ATQUA – Advancing the Theory of Quantitative Universal Algebra

Country	University or Institution	Last name	First name	Current Position	Role in the project	PM[1]
France	CNRS & ENS Lyon[2]	Mio	Matteo	CRCN	coordinator at ENS Lyon	18
France	ENS Lyon	Pagani	Michele	Professor	participant	4
France	ENS Lyon	Pistone	Paolo	Prof. Junior	participant	15
France	CNRS & ENS Lyon	Pous	Damien	DR2	participant	8
France	ENS Lyon	Riba	Colin	MdC	participant	12
France	CNRS & ENS Lyon	Vignudelli	Valeria	CRCN	participant	4
France	ENS Lyon	Postdoc 1			postdoctoral researcher	12
France	ENS Lyon	Postdoc 2			postdoctoral researcher	12
France	ENS Lyon					Total = 85
Germany	FAU[3]	Milius	Stefan	Professor	coordinator at FAU	18
Germany	FAU	Schröder	Lutz	Professor	participant	6
Germany	FAU	Urbat	Henning	Research Associate[4]	participant	18
Germany	FAU	Wild	Paul	Lecturer[5]	participant	6
Germany	FAU	PhD			doctoral researcher	36
Germany	FAU					Total = 84

1. PM is the involvement of a member in person-months over the 36 months period of the project.
2. ENS Lyon is a shorthand for: École normale supérieure de Lyon.
3. FAU is a shorthand for: Friedrich-Alexander-Universität Erlangen-Nürnberg.
4. Independent research position ('Eigene Stelle') funded by DFG until October 2025, afterwards Lecturer.
5. German 'Akademischer Rat'.

1 Context Positioning and Objectives

Program equations, expressing behavioural equality between syntactically different programs, are ubiquitous in computer science. They are at the heart of program semantic definitions, transformations, optimizations, etc. The mathematical field of universal algebra (UA) constitutes a theoretical background for these topics.

However, in many applications a more fine grained reasoning that allows to express that two programs, while behaviourally different, have very similar behaviours is more desirable. Quantitative universal algebra (QUA), a generalization of UA recently proposed by Mardare, Panangaden and Plotkin in influential work [50], provides the mathematical foundations for this reasoning via quantitative program equations. In QUA, informal similarity is captured quantitatively and numerically by the mathematical concept of metric distance. For example, one could take a program and start replacing some lines of code with different, but sufficiently near ones under an appropriate metric, and expect the final outcome to be sufficiently near to the original program in the metric. Quantitative algebraic methods also appear to be very promising in the context of AI or IoT. In these settings, it is desirable to expect reliable results even when inputs, parameters, sensor data, etc., are affected by some small perturbations, e.g. due to noise.

Since the initial paper [50], QUA has received considerable attention and is now an active topic of research in theoretical computer science [1, 2, 5–8, 32, 33, 51, 78]. The two research groups involved in this ANR PRCI project have both contributed, separately, with a number of technical advancements presented at top conferences in the field [22, 31, 36, 41, 46, 59, 61–64, 86, 87].

The main goal of ATQUA is to stimulate a synergy between the two research groups and advance the theory and applications of quantitative universal algebra. Our main objectives are outlined in Section 1.a below, and the project is organized into six work packages (WPs A–F). The current state of the art, which constitutes the basis of the project, is presented in Section 1.b.

The methodologies, the planning and the risk management techniques required to achieve the goals of the project are described in Section 1.c. Besides the activity of the permanent researchers, the research will be carried out by one or two postdocs (for a total of 24 months) at ENS Lyon and one PhD student at FAU (for a total

of 36 months). The costs associated with hiring the PhD student and postdocs constitute the vast majority of the budget requested for the project. All spending lines are presented in detail in Section 2.b.

1.a Objectives and Research Hypothesis

The fundamental idea behind QUA, first presented by Mardare et al. [50], is to replace the standard equality relation $(=)$, which is at the core of UA, with a collection of binary relations $=_\varepsilon$, one for each $\varepsilon \in \mathbb{R}_{\geq 0}$, with the interpretation that $x =_\varepsilon y$ means that the 'distance' between x and y is at most ε (i.e. $d(x, y) \leq \varepsilon$). Indeed, models of QUA are algebras $\mathbb{A} = (A, \{f^{\mathbb{A}}\}_{f\in\Sigma})$ for a signature Σ of function symbols, just as in UA, further endowed with an (extended) metric $d_A \colon A \times A \to \mathbb{R}_{\geq 0} \cup \infty$ that is suitably compatible with the operations. Judgments between terms of the form $s =_\varepsilon t$ are called *quantitative equations*. The key contributions of op. cit. are a sound and complete deductive system for reasoning about quantitative equations, similar to Birkhoff's classical equational logic [13], and a proof that free quantitative algebras exist, for every quantitative equational theory, again generalizing a fundamental standard result of UA. In addition, several examples of quantitative theories inducing interesting free quantitative algebras are given.

Since that seminal work by Mardare et al. [50], a significant amount of research has explored this apparatus [1, 2, 5–8, 22, 31–33, 36, 41, 46, 51, 59, 61–64, 78, 86, 87]. In particular, the applicants' groups at ENS Lyon and FAU have, separately, worked out generalizations of the framework of QUA. For instance: (1) the distances d_A are not required to be metrics but can be arbitrary fuzzy relations and the interpretations $f^{\mathbb{A}}$ of the operations are not required to be nonexpansive [63], (2) the distances can take values in arbitrary lattices (e.g. quantales) rather than just the familiar $[0, 1]$ or $\mathbb{R}_{\geq 0} \cup \infty$ [78], (3) in fact, rather than distances, it is possible to consider arbitrary relations whose properties are axiomatized by an infinitary Horn theory [29, 41]. These generalizations, besides widening the applicability of QUA, also highlight and isolate the key mathematical ideas from what are merely *ad hoc* or *parametric* choices, such as working with distances valued in $[0, 1]$ or $\mathbb{R}_{\geq 0} \cup \infty$.

The objective of ATQUA is to advance the theory of quantitative algebra, from a foundational point of view but also with a look towards applications in computer science, especially in the area of program semantics and formal verification. In more detail, our aims are organizes along the following three axes; each of the axes is associated with two work packages of the project:

Syntactical Aspects of Quantitative Algebra Here we are interested in formal deductive systems to reason about quantitative equations with a particular focus towards mechanization, reasoning algorithms, and proof techniques. As we have mentioned already, a sound and complete deductive system for QUA is known. However, it contains an infinitary proof rule, which is problematic for mechanization and algorithmic treatments. One aim of ATQUA is thus to investigate *compact* theories in which the infinitary rule is not needed (WP A). Another aim along this axis concerns logical definability in quantitative algebra (WP B). In this strand of work we will develop axiomatizations by quantitative equations of certain properties of (operations of) quantitative algebras, e.g. types of continuity or properties of the underlying metric space such as compactness or convexity.

Categorical Aspects of Quantitative Algebra As the backbone for our theory of quantitative algebra we will develop flexible and generic mathematical foundations. This will be achieved by using the methods and language of category theory. In particular, the category-theoretic notion of a *monad* plays a fundamental role (Section 1.b.2). Studying quantitative algebras as algebras for a monad on the category metric spaces will allow us to draw from results in categorical universal algebra to advance the theory of quantitative algebra, but also to obtain generic results which will generalize beyond the metric setting, e.g. to algebras over sets equipped with a relational structure (WP C). In addition, structural operational semantics, that is, the specification of program constructs by well-behaved operational rules, has a categorical foundation in Turi and Plotkin's mathematical operational semantics (Section 1.b.3). Our objective is to build on and extend these foundations towards a theory for the specification of well-behaved program constructs for quantitative systems (WP D).

