MÉMOIRE D'HABILITATION À DIRIGER DES RECHERCHES

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Recent Results on Quantitative Equational Logic

Résultats Récents sur la Logique Équationnelle Quantitative

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Preface

Starting with my PhD thesis [Mio12a] in 2012, the main focus on my research in the last 12 years has been the design and investigation of formal methods for reasoning about quantities, mostly in the form of probabilities. Within this general topic, my contributions can be roughly classified into three main groups:

1. modal and temporal logics with a real-valued semantics: e.g., [Mio12b, Mio12c, MS13, Mio14, MS17, MFM17, Mio18, LM19, LM21, LM22],

2. quantitative and numerical aspects of automata theory, mostly over infinite trees: e.g., [MM15, GMMS17, MSM18],

3. quantitative equational logic, quantitative algebras and monads in categories of metric spaces: e.g., [MV20, MSV21, MSV22, MSV23].

Of course, all three topics above are deeply correlated. For example, a question originating from (Topic 1) the *game semantics* of a real-valued logic [Mio12c] motivated the (Topic 2) study of measure-theoretic properties of regular languages of infinite tress [GMMS17]. The (Topic 1) real-valued logical characterisation of bisimulation for probabilistic nondeterministic system of [Mio13] is closely related to (Topic 3) the quantitative–equational presentation of the monad (in the category of metric spaces) of convex sets of probability distributions [MV20]. *Etcetera*.

This document presents some recent results regarding Topic 3: quantitative equational logic, quantitative algebras and monads. These have emerged in a series of joint works, starting around 2019, with my colleague Valeria Vignudelli (CNRS and ENS-Lyon) and my PhD student Ralph Sarkis (ENS-Lyon): [MV20, MSV21, MSV22]. An extended version of this document is available as [MSV23].

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1 Introduction

Equational reasoning and algebraic methods are widespread in all areas of computer science, and in particular in program semantics. Indeed, initial algebra semantics and monads are cornerstones of the modern theory of functional programming and are used to reason about inductive definitions, computational effects and specifications in a formal way (see, e.g., [PP03, RT93, Mog91, HP07]).

In the last few decades, with the growth of quantitative methods in computing (e.g., from artificial intelligence, probabilistic programming, cyber-physical systems, *etc.*) it has become evident that traditional program equations:

$$P = Q \quad \Leftrightarrow \quad \text{the programs } P \text{ and } Q \text{ have the same behavior}$$

are not always adequate to reason about the behaviour of programs that are similar, in a certain quantitative sense, but that, strictly speaking, have different behavior. Examples include programs that differ only by small perturbations in some of their numeric constants such as probabilities, values measured from noisy sensors, scalars in a neural network, *etc.* The intuitive notion of "similar in a quantitative sense", has been formally captured in many works by means of program distances $d(P, Q) \in [0, \infty]$ expressing numerically the divergence in behavior. See [GJS90, DGJP99, DJGP02, DEP02, vBW01b, vBW01a, vBW05] for a selection of seminal papers.

In a recent work [MPP16], Mardare, Panangaden and Plotkin introduced a novel abstract mathematical "framework", called *Quantitative Algebra*, which extends ordinary Universal Algebra (see., e.g., [BS81, Wec92]) and is designed to reason about distances that are metrics.[2] The standard equality judgment $(s = t)$ of Universal Algebra is replaced by *quantitative equations* $(s =_\epsilon t)$, intuitively expressing that $d(s, t) \leq \epsilon$. In the program semantics context, we thus have:

$$P =_\epsilon Q \quad \Leftrightarrow \quad \text{the difference in behavior between } P \text{ and } Q \text{ is at most } \epsilon.$$

The usual notion from Universal Algebra of *algebra* $(A, \{op^A\}_{op\in\Sigma})$ for a signature Σ, that is a carrier set A together with interpretations op^A for all function symbols $op \in \Sigma$, is replaced by that of *quantitative algebra*:

$$\big((A, d_A)\ ,\ \{op^A\}_{op\in\Sigma}\big)$$

[2] Precisely, they consider *extended metrics*, which are distances $d : X^2 \to [0, \infty]$ satisfying the following constraints for all $x, y, z \in X$: $d(x, y) = 0 \Leftrightarrow x = y$, $d(x, y) = d(y, x)$ and $d(x, z) \leq d(x, y) + d(y, z)$.

where the carrier is a metric space (A, d_A) and the interpretations op^A, for all $op \in \Sigma$, are nonexpansive maps[3].

