega$ for each $s \in (-\epsilon,\epsilon)$.

Solution. We denote by $Y := \partial_s \Gamma_s|_{s=0}$ the vector field along γ associated with the homotopy Γ. Notice that $Y(0) = 0$ and $Y(1) \in \mathrm{T}_{\gamma(1)}N$. We compute the derivative

$$\begin{aligned} 0 = \tfrac{d}{ds}\big|_{s=0} E(\Gamma_s) &= \int_0^1 g(\nabla_s \partial_t \Gamma_0(t)|_{s=0}, \dot\gamma(t))\,dt = \int_0^1 g(\nabla_t Y, \dot\gamma(t))\,dt \\ &= \int_0^1 \Big(\tfrac{\mathrm{d}}{dt} g(Y(t),\dot\gamma(t)) - g(Y(t),\nabla_t\dot\gamma)\Big)\,dt = g(Y(1),\dot\gamma(1)) - \int_0^1 g(Y(t),\nabla_t\dot\gamma)\,dt. \end{aligned} \tag{1}$$

This holds for all homotopies Γ with the properties stated in the exercise, and therefore for every vector field Y along γ such that $Y(0) = 0$ and $Y(1) \in \mathrm{T}_{\gamma(1)}N$. In particular, since the equality holds for all such Y so that $Y(1) = 0$, we readily see that

$$\nabla_t \dot\gamma \equiv 0. \tag{2}$$

Therefore, equality (1) also implies that

$$g(w,\dot\gamma(1)) = 0, \qquad \forall w \in \mathrm{T}_{\gamma(1)}N. \tag{3}$$

Conversely, (2) and (3) readily imply equality (1). We conclude that the critical points of $E : \Omega \to [0,\infty)$ are precisely the geodesics $\gamma : [0,1] \to M$ such that $\dot\gamma(1)$ is orthogonal to the submanifold N.

Exercise 6.

(i) Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function with a critical point at the origin, i.e. $du(0) = 0$. Consider the graph of u, which is the surface $\Gamma_u = \{(z, u(z)) \mid z \in \mathbb{R}^2\}$ embedded in the Euclidean space $\mathbb{R}^3$. Compute the Gaussian curvature of Γ_u at the origin in terms of the function u.

(ii) Let M be a closed surface embedded in $\mathbb{R}^3$, and $r > 0$ the smallest radius such that the round ball $B_r = \{q \in \mathbb{R}^3 \mid \|q\| \leq r\}$ contains M. Provide a lower bound for the Gaussian curvature of M at any point $q \in M \cap \partial B_r$.

Solution.

(i) If $q = (z, u(z)) \in \Gamma_u$, the tangent space $\mathrm{T}_q\Gamma_u$ is precisely the graph

$$\Gamma_{du(z)} = \big\{(v, du(z)v) \bigm| v \in \mathbb{R}^2\big\}.$$

The upward-pointing unit-normal vector field $N = (N_1, N_2, N_3) : \Gamma_u \to \mathbb{R}^3$ is uniquely defined by $N_3 > 0$, $\|N\| \equiv 1$, and $N_1(q)v_1 + N_2(q)v_2 + N_3(q)du(q)v = 0$ for all $v = (v_1, v_2) \in \mathbb{R}^2$, which gives

$$N(q) = \Big(-\tfrac{\nabla u(z)}{(1+\|\nabla u(z)\|^2)^{1/2}}, \tfrac{1}{(1+\|\nabla u(z)\|^2)^{1/2}}, \Big),$$

where ∇u denotes the gradient of u. Consider the diffeomorphism $\iota : \mathbb{R}^2 \to \Gamma_u$, $\iota(x, y) = (x, y, u(x, y))$. Since $du(0) = 0$, the matrix of $d(N \circ \iota)(0)$ is given by

$$d(N \circ \iota)(0) = \begin{pmatrix} -\partial_{xx}u(0) & -\partial_{xy}u(0) \\ -\partial_{yx}u(0) & -\partial_{yy}u(0) \\ 0 & 0 \end{pmatrix}.$$

The curvature of Γ_u at the origin is given by the determinant of the Hessian of u at 0, i.e. $K_{\Gamma_u}(0) = \partial_{xx}u(0)\partial_{yy}u(0) - \partial_{xy}u(0)^2$.

(ii) We consider the round sphere $S_r := \partial B_r$. Since $r > 0$ is the smallest radius such that $M \subset B_r$, at every intersection point $q \in M \cap S_r$ we must have $\mathrm{T}_qM = \mathrm{T}_qS_r$. Let us fix one such intersection $q \in M \cap S_r$. Without loss of generality (up to applying an isometry of $\mathbb{R}^3$), we can assume that $q = (0, 0, -r)$ is the south pole of S_r, so that $\mathrm{T}_qM = \mathrm{T}_qS_r = \mathrm{span}\{\partial_x, \partial_y\}$. On a neighborhood of the point q, the sphere S_r coincides with the graph Γ_u of the function

$$u(x, y) = -\sqrt{r^2 - x^2 - y^2},$$

whereas the surface M must coincide with the graph Γ_v of some function v such that $v(q) = u(q) = -r$ and $v(p) \geq u(p)$ for all p on a neighborhood of the origin. We already know that the sphere S_r has constant curvature r^{-2}; we can also reobtain this result at the point q by means of point (i): the Hessian matrix of u at the origin is given by

$$\begin{pmatrix} r^{-1} & 0 \\ 0 & r^{-1} \end{pmatrix},$$

and the curvature is its determinant, i.e. $K_{S_r}(q) = r^{-2}$. Since $v(0) = u(0)$ and $dv(0) = du(0) = 0$, we have

$$v(q) = \tfrac{1}{2}\mathrm{d}^2v(0)[q, q] + o(|q|^2), \quad u(q) = \tfrac{1}{2}\mathrm{d}^2u(0)[q, q] + o(|q|^2), \quad \text{as } q \to 0.$$

Since $v \geq u$, we must have

$$\mathrm{d}^2 v(0)[q,q] \geq \mathrm{d}^2 u(0)[q,q] = r^{-1}\|q\|^2. \tag{4}$$

Let H be the Hessian matrix of v at 0, i.e. the symmetric 2×2 matrix defined by $\langle H\cdot,\cdot\rangle = \mathrm{d}^2 v(0)$. Equation (4) implies that both the eigenvalues λ_1 and λ_2 of H are larger than or equal to r^{-1}, and therefore

$$K_M(q) = \det H = \lambda_1 \lambda_2 \geq r^{-2}.$$