Applications to Semantics and Verification With the mathematical foundations for the semantical structures in place and the syntactic calculi at hand, we will move towards applications of our theoretical framework for quantitative algebra. One focus is the development of up-to techniques, which entail more efficient methods for reasoning about behavioural distances (WP E). Within our new quantitative mathematical operational semantics this will lead to easily checked rule formats that entail more efficient proof systems for checking upper bounds on behavioural distances of programs (WP D). In addition, work on quantitative apartness will yield proof systems for establishing lower bounds on behavioural distances with applications to explainability for quantitative systems and approximate learning of quantitative automata (WP F).

In summary, we aim for a comprehensive body of results and applications that will turn quantitative algebra into a fully fledged theory offering a flexible and generic category-theoretic mathematical foundation, a toolbox for the mechanizable deductive and algorithmic treatment of specifications with quantitative aspects, as well as support for specifying well-behaved programming languages for systems with metric structure and efficient reasoning about the upper and lower bound for the distance of the behaviour of programs.

1.b Position of the Project as it Relates to the State of the Art

Before we describe our work packages in detail in Section 1.c, we proceed to discuss the relevant state of the art.

1.b.1 Syntactical Aspects of Quantitative Algebra

The very first article on quantitative universal algebra [50] includes a sound and complete proof system for reasoning about quantitative equations. More precisely, the judgments dealt with by this proof system are called *quantitative inferences* in [50] and are Horn implications between quantitative equations. Subsequently, narrower proof systems only dealing with *basic* quantitative inferences (i.e., quantitative equations where variables are constrained by given distances) have been developed [29, 63] by researchers of both ENS Lyon and FAU. These narrowed down proof systems can actually be used to reason about distances that are not metrics such as, for instance, diffuse metric spaces where points can have strictly positive self-distance $d(x, x) > 0$.

These proof systems serve as a fundamental tool for establishing interesting axiomatizations. For example, in [50] a quantitative axiomatization of the all-important Kantorovich metric (on finite probability distributions) and of the Hausdorff metric (on finite subsets) is given. Subsequently, ENS Lyon members have provided axiomatizations for the Kantorovich-Hausdorff metric [64] (on finitely generated convex sets of distributions) and natural variants [61] (for example, involving subdistributions rather than distributions). Bacci et al. [5] have demonstrated how from axiomatizations such as the above, it is possible to derive axiomatizations of simple process algebras (e.g., for probabilistic transition systems) involving recursion. Bacci et al. [8] have also investigated how to combine such axiomatizations, for example in the case where they have disjoint signatures. In another related direction, Gavazzo et al. [33] have started developing a theory of metric rewriting systems, refining cornerstone results of classical rewriting theory such as the Church-Rosser theorem to a quantitative setting.

Additionally, foundational results in the model theory of quantitative algebras have been achieved. Mardare et al. [51] proved a quantitative version of Birkhoff's classical variety theorem [13], which characterizes classes of quantitative algebras axiomatizable by various forms of quantitative equations by suitable closure properties. This result has been extended and refined by two of the applicants (Milius and Urbat) and Jurka [41] to algebras over arbitrary relational structures given by an infinitary Horn theory, with quantitative algebras as a special case.

1.b.2 Categorical Aspects of Quantitative Algebra

In the 1960s, researchers in category theory discovered that core concepts of classical universal algebra (such as algebraic structures, terms, equational theories, varieties) admit a convenient abstraction based on the notions of *Lawvere theories* [4, 47] and *monads* [26, 49]. The latter emphasize a semantic view of universal algebra, abstracting away from unwieldy syntactic manipulations at the level of terms and equations. For example, for every regular cardinal κ, equationally axiomatizable classes of κ-ary algebras (those with operations of arity less than κ) correspond precisely to κ-ary monads (those preserving κ-directed colimits) on the category of sets. Similarly, κ-ary algebras presentable by less then κ generators and relations can be characterized as those algebras whose covariant hom functor into the category of sets is κ-ary. Such correspondence results pave the way to studying properties of algebras, or classes of algebras, by purely categorical means. The tight interplay between monads and equational theories has proven highly useful in the study of computational effects [65, 69].

It has been recognized early on that the theory of quantitative algebras can benefit from a monad-based perspective in the same way. For example, the work on combining quantitative axiomatizations [8] mentioned in Section 1.b.1 is elegantly approached in terms of sums and tensors of quantitative monads, that is, monads over the category of metric spaces. Overall, the theory of quantitative monads is far more involved than its classical counterpart, and the relation between quantitative monads and equational theories is only partially understood. In the quantitative world, equational theories are generalized to *κ-basic quantitative equational theories* (given by axioms of the form $x_i =_{\varepsilon_i} y_i \ (i < \kappa) \vdash s =_\varepsilon t$ where x_i, y_i are variables), which are interpreted over κ-ary quantitative algebras. Similar to classical universal algebra, the key goal is a full characterization of such theories

in terms of categorical properties of their induced free-algebra monads. So far, this has only been achieved for the case $\kappa = \omega_1$: there is a bijective correspondence between ω_1-basic theories and quantitative monads that are metric-enriched, ω_1-ary, and preserve surjections [1]. In fact, this correspondence generalizes to enriched monads over symmetric monoidal closed categories that are locally κ-presentable as a closed category [74].

However, most quantitative equational theories of practical importance are *finitary*: they are interpreted over finitary quantitative algebras and involve ω-basic quantitative equations. While it is known that ω-basic theories induce quantitative monads [6], the precise characterization of those monads is an open problem; only the restricted special case of monads over *ultra*metric spaces has been settled [1, 2]. The general case can be expected to be very subtle and challenging – for instance, monads induced by ω-basic theories need not even be finitary [1, 29]. From a categorical perspective, the difficulty of handling finitary quantitative equational theories originates in the fact that unlike the category of sets, the category of metric spaces is not locally finitely presentable. This means that the techniques used in the analysis of monads for ω_1-basic theories do not apply.

1.b.3 Mathematical Operational Semantics

Structural operational semantics (SOS) is concerned with the specification of transition system semantics for programming languages and for process algebras by means of operational rules of a certain shape. A key prerequisite for modular reasoning about programs is *compositionality*: A semantics is compositional if the associated semantic equivalence forms a congruence, that is, subterms of a given program term or process may be replaced with equivalent subterms without affecting the overall denotational meaning of the term. For instance, the classical GSOS format by Bloom et al. [14] for rules specifying process languages interpreted over non-deterministic labelled transition systems guarantees that (strong) bisimilarity is compositional. More abstractly, GSOS is captured in Turi and Plotkin's bialgebraic framework of *mathematical operational semantics* [81], in which sets of operational rules are represented as distributive laws of a monad representing the syntax of a given language over a copointed functor representing the next-step behaviour; this is also termed *abstract GSOS*. The generic category-theoretic setting allows one to extend the principles of (G)SOS to other semantic settings than the one of classical process algebra, such as weighted and stochastic transition systems [42], name passing process calculi [28], and calculi combining non-determinism with weights [56].

In previous work, one of the applicants (Milius) has done fundamental work on modular recursive specifications within the framework of abstract GSOS with Moss and Schwencke [58], as well as work on specifications of operations (or program constructs) which preserve regularity of behaviour with Bonsangue et al. [18, 57]. Recently, a highly successful line of investigation into extensions of mathematical operational semantics has been performed by two of the applicants (Milius and Urbat) and others, on the one hand extending the principles of abstract GSOS to imperative programming languages with variable store [38] and on the other hand to higher-order languages such as combinatory logics and λ-calculi [39, 83], the latter solving a long-standing open problem.

1.b.4 From Behavioural Equivalence to Behavioural Distances

The theory of state-based systems of various types can be treated uniformly within the categorical framework of *universal coalgebra* [77]. In this framework, the type of transitions of a state-based system is encapsulated by a set functor, or, more generally, by an endofunctor on a suitable base category. By varying the functor, the abstract notion of a coalgebra instantiates to a wide range of state-based systems, including classical labelled transition systems and (non-)deterministic, weighted, probabilistic, or nominal automata, as well as to systems mixing several effects, such as Markov decision processes, which combine non-determinism and probabilistic branching. Coalgebras come with a canonical notion of *behavioural equivalence* defined by identifiability of states under coalgebra morphisms; this instantiates to well-known forms of behavioural equivalence such as Milner [60] and Park's [67] strong bisimilarity for labelled transition systems, its weighted or probabilistic analogues, language equivalence for deterministic automata, etc.