A number of recent works have built on top of the results of the seminal [MPP16]. For a non-exhaustive list, see, e.g., [MPP17, BMPP18, MV20, MSV22, MSV21, MPP21, FMS21, BMPP21, MSV22, Adá22, DLHLP22, ADV23, GF23]. Key theoretical results in quantitative algebra include: sound and complete deductive systems, existence of free quantitative algebras generated by metric spaces, monads and composition techniques for monads in the category **Met** of metric spaces and nonexpansive maps, completion results, variety "HSP-type" theorems, *etc.* Applications of this framework can be found in the identification of useful monads in **Met** as "free quantitative algebra" monads (see, e.g., [MPP16, MV20, MSV21, MSV22]) and in the quantitative axiomatisation of behavioral metrics [BMPP18, BBLM18b, BBLM18a, MSV21].

Furthermore, some works have proposed extensions or modifications of the framework of [MPP16]. For instance, [MSV22] has considered quantitative algebras $\big((A, d_A)\ ,\ \{op^A\}_{op\in\Sigma}\big)$ where (A, d_A) is not necessarily a metric space but, more generally, a *generalised metric space*[4] (e.g., pseudometrics, quasimetrics [Wil31a], ultrametrics [BvBR98], semimetrics [Wil31b], diffuse metrics [HS00, CKPR21], rectangular metrics [Bra00], b-metrics [Cze93]). In [FMS21] this type of generalisation is pushed even further, allowing the carrier to be an arbitrary relational structure. In a different direction, in [MSV22] (see also [BBLM18b] and, in the different context of ordered algebras, [ADV22] and [AFMS21]) the authors have considered quantitative algebras where the interpretations op^A, of all $op \in \Sigma$, are not necessarily nonexpansive maps. This extends considerably the application of the theory, as witnessed by interesting examples (e.g., from concurrency theory in [BBLM18b] and artificial intelligence in [MSV22]).

1.1 Contributions

The main goal of this document is to present a generalisation of the framework of [MPP16] in a self-contained and coherent way and to prove some of the fundamental results. An extended version, containing all proofs, is available as [MSV23].

[3] More precisely, the interpretation $op^A : (A^n, d_A^n) \to (A, d_A)$ of $op \in \Sigma$ is nonexpansive, where d_A^n is the (categorical) product metric on A^n. See [MPP16] and Section 9 for a detailed discussion.

[4] The terminology "generalised metric space" has already appeared in the literature (see, e.g., [BvBR98]) with a slightly different meaning. Our notion is more general, and subsumes that of [BvBR98]. See Remark 2.33 in Section 2.3.

Following and further developing ideas from a previous paper [MSV22], we extend [MPP16] along two orthogonal lines, by considering quantitative algebras $\big((A, d_A)\ ,\ \{op^A\}_{op\in\Sigma}\big)$ where:

(1st line): the carrier (A, d_A) is an arbitrary fuzzy relation space [Zad71], that is, a set A together with an arbitrary map $d_A : A^2 \to [0, 1]$,

(2nd line): the interpretations op^A, for $op \in \Sigma$, are arbitrary set-theoretic functions, and not required to be nonexpansive.

Regarding the first line of extension, the choice of considering fuzzy relations $d_A : A^2 \to [0, 1]$ allows us to have a concrete notion of numeric distance, while still being general enough to include, e.g., all the generalised metric spaces listed above. Regarding the second line of extension, allowing op^A to be an arbitrary set-theoretic function results in greater generality in the definition.