However, it has long been realized that for quantitative systems, behavioural equivalence is too strong, and one seeks for quantitative comparisons between of states in the form of *behavioural distances*, expressing how much the observable behaviour of two states differs. Behavioural distances have been studied, for instance, for probabilistic transition systems [25, 35, 85], systems combining nondeterminism and probability [19], or metric transition systems [23, 27]. In the coalgebraic framework, one systematic approach to behavioural distances uses liftings of set functors to the category of pseudometric spaces, briefly *metric liftings* [9]. The idea here is that

for every metric lifting $\bar{F}$ of a set functor F, the forgetful functor from the category of all $\bar{F}$-coalgebras to that of all F-coalgebras has a right-adjoint which equips a given F-coalgebra with a behavioural pseudometric. An alternative coalgebraic approach to behavioural distance uses *lax extensions* [45,80], which extend the action of a set functor to quantitative relations such that certain axioms are satisfied. Every lax extension gives rise to a notion of *quantitative bisimulation*, generalizing the standard notion of bisimulation as a relation that is compatible with the transition structure [32, 76, 86, 88]. The notion of behavioural distance then arises as the greatest quantitative bisimulation. The two approaches via metric lifting and lax extensions are closely related [36].

In practice, actually proving that two system states are behaviourally equivalent, or more generally proving upper bounds to their behavioural distance, is usually a non-trivial task, and substantial research has been devoted to developing effective proof methods. One highly successful approach is provided by *up-to* techniques [70]. For instance, in a transition system carrying algebraic structure (such as a transition system of terms of a process algebra or programming language), a bisimulation R *up to context* only requires that related states transition into states related by the context closure of R, rather than R itself. Such relations are potentially much smaller and easier to construct than full bisimulations. The prequisite to using any up-to technique is its soundness, which ensures that every bisimulation up-to extends to a bisimulation.

In the quantitative setting of behavioural *distances*, up-to techniques are far less explored. Chatzikokolakis et al. [20] introduce quantitative up-to techniques for probabilistic automata. Bonchi et al. [15] study up-to techniques for coalgebraic behavioural distances at the generality of quantale-valued relations, building on earlier work on coalgebraic fibrational bisimulations [16, 75]. Their framework applies to several quantitative up-to techniques (such as behavioural distances up to metric closure or up to context) and gives abstract criteria for their soundness. Due to their fairly technical nature, verifying those criteria in a concrete instance can be challenging in itself.

1.c Methodology and Risk Management

The goal of the ATQUA project is to advance the theory and the applications of quantitative algebra and to stimulate the synergy between the research groups at ENS Lyon and FAU, which collectively include 10 permanent researchers. In order to achieve this, we have split the research into six work packages (WPs) which we have allocated to researchers on both the French and German side.

The tables in Sections 2.a.1 and 2.a.2 indicate the engagement of the members of the ENS Lyon and FAU teams, respectively, in each of the work packages. By design, no single work package will be investigated by only one of the French and German teams. Collaborations will mostly be carried out remotely, but we will organize regular mutual research visits. We plan for a yearly meeting involving the funded (post-)doctoral researchers and some of the permanent researchers of about one week length. In addition, we envisage longer research stays, of up to six weeks, of the funded doctoral researcher (at FAU) to ENS Lyon and of the postdocs (at ENS Lyon) to FAU. We will also organize, towards the end of the project, an international workshop to disseminate our results to the research community. This workshop will naturally also serve as the final meeting event.

Work Package A: Finite Proofs and Compactness

At the core of universal algebra is a sound and complete deductive system, due to Birkhoff [13], which allows to derive judgments of the form $E \vdash s = t$ where E is a set of equations, with the meaning 'every algebra that satisfies all equations in E also satisfies the equation $s = t$'. Since Birkhoff's proof system can be seen as a fragment of the deductive apparatus of classical first order logic, every proof is a finite (both in width and depth) tree. This is a key property, and it is fundamental when it comes to applications of universal algebra in computer science: proofs, being finite objects, can be represented and manipulated by machines, they can be enumerated, and there are effective procedures to check whether a given derivation tree is a valid proof or not.

Since the very first article on quantitative universal algebra [50], a sound and complete proof system, extending that of Birkhoff, has been investigated for deriving judgments of the form $E \vdash s =_\varepsilon t$, that involve quantitative equations[6] (see Section 1.a) in lieu of equations. However, unlike Birkhoff's system, the deductive system for quantitative algebras is based on the infinitary first order logic $L_{\omega_1,\omega}$ [40] which, crucially, allows countable conjuctions. As a consequence, a proof in quantitative algebra is, in general, not a finite object: it is a well founded

[6] This is a simplification sufficient to explain the intents of this WP, avoiding technicalities. More precisely, as discussed in Section 1.b.1, the judgments of the proof system are called *quantitative inferences* [50] and are Horn implications involving quantitative equations.

tree of countable width. The source of proofs of infinite width is the presence of the infintary rule

$$\frac{E \vdash s =_{\varepsilon_0} t \qquad E \vdash s =_{\varepsilon_1} t \qquad \dots \qquad E \vdash s =_{\varepsilon_n} t \qquad \dots}{E \vdash s =_{\varepsilon} t} \; \varepsilon = \inf\{\varepsilon_i\}_{i \in \mathbb{N}} \tag{1}$$

which states that if the distance between s and t is bounded above by ε_i (i.e. $d(s,t) \leq \varepsilon_i$), for all $i \in \mathbb{N}$, then it is necessarily also bounded by the infimum of all bounds ($d(s,t) \leq \varepsilon$). The non-finiteness of proofs in quantitative equational logic is an obstacle to its mechanization.

It can be shown that the presence of an infinitary rule, such as the above rule (1), is required in any sound and complete proof system for quantitative algebra. From a categorical perspective, this is due to the category $\mathbf{Met}$ of metric spaces not being locally finitely presentable [3]. However, it is still possible for some well-behaved quantitative equational theories E to satisfy the following property:

$$E \vdash s =_{\varepsilon} t \qquad \text{iff} \qquad \text{there is a finite proof not using the infinitary rule (1).}$$

We refer to such quantitative equational theories E as *compact*. A variant of this notion, which is also worth exploring, is given by theories whose proofs may use the infinitary rule but in a 'recursively uniform' way, i.e. where there is a recursive scheme that describes all countably many subproofs of the premises. Perhaps surprisingly, several of the main examples of quantitative equational theories that have appeared in the literature are compact. One example is the quantitative theory of *interpolative barycentric algebras* E_{IB} [50, §10], also known as *convex algebras*. Under the axioms E_{IB} of this theory, terms s and t correspond to finitely supported probability distributions Δ_s and Δ_t over a metric space (A, d_A) and the distance between terms being axiomatized is the Kantorovich lifting $K(d_A)$ of d_A [24]. It is well-known that an upper bound $K(d_A)(\Delta_s, \Delta_t) \leq \varepsilon$ on the Kantorovich distance is witnessed by a coupling Δ of Δ_s and Δ_t; importantly, couplings are finite objects (because they are finitely supported probability distributions) and an optimal coupling always exists. Informally, the theory E_{IB} is compact because, for any given terms s and t and an optimal coupling Δ of Δ_s and Δ_t, it is possible to construct a finite derivation tree whose content is the finite information provided by Δ; thus, the derivation does not need the rule (1). Further examples of compact theories are the quantitative theories of semilattices [50, §9] and convex semilattices [64], for analogous reasons.

We aim to explore the theory of compact quantitative equational theories. Despite the naturalness of this class of theories, and its clear significance in the context of computer science, to the best of our knowledge, no systematic study has been done so far. We will investigate the following key questions:

(1) Can we identify syntactic conditions on the axioms of a theory E that ensure compactness of E?

(2) What are the relations between logical compactness (as we have formulated it for theories E) and metric/topological compactness? For example, observe that for the compact theory E_{IB} of interpolative barycentric algebras mentioned above, the corresponding Kantorovich monad $M_{E_{IB}}$ has the well-known property of mapping compact metric spaces to compact metric spaces. This property is shared also by the Hausdorff monad [50] and convex powerset monad [50], corresponding to the quantitative theories of semilattices and convex semilattices.

Work Package B: Logical Definability in Quantitative Algebra

A significant amount of research on quantitative algebra has so far focused on identifying quantitative theories whose free algebras are naturally interesting mathematical objects and useful in the context of program semantics. For example, free algebras of the quantitative theory of interpolative barycentric algebras [50, §10] can be identified as the metric spaces of finitely supported probability distributions endowed with the Kantorovich metric.

Less attention has been given to the study of the expressive power, in the context of quantitative algebra, of different logical formalisms such as quantitative equations, the infinitary logic $L_{\omega_1,\omega}$, its fragments (e.g. finitary first-order logic or the existential fragment), or (monadic) second order logics. A number of initial results have already been obtained in this line of research, including Birkhoff-type variety theorems by Mardare at al. [51] as well as by Jurka and two of the applicants [41] that characterize those classes of quantitative algebras that are axiomatizable by various types of quantitative equations (Section 1.b.1).

The goal of this work package is to investigate questions related to logical definability. The questions we will address include:

(1) What type of continuity properties of the algebraic operations can be axiomatized by quantitative equations? Since the first works on quantitative algebras, starting with that by Mardare at al. [50], it has been observed

that nonexpansiveness of the algebraic operations can be imposed by basic quantitative equations of the form $x_1 =_\varepsilon y_1, \ldots, x_n =_\varepsilon y_n \vdash f(x_1, \ldots, x_n) =_\varepsilon f(y_1, \ldots, y_n)$. In a recent article by researchers of ENS Lyon [63], the apparatus of quantitative algebras has been generalized to allow for more permissive, not necessarily nonexpansive operations. For example, the class of c-Lipschitz continuous functions, for a real constant $c > 0$, can be expressed by quantitative equations of the form $x_1 =_\varepsilon y_1, \ldots, x_n =_\varepsilon y_n \vdash f(x_1, \ldots, x_n) =_{c\cdot\varepsilon} f(y_1, \ldots, y_n)$. Even more generally, Hölder continuity can be expressed in a similar way.

What is the most general continuity condition definable by quantitative equations, or by formulas of standard or infinitary first order logic $L_{\omega_1,\omega}$? While proving that a given property is definable just requires a witness formula, some more advanced techniques are needed to prove impossibility results. Standard methods from classical model theory based on the compactness theorem are not immediately applicable, since the underlying theory of quantitative algebras is not compact. We plan to investigate and develop such methods; see also (2) below.

(2) The machinery of two-player games, such as Ehrenfeucht-Fraïssé and bisimulation games, is a central tool in model theory. It can be used, for example, to establish that two models are indistinguishable by a certain class of formulas or that a certain property is not expressible. Furthermore, games underpin fundamental results in automata theory and model checking. We will investigate the applicability of games in the context of quantitative algebras, for instance, to establish apartness relations (see also (4) below and WP F) or to prove that certain properties, like the continuity conditions explored in (1), are not expressible by quantitative equations. From a mathematical point of view, this line of research is enabled by the fact that the theory of two-player games is well behaved in the context of the infinitary logic $L_{\omega_1,\omega}$ (see e.g. Hodges [40, §3.5] and Libkin [48, §8] for applications in finite model theory) which underlies quantitative algebra. In addition to two-player games, we will also explore several other classical techniques and ideas from model theory in the context of quantitative algebra. For instance, quantifier elimination and categoricity are useful tools for establishing decidability of logical theories. Stability theory, types, and companions are other examples [40].

(3) We will use the results from (2) above to establish definability results relevant in quantitative algebra. For example, is it possible to axiomatize the class of quantitative algebras whose underlying metric space satisfies some property, e.g. is compact, connected, path-connected, convex, injective, or exponentiable?

(4) Is it possible to axiomatize the negation of the basic $x =_\varepsilon y$ relation? This negation expresses the constraint that $d(x, y) > \varepsilon$. Clearly, $\neg(x =_\varepsilon y)$ is axiomatizable in the full infinitary logic $L_{\omega_1,\omega}$, since that logic has negation, and $=_\varepsilon$ is axiomatizable in the Horn fragment of $L_{\omega_1,\omega}$. It is, however, desirable to find finer characterizations and ideally design a sound and complete proof system specifically tailored to the quantitative setting. A key application of the proof system will be the study of quantitative apartness developed in WP F.

(5) Most research on quantitative algebra has been based on a classical, boolean-valued logical apparatus. However, recently it has been argued [7] that a real-valued logical foundation for quantitative algebra could be convenient. In a similar spirit, a line of research in model theory has focused on *continuous logic* (see Yaakov and Usvyatsov [89] for an introduction). Continuous logic features formulas valued in $[0, 1]$, and its models are complete metric spaces. We plan to explore the applicability of continuous logic towards the development of quantitative algebra, along the research directions discussed above in the context of classical logic.

At a higher level, the goal of this work package is to better understand what kind of logical languages are needed, useful, and convenient for reasoning about quantities and metric properties. This line of 'linguistic' research might turn out to be relevant in the currently very active topic of *explainability* in AI, where the goal is to describe (by means of a sentence in some carefully crafted language) how an AI model makes its decisions.

Work Package C: Monads and General (Category) Theory

As highlighted in Section 1.b.2, many of the foundational 'syntactical' results in the theory of quantitative algebras and logics, such as completeness of proof systems or variety theorems, come naturally accompanied with a more abstract point of view provided by monads over the category of metric spaces, or related structures such as complete metric spaces, generalized metric spaces, or quantale-valued relations. One important goal of ATQUA is to advance the theory of such monads. Notably, we expect that the investigation of the research problems stated in WPs A and B will strongly benefit from a categorical (as opposed to purely proof-theoretic or model-theoretic) approach. The core type of questions we will pursue in this WP are:

> Given *syntactical/logical conditions* on the equations of a quantitative theory E, what properties of the induced free-algebra monad M_E do they correspond to?

One relevant instance arises in the context of our planned investigation of compact quantitative equational theories. Since compactness is a finiteness property at the syntactic level of proof trees, it should be reflected by suitable finiteness properties at the level of monads. For example, if E is a compact theory, then what are sufficient conditions (on top of compactness) on E to ensure that the monad M_E is finitary, or satisfies some related property such as preservation of directed unions of isometric embeddings? One hint in this direction is given by the result by two of the applicants (Milius and Urbat) with Adámek and Moss [3] that the Hausdorff monad $H\colon \mathbf{CMet} \to \mathbf{CMet}$ (where $\mathbf{CMet}$ is the category of complete metric spaces) is finitary and that H can be obtained by the 'completion' of the quantitative theory of semilattices [50, §9], which is compact. We believe that this result is no coincidence and aim to extend it to general compact theories.

Another interesting instance of the above question is related to the difficult open problem of classifying the monads corresponding to ω-basic theories. These monads are known to be finitary in the special case of 1-basic (that is, unconditional) equations, but need not be finitary for general ω-basic equations [29]. A simple counterexample, discovered by one of the applicants (Urbat) and mentioned in *op. cit.*, is given by the theory consisting of the single equation $x =_1 y \vdash x =_0 y$. The non-finitarity of the induced monad is caused by the discontinuity introduced by the 'jump' from distance 1 to distance 0 together with the fact that no strengthening $x =_{1+\varepsilon} y \vdash x =_0 y$ of the equation is derivable in the theory. This suggests to explore the class of *continuous equation schemes* introduced in the original paper on quantitative algebra [50], which are essentially quantitative equational theories without jumps. Our initial conjecture is that continuous schemes generate finitary monads.

We will also approach the finitarity problem from the opposite direction: Is there a form of equational presentation that captures *precisely* the (enriched) finitary monads over metric spaces, or natural subclasses of them? Previous work on ω_1-ary monads over relational structures [29], and more generally κ-ary monads over locally κ-presentable categories [73], suggests that such a presentation requires departing from quantitative algebras with operations of discrete arities (natural numbers) to operations whose arities are arbitrary finite metric spaces X. An X-ary operation is interpreted over an algebra A as a map of type $\mathbf{Met}(X, A) \to A$, that is, as a partial n-ary operation (for $n = |X|$) that is only defined on arguments of suitable distance. Non-discrete ω-basic theories always generate enriched monads, but like in the discrete case the monads are generally only ω_1-ary. It seems conceivable that the gap between ω and ω_1 is given precisely by continuity: Are finitary enriched monads those presentable by ω-ary non-discrete continuous equation schemes? A positive answer would be a substantial step towards the characterization of monads for general ω-basic theories.

All the above questions can be studied beyond the setting of metric spaces, namely for categories of relational structures or even for general enriched categories. Addressing them at this level of abstraction might not only help isolate the underlying categorical patterns, but also extend the scope and possible applications of our prospective results. For example, a more abstract approach may encompass structures such as *partial metric spaces* [52], which support non-exact quantitative reasoning and are not instances of generalized metric spaces earlier studied by some of the applicants (Mio and Vignudelli [62, 63]). Additionally, we aim to capture notions of quantitative algebras where distances are not given by real numbers, but more complex objects such as elements of a quantale [72], probability distributions [54], or differentials [21]. The latter arise in the theory of behavioural distances and differential analysis for programs in effectful higher-order languages.

Work Package D: Quantitative Mathematical Operational Semantics

Here we will depart from purely algebraic quantitative reasoning and move to a *bi*algebraic setting in which the relation between syntactic (algebraic) and semantic (coalgebraic) aspects of transition systems and programming languages with quantitative features is studied. The core objective is the development of a quantitative version of mathematical operations semantics. The original framework of mathematical operational semantics (Section 1.b.3) gives a categorical account of rule formats, dubbed abstract GSOS, for inductively specifying transition systems on program or process terms. All specifications in abstract GSOS are automatically compositional, which means that behavioural equivalence of terms (corresponding to strong bisimilarity in the case of labelled transition systems) is a congruence.

In the quantitative setting, the focus shifts from the behavioural equivalence to behavioural distances of states (Section 1.b.4). Moreover, the role of congruence relations on a quantitative algebra (A, d_A) is typically played by pseudometrics p on A such that all operations are non-expansive w.r.t. p and such that $p(a, a') \leq d_A(a, a')$ for every pair a, a' of elements of A. In some applications, however, more flexible notions of quantitative congruence might be desirable, such as Lipschitz continuity instead of nonexpansivity w.r.t. p, or a relation between p and d_A

of the form $p(a, a') \leq f(d_A(a, a'))$ for some fixed and suitably continuous map f on the reals. We will introduce a metric GSOS format guided by the principle that well-behaved sets of operational rules correspond to distributive laws, in this case between (lifted) monads and copointed functors on the category of pseudometric spaces; the use of *pseudo*metrics accounts for the fact that syntactically different terms may have distance 0. While for standard GSOS over the category of sets the correspondence between distributive laws and concrete syntactic rule formats ultimately boils down to the Yoneda lemma, the metric case will be more intricate. The previous work on the equational presentation of quantitative monads [41, 62, 63] along with the contributions of WP C will provide the key technical tools required for the analysis of distributive laws involving quantitative monads. In fact, paralleling the goals of WP C, we will develop the theory of distributive laws not only at the level of pseudometric spaces, but for base categories that allow for more permissive notions of behavioural distance, such as generalized metric spaces [62, 63] or even general relational structures [41]. We then expect to obtain general compositionality results guaranteeing that operations on program terms are non-expansive with respect to the behavioural distance on the coalgebra on program terms induced by a metric GSOS specification.

The compositionality of metric GSOS has the potential to greatly simplify and systematize reasoning about behavioural distances of given terms. It enables a proof calculus where metric GSOS rules induce the quantitative algebra on program terms (with nonexpansivity properties of the operations ensured by compositionality of the metric GSOS format) and the apparatus of quantitative equational logic provides generic proof rules used for establishing upper bounds on behavioural distances. This separation between statements provided by the rule format and statements derivable in the logic will contribute towards a modular and mechanizable reasoning. In particular, since nonexpansivity properties now come for free and no longer need to be proven from scratch, this can lead to shorter and cleaner proofs.

The results on metric GSOS may also pave the way to further investigations into coarser notions of behavioural distance. Specifically, we aim to look into quantitative versions of trace semantics (observing the sequence of outputs generated by a system) or termination semantics (observing only the final output of a system). This line of work will be inspired by recent advances in non-quantitative mathematical operation semantics [38, 39, 83], which provide new and highly parametric methodology for establishing congruence properties of weak (bi)simulations, and demonstrate that trace and termination semantics can be treated using the principles of abstract GSOS.

Work Package E: Quantitative Up-To Techniques

In addition to the logical approach in WP D, another approach to proving upper bounds to behavioural distances is given by quantitative up-to techniques. While such techniques are a well-studied and classical concept in the non-quantitative setting, the quantitative extension is still in its infancy (Section 1.b.4). We will apply the quantitative (bi)algebraic techniques developed in previous WPs to improve and simplify the development of quantitative up-to techniques, partly building on the categorical foundations laid by Bonchi et al. [15].

The existing work on coalgebraic up-to techniques [15, 16, 75] so far has been based on *liftings* of behaviour functors to categories of relational structures (e.g. pseudometric spaces or sets equipped with a binary relation) and structure-preserving maps. However, for many behaviour functors of interest, this is more general than necessary: their liftings are induced by *lax extensions* to categories whose objects are sets and whose morphisms are (certain) relations [36]. For example, this is the case for the power set functor with its Hausdorff lifting, and for the distribution functor with its Kantorovich lifting. Studying bisimulations at the level of lax extensions bears some advantages; notably, it allows considering bisimulations between different systems, and even systems of different types [66], and more importantly for our purposes, it may facilitate the study of up-to techniques. For instance, it is known that for lax extensions to the category of non-quantitative relations over sets, coinduction up to transitive closure is always sound [37], unlike the lifting-based setting where this is a non-trivial property. Starting from this observation, we will extend this result to the metric setting and to up-to techniques beyond transitive closure. We strive for abstract soundness criteria that connect to those of Bonchi et al. [15] for liftings but are easier to verify.

Among the up-to techniques studied in the literature, *up to context* is arguably the most powerful, and its soundness tends to be the hardest to establish. From a categorical perspective, the soundness property amounts to proving that the distributive law that connects the syntax and semantics of a bialgebra lifts to quantitative relations. This condition is categorically simple and natural but is usually not easy to verify in concrete settings. Towards a simplification, we will explore equational aspects of lifted behaviour functors. Mio et al. [63] have shown that every monad on generalized metric spaces that lifts a finitary monad over sets has a quantitative equational presentation. An analogous result can be expected to hold for lifted finitary functors (rather than monads) by

restricting to *flat* equational presentations, which only use terms of height 1. Building on this, we will develop a theory of quantitative equational presentation of distributive laws over metric spaces, taking inspiration from the work of Bonsangue et al. [17] on non-quantitative distributive laws. Such presentations may greatly facilitate the construction of liftings of non-quantitative to quantitative distributive laws and thus facilitate proving the soundness of behavioural metrics up to context in the abstract coalgebraic framework.

One relevant application of these results lies in the bialgebraic setting of quantitative mathematical operational semantics (WP D), where the semantic property of soundness of up-to techniques (in particular of up-to context) should be reflected by syntactic properties of the rules of a metric GSOS specification. Our goal is to identify suitable syntactic restrictions on the rule format that guarantee soundness from the outset, e.g. by satisfying the abstract soundness criteria of Bonchi et al. [15]. For instance, in classical GSOS, one such restriction is given by the *cool* rule format, which from a categorical point of view enables up-to-context reasoning for *weak* bisimulations because it induces a lax bialgebra [16]. We will take cool GSOS as a starting point and explore its quantitative extensions. Overall, we strive for a family of syntactic rule formats that are well-behaved w.r.t. quantitative up-to methods, derived in a principled way from categorical foundations.

Work Package F: Quantitative Apartness

Substantial work in programming language theory and process algebra (and in the more abstract world of coalgebras) has been devoted to come up with effective methods for proving behavioural equivalence. But how to systematically prove *non*-equivalence? Geuvers and Jacobs [34] have begun a study of the theory of *apartness*, a notion that is dual to bisimilarity, for classical labelled transition systems and, more generally, for coalgebras. Roughly, the idea is that two states in a transition system are apart iff they are not bisimilar. The point is that the ensuing notion of cobisimulation and apartness (i.e. the smallest cobisimulation) are of an inductive nature and so lead to finite proofs of apartness of states. Turkenburg et al. [82] have taken this idea a step further and given an apartness definition for coalgebraic behavioural equivalence (in lieu of bisimilarity). Note here that bisimilarity and behavioural equivalence are often equivalent; however, in cases where they are not, the latter is the more well-behaved notion. In fact, for every finitary set functor, it leads to a finitary proof system for apartness: states can be shown to be behaviourally apart using a finite proof tree [82]. Apartness has also been used by Vaandrager et al. [84] for a new efficient approach to active automata learning of Mealy machines.

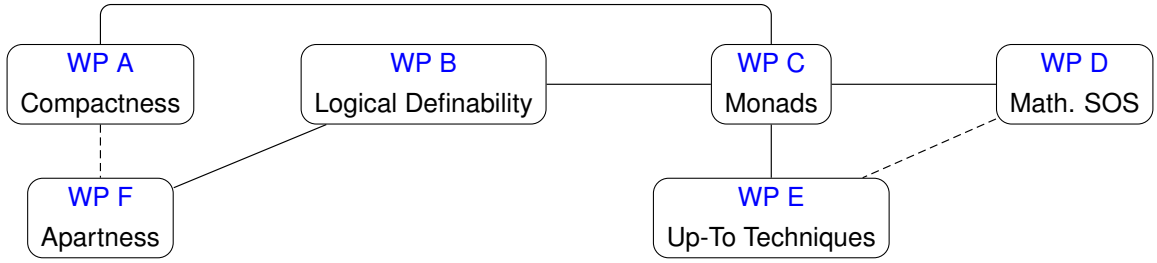
Here we will move to quantitative apartness. While proving behavioural equivalence generalizes to proving upper bounds on the behavioural distance in the quantitative setting, apartness amounts to proving lower bounds: given two states x and y of a quantitative transition system, one aims to show statements like $d(x, y) \geq \varepsilon$ or $d(x, y) > \varepsilon$ for the behavioural metric d. As in the non-quantitative setting, we expect that quantitative apartness has an inductive description as a least fixed point. This will lead to a finitary proof system for quantitative apartness, based on the proof system for negated assertions $\neg(x =_\varepsilon y)$ developed in WP B. An interesting question concerns the connection of this proof system to coalgebraic modal logics [68, 79], the standard specification logics for coalgebras, which generalize the standard Hennessy-Milner logic for Kripke models and its expressive completeness w.r.t. bisimilarity. Coalgebraic modal logics have recently been extended to capture behavioural distances [30, 36, 43, 44, 86]. Even in the non-quantitative setting, it is presently unclear how exactly the proof system for apartness by Turkenburg et al. [82] relates to coalgebraic modal logic, specifically, how a finite proof of apartness can be used to construct a distinguishing modal formula. We will study that relation and then the ensuing question concerning quantitative coalgebraic modal logics and apartness proofs.

As an important application of our quantitative apartness calculus and its relation to quantitative modal logic, we will study how it supports *explainability* in the context of quantitative systems. For labelled Markov chains, van Breugel et al. [71] proposed an approach to explainability based on modal logic: lower bounds on behavioural distances of states are justified in terms of an approximating sequence of distinguishing modal formulae. Our proof system for quantitative apartness my serve as an alternative basis for constructing (potentially shorter and simpler) formal arguments for why two states in a system behave differently, and how differently they behave.

A further possible application of quantitative apartness lies in the approximate learning of weighted or probabilistic automata. In approximate learning, the goal is to derive a model for a quantitative black-box system that is approximately correct, in the sense that it models the system behaviour up to a small error with respect to a given notion of behavioural distance. While the related subject of approximate minimization is well-studied in the context of weighted automata, e.g. [10–12], the theory of approximate learning is limited so far to fairly restricted models [53, 55]. We will look into active learning algorithms for quantitative automata from the perspective of quantitative apartness, building on the ideas by Vaandrager et al. [84].

Relationship Between the Work Packages

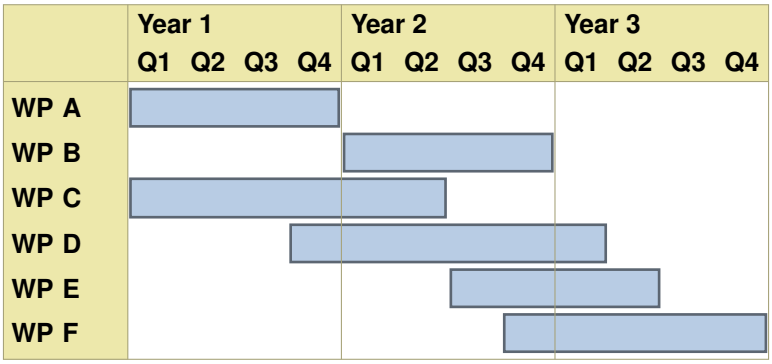
The following graph illustrates the direct relationships between the six WPs as described above:



Solid lines indicate that the results of one WP may be used for the planned work in another WP, while the dashed lines indicate a more loose coupling, e.g. via an application. However, none of these couplings is so strong that delayed progress in any of the WPs would completely block work in a different WP (cf. our risk management).

Time Planning

There are no strict dependencies between the WPs and this means that they can be executed with a high degree of parallelism. However we plan to adhere to the following Gantt chart based on the current status of preliminary ideas.



Work will begin with WPs A and C, while WPs B and D will start later during the course of the project. The WPs E and F, which pertain more to applications and might use some results of previous WPs, start latest and range into the last project year or up to the end of the project, respectively.

Risk Management

Of course, for a foundational project such as this one, it is impossible to predict which theorems will be proved and which algorithms discovered. However, the feasibility of the intended approach has clearly been demonstrated by the preliminary work leading to this project, including the previous publications by the applicants. In addition, each work package has been designed to be sufficiently independent so that, even in the improbable case of one WP failing, substantial results are still to be expected from other ones. This also allows for a good degree of parallelism and ensures that, at all times, each project participant can actively work on their research tasks. Moreover, the strategy with which work packages are allocated to the project participants will mitigate the project risks while retaining flexibility in the fine planning,. At the same time, we expect that results from one WP will provide useful results, methods, and tools for the others. This will be fostered by regular meetings between the researchers in both groups individually and the two groups together, respectively, which will allow the participants of ATQUA to interactively share ideas and results.

1.d Ability of the Project to Address the Research Issues Covered by the Chosen Research Theme

The research theme of the ATQUA project is **E.01**. According to the official ANR call, we have:

- Title: *foundations of digital sciences: computer science, automatic control, signal and image processing.*

- Short description: *this theme concerns fundamental research striving for excellence and breakthroughs in the fields of computer science, automatic control, and signal and image processing.*
- Relevant keywords: *theoretical computer science; languages and semantics; logic; formal methods; computational models; proof theory; systems theory and modelling.*

The main topic of research of the ATQUA project, *quantitative algebra*, is an active area of research in theoretical computer science with numerous recent contributions in top conferences and journals related to the keywords listed above. Furthermore, both research groups at ENS Lyon and FAU have already contributed to this area of research in a number of works, again in top conferences and journals of the field. We therefore argue that ATQUA fits the criteria of the E.01 research theme.

Furthermore, we highlight the fact that several of the WPs of ATQUA project, while theoretical, are explicitly oriented towards making the theory of quantitative algebra more practical: for example, the focus on theories allowing for finite proofs in WP A, the design of convenient structural operational rule formats for handling behavioural distances (WP D), and proof techniques for establishing apartness (WP F).

2 Organization and Implementation of the Project

The consortium is composed of two partners: École normale supérieure de Lyon (ENS Lyon) in France and Friedrich-Alexander-Universität Erlangen-Nürnberg (FAU) in Germany.

Both partners have, separately, contributed on topics related to the ATQUA project with a number of technical advancements presented at top conferences and journals in the field [22, 31, 36, 41, 46, 59, 61–64, 86, 87]. The synergy enabled by the ATQUA project will amplify the individual potential of both research teams.

2.a Scientific Coordinators and their Consortium/their Team

2.a.1 Partner 1 – ENS Lyon

The participants from the ENS Lyon side, listed below, are members of the PLUME team of the LIP laboratory of ENS Lyon, in France. Their expected relative participation in the ATQUA project is expressed in the list below (see also the table on p. 1) in terms of person-months (PM) over the course of the 36 months duration of the project.

Coordinator:
Matteo Mio (CNRS CRCN, HdR, 18 PM) has joined the CNRS and ENS Lyon in 2014 and is an expert in probabilistic logics, quantitative semantics and related formal method techniques. Some of his recent publications in top conferences are directly related to ATQUA. He has been the local coordinator (ENS Lyon site) of the ANR project REPAS (2016–2020, 166 813€) and the principal investigator of the ANR JCJC project QUAREME (2020–2025, 132 702€). He has supervised two PhD students (C. Lucas, 2019-2022, R. Sarkis 2021-2024), two postdocs (funded by ANR projects, V. Vignudelli 2020-2021 and A. Goy 2021-2022) and several master students. He has successfully obtained his HdR at ENS Lyon in April 2024. He is currently involved in two ongoing ANR projects, both ending in 2025.

(1) 2020–2025: JCJC QUAREME, involvement 80% (38.4 PM over 48 months), principal investigator.
The project has funded the organization of the QUALOG 2023 workshop, affiliated with LICS 2023, on topics related to quantitative algebra.

(2) 2020–2025: RECIPROG, involvement 10% (4.8 PM over 48 months), participant.

Other participants (permanent members):
Michele Pagani (Prof., HdR, 4 PM): expert on the theory of (probabilistic) programming languages, type systems and automatic differentiation. Participant of the ANR project Probabilistic Programming Semantics (2020–2024).

Paolo Pistone (Prof. Junior, 15 PM): expert on the theory of (probabilistic) programming languages. Junior Professor Chair at ENS-Lyon (2023-2027, ANR-23-CPJ1-0054-01). Recently used quantitative algebra to axiomatize behavioural metrics for the lambda calculus.

Damien Pous (CNRS DR2, HdR, 8 PM): expert on (co)algebra, automata, up-to techniques and related formal methods. Principal investigator of the CoVeCe ERC project (2016–2021).

Colin Riba (MdC, HdR, 12 PM): expert on logic, realizability theory, automata theory and semantics of programming languages.

Valeria Vignudelli (CNRS CRCN, 4 PM): expert on probabilistic concurrent systems and semantics. Co-authored with M. Mio several works on topics related to the ATQUA project.

Other participants (non-permanent members):
In addition to the permanent members, the fundings of the ATQUA project will be used to hire two full time (1-year) postdocs, or one full-time (2-year) postdoc, that is, 24 PM.

The expected involvement of the permanent members of the working group in the six work packages (WPs A–F) is summarized as follows (numbers in brackets are in PM):

	WP A	WP B	WP C	WP D	WP E	WP F
ENS Lyon members	Mio (8) Riba (4)	Mio (4) Pistone (3) Riba (4)	Mio (4) Pistone (3) Riba (4) Vignudelli (2)	Pagani (2) Pistone (6) Pous (2)	Pous (4) Vignudelli (2)	Mio (2) Pagani (2) Pistone (3) Pous (2)

The focus of the two postdocs within the six WPs will be determined during the execution of the project, based on their interests and skills. Naturally, the research activity at ENS Lyon will greatly benefit from interactions with the other members of the PLUME team not directly involved in the project.

2.a.2 Partner 2 – FAU

The German participants are members of the Chair for Theoretical Computer Science at the Department of Computer Science of the Technical Faculty of FAU. Their expected relative participation in the ATQUA project is expressed in the list below (see also the table on p. 1) in terms of person-months (PM) over the course of the 36 months duration of the project.

Coordinator:

Stefan Milius (Prof. Dr., 18 PM) joined the Theoretical Computer Science group at FAU in 2012 and is a tenured senior lecturer (*Akademischer Direktor*). He is an expert in the theory of coalgebras and he is known for his work on the category-theoretic approach to the semantics of iteration and recursion, in particular the category-theoretic solution of recursive program schemes. Moreover, he is one of the inventors of the categorical approach to algebraic language theory. He has published over 130 research articles in competitive refereed conferences, workshops, and journals. Some of his recent publications in top conferences are directly related to ATQUA. He has supervised five PhD students and co-supervised two postdoctoral researchers. He has been the principal investigator of several research projects funded by the German Research Foundation (DFG), the Federal Ministry of Education and Research (BMBF), and the German Academic Exchange Service (DAAD) – total coordinated volume: €3 222 000 (312 PM for funded (post-)doctoral researchers). He is currently involved as principal investigator in DFG projects as follows:

(1) 2024–2027: DFG project *Coalgebraic Nominal Automata with Name Allocation* (CoNAN), one doctoral researcher (jointly with L. Schröder, FAU)

(2) 2022–2025: DFG project *Categorical Theory of Automata* (CATHY), one doctoral researcher (jointly with H. Urbat whose current position is funded by the project, FAU)

(3) 2019–2028: DFG Research Training Group 2475 *Cybercrime and Forensic Computing* (coordinated by F. Freiling, FAU), one doctoral researcher for the duration of the RTG (nine years)

Other participants:

Lutz Schröder (Prof. Dr., 6 PM) is a full professor (W3, tenured) and the head of the Theoretical Computer Science group at FAU. He is one of the inventors of coalgebraic modal logic. Recently he has extensively worked on behavioural metrics, the semantics of metric systems using graded monads and connections to modal logics.

Henning Urbat (Dr., 18 PM) is a research associate currently funded as PI by his own DFG project until October 2025; subsequently, he will hold a fixed-term position as lecturer (*Akademischer Rat*) in the applicants' working group until October 2028, to be subsequently extended for a period of three years. He is one of the inventors of the category-theoretic approach to algebraic language theory and an expert in categorical algebra and its applications to automata theory and program semantics. In particular, he has recently worked on extensions of mathematical operational semantics.

Paul Wild (Dr., 6 PM) currently holds a fixed-term position as lecturer (*Akademischer Rat*) until June 2027, to be subsequently extended for a period of three years. He is an expert on semantics of metric systems and behavioural metrics and their connections to modal logic. For example, he has worked on metric versions of the van Benthem-Rosen as well as the Hennessy-Milner theorems.

In addition to the above group members, the funding of the ATQUA project will be used to hire a full time doctoral researcher for three years (36 PM) and a student assistant with a bachelor degree working 10 hours per week for the full duration of the project.

The expected involvement of the current members of the working group in the six work packages (WPs A–F) is summarized as follows (numbers in brackets are in PM):

	WP A	WP B	WP C	WP D	WP E	WP F
FAU members	Milius (2) Urbat (2)	Urbat (2) Wild (2)	Milius (6) Urbat (4)	Milius (4) Schröder (2) Urbat (6)	Schröder (4) Wild (4)	Milius (6) Urbat (4)

The thematic focus of the doctoral researcher will be determined during the course of the project, based on interest and skills. The student assistant is expected to work on case studies concerning compositional quantitative programming languages (WP D) and prototypical implementation and evaluation of learning algorithms (WP F). In addition, the research work at FAU will benefit from interactions with Dr. Thorsten Wißmann who holds a lecturer position within the group at FAU.

2.b Implemented and Requested Resources to Reach the Objectives

The total requested funding, as specified at the top-right corner of each page of this document, is calculated as:

$$\text{(ENS Lyon) €265 022,50} \quad + \quad \text{(FAU) €307 176} \quad = \quad \text{€572 198,50}$$

and is detailed, for each of the two partners, below.

2.b.1 Partner 1: ENS Lyon

Since most PhD students at ENS Lyon have their own fundings, the financial resources of the project will be used to hire two full-time (1-year) postdocs, or one full-time (2-year) postdoc. In the table below we have indicated the salary (as of March 2025) of postdocs, assuming they have between 2 and 7 years of research experience after their PhD. In case a suitable postdoc with less experience is hired, the salary will be lower and the difference will then be used to cover other expenses.

Beside the postdoc salaries, which constitute the vast majority of the demanded financial resources, the rest of the budget will be used to:

(1) partially cover the expenses related to missions of the 6 members and the 2 postdocs (e.g. travel costs, registration to conferences, etc), including research visits to partners at FAU. This line of budget will also partially fund (together with FAU's budget) the organization of a workshop, towards the end of the ATQUA project, to help disseminate our scientific results. €14 000/year = **€42 000**

(2) cover for "libre and open access fees" for articles accepted at conferences [7] and journals adopting publication venues free of restrictions on both access and (re-)use. €1 400/year = **€4 200**

(3) buy equipment for the 6 members and the 2 postdocs (e.g. laptops). €5 500/year = **€16 500**

(4) support the monthly "Chocola" seminars regularly taking place at ENS Lyon since 2011. Such seminars provide a great opportunity to invite external researchers and disseminate our work. €2 000/year = **€6 000**

We do not plan any other costs, such as "Building/Ground costs" or "Outsourcing/Subcontracting costs".

The following table details the exact amounts requested for each line of expenditure, and also includes the total request including the administrative costs.

All numbers have been generated using the official CNRS application (https://webcontrat.cnrs.fr/) and approved by the "service partenariat et valorisation (SPV)" of the CNRS (responsible: mohamed.hajaji@cnrs.fr).

[7] For example, the ACM (Association for Computing Machinery) has announced (see https://www.acm.org/publications/openaccess) that all papers in the Digital Library will be freely available from Jan 2026. This move will require novel article processing charges of 700 USD on published papers (e.g., at conferences such as POPL), for non-members of ACM OPEN.

12months postdoc 1 (2–7 years after PhD):	€82 400
12months postdoc 2 (2–7 years after PhD):	€82 400
Travel/missions for 6 permanent researchers and 2 postdocs and organization of a final workshop:	€42 000
Equipment and consumables for 6 permanent researchers and 2 postdocs:	€16 500
Registration to conferences and publication costs:	€4 200
Support for monthly TCS workshop “Chocolat” in Lyon:	€6 000
Total without administrative costs:	€233 500
Administrative costs	€31 522,50
Total requested funding for ENS Lyon	**€265 022,50**

2.b.2 Partner 2: FAU

The volume of the work as detailed in the work packages (WPs A–F) calls for one doctoral researcher and a student assistant (with a bachelors degree) on the FAU side funded by the DFG for the full duration of the project (36 months). For the doctoral student we calculate costs of €81 600 p.a. according to DFG personnel rates for 2025, in total $3 \times €81\,600 = \mathbf{€244\,800}$

The student assistant is expected to work for 10 hours per week; we estimate monthly costs of €842 (including social insurance) leading to total expenses of $36 \times €842 = \mathbf{€30\,312}$

In addition to staff expenses we budget for travel expenses and visiting researchers. Concerning travel expenses, we expect three European and three overseas conference trips in the course of the project (DFG only funds travel expenses for funded researchers, i.e. for the doctoral researcher), estimated at €2 050 for European trips (flight €500, five overnight stays €750, conference fee €600, per diem €200) and at €2 850 for overseas trips (flight €1 300, remaining calculation as for European trips). We will target conferences in the areas of logic in computer science (LICS, CSL, FoSSaCS, CALCO, ICALP, STACS, MFCS), concurrency (CONCUR), and programming languages (ESOP, POPL). The travel costs thus sum up to $3 \times €2\,050 + 3 \times €2\,850 = \mathbf{€14\,700}$

Concerning visiting researchers, in addition to regular visits between the French and German project partners, we will invite international collaborators for research visits of a duration of 6 days on average with costs estimated at €1 644 per visit (travel €700, accommodation €800, per diem €144). We request funding for six visits for the duration of the project, in total $6 \times €1\,644 = \mathbf{€9\,864}$

Finally, we plan to divide the costs for the final workshop between the French and German partners. Therefore we ask for financial support to cover a part of the travel expenses of invited speakers and to reduce or waive the registration costs of participants imposed by the hosting conference, in particular for participants in early career phases. In total, we ask for **€7 500**

The project costs can thus be summarized as follows:

Staff expenses	doctoral researcher	€244 800
	student assistant	€30 312
Travel expenses		€14 700
Visiting researchers		€9 864
Final workshop		€7 500
Total requested funding for FAU		**€307 176**

Note that we budget no costs for instruments and materials since, according to DFG regulations, instruments and materials that constitute core support in the relevant subject area cannot be funded (e.g. laptops for the funded researchers). Further note that, according to DFG regulations, the DFG automatically ads an overhead of 22% (*Programmpauschale*) on top of the total funded project costs requested in the proposal. Adding administrative management & structure costs (as according to French regulations) to the requested funding therefore does not count in the total funding requested by the German partners from the DFG.

3 Impact and Benefits of the Project

The research proposed in ATQUA is very timely, in fact, it is hard to imagine a more propitious time to carry out this project. Basic mathematical foundations for our work have been laid with the successful work by the two groups at ENS Lyon and FAU and others. The more widespread need for methods based on quantitative algebra has been realized by researchers and practitioners in many application domains. So the time is ripe to provide the corresponding toolbox to them. The results which will be obtained in ATQUA will make a substantial contribution to the field of quantitative algebra and its applications: a comprehensive body of methods, constructions, and a support for mechanization and algorithmic treatment, as laid out in the research objectives (Section 1.a). A particular focus will be the development of solutions that enable the wider applicability of quantitative algebra in areas such as programming language design, formal verification, and explainable artificial intelligence. We therefore expect that the research of ATQUA will generate a considerable impact within our research community as well as giving impulses beyond.

Another key benefit of the ATQUA project is to create and intensify a close collaboration which will lead to a sustained synergy between the two research groups at ENS Lyon and FAU. We expect that this will continue well beyond the 36 months duration of the project. Moreover, the systematic scientific collaboration and mutual in person visits of the two research groups during the course of the project will in particular help the non-permanent funded researchers (the two postdocs at ENS Lyon and the doctoral reasearcher at FAU) to start building a scientific network and international contacts and collaborations in addition to their regular participation at international competitive conferences.

Furthermore, ATQUA will also help to initiate and train students in doing research work through the supervision of Master level projects and theses on topics related to the work program.

Concerning the dissemination strategy: we aim to publish all results in top tier conferences (such as LICS, POPL, CONCUR, FoSSaCS, and ICALP) and competitive journals (such as Logical Methods in Computer Science, TheoretiCS and ACM ToCL). This is consistent with the choice of publication venue by both ENS Lyon and FAU members for previous results on quantitative algebra [22, 31, 36, 41, 46, 59, 61–64, 86, 87]. A clear preference will be given to *open access* venues. Prototypical tools will be made available via open repositories, e.g. Zenodo. We will also organize a workshop[8] towards the end of the project to share our results with the research community.

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