From a logical point of view, since we work with arbitrary fuzzy relations (for which the property $x = y \Leftrightarrow d(x, y) = 0$ might not hold), we have to decouple the notion of equality from that of distance. As a result, our theory of quantitative algebras deals with two types of formal judgments: *equations* and *quantitative equations*, respectively of the form:

$$\forall(X, d_X).s = t \qquad\qquad \forall(X, d_X).s =_\epsilon t$$

where $\epsilon \in [0, 1]$ and $s, t \in \mathrm{Terms}_\Sigma(X)$ are terms built from a set of variables X which is endowed with a fuzzy relation $d_X : X^2 \to [0, 1]$. We have followed the Universal Algebra textbook [Wec92] in using the "$\forall$" symbol, in the formal judgments, to explicitly remind that the stated equality (equation) or bound on distance (quantitative equation) is universally quantified over all interpretations $\tau : X \to A$ of the variables in X.

Crucially, as the set of variables is endowed with a fuzzy relation d_X, an interpretation is required to be a nonexpansive map of type $\tau : (X, d_X) \to (A, d_A)$. The idea of restricting attention to interpretations that "preserve structure" has already appeared in literature (e.g., in [AFMS21] in the study of ordered algebras) but appears to be new in the literature about quantitative algebras. This has important consequences. Consider for example the following "gap" property:

For all x, y, if the distance between x and y is $\leq \dfrac{1}{2}$, then the distance is $\leq \dfrac{1}{4}$.

In the seminal paper [MPP16] and most subsequent works, to express the above property one needs to consider Horn implications[5] between quantitative equations:

$$x =_{\frac{1}{2}} y \Rightarrow x =_{\frac{1}{4}} y$$

and indeed the syntactic deductive apparatus presented in [MPP16] is designed to manipulate Horn implications between quantitative equations, and not just quantitative equations. In our setting, instead, the "gap" property is directly expressed by the quantitative equation:

$$\forall(\{x,y\}, d_X).x =_{\frac{1}{4}} y$$

where the fuzzy relation d_X on the set $\{x, y\}$ of variables assigns value $\frac{1}{2}$ to (x, y):

$$d_X(x,x) = 1,\ d_X(x,y) = \tfrac{1}{2},\ d_X(y,x) = 1,\ d_X(y,x) = 1.$$

Requiring interpretations to be nonexpansive then corresponds precisely to requiring that the premise of the Horn implication is satisfied. As a consequence, we are able to work just with equations and quantitative equations, thus avoiding higher-level logical concepts such as Horn implications. This is a novelty with respect to both [MPP16] and [MSV22]. See Section 9 for a detailed comparison with [MPP16].

In this new setting, we can recover the framework of Mardare, Panangaden and Plotkin as a specific "quantitative equationally" defined subclass of quantitative algebras. For instance, we can now define, by means of equations and quantitative equations, the subclass of quantitative algebras whose d_A satisfy the constraints of a chosen generalised metric (including, e.g., the standard conditions of metrics, as in Footnote 2), in the same way that, in Universal Algebra, the subclass of abelian groups can be defined equationally from groups. At the same time, it is possible to define by means of equations and quantitative equations the subclass of quantitative algebras whose op^A is nonexpansive or, more generally, Lipschitz with constant $\alpha > 1$, or other useful notions.

The following is a list of our main results (Item 1–Item 5) concerning the generalised quantitative algebra theory presented in this work:

1. We present a sound and complete "Birkhoff-style" deductive system to derive valid equations and quantitative equations. The novelty of such proof system is that it only manipulates equations and quantitative equations, rather than Horn implications.[6]

[5] In [MPP16] such implications are referred to as "quantitative inferences" and denoted by $x =_{\frac{1}{2}} y \vdash x =_{\frac{1}{4}} y$.

[6] We expand the discussion on the novelties of our system with full technical details in Section 9.

2. We show that, for any class of quantitative algebras defined by equations and quantitative equations, the free quantitative algebra generated by a fuzzy relation space (A, d_A) always exists in the class, and we give an explicit construction.

3. We prove that the adjunction induced by the free construction above is *strictly monadic*. Strict monadicity is a key property in the context of Universal Algebra, and the fact that it holds in our theory of quantitative algebras suggests that we have indeed identified an "equational" (in a categorical sense) quantitative setting.

4. We show that all monads on **FRel** which are liftings of finitary **Set** monads, i.e., **Set** monads with an equational presentation, can be presented by a given set of equations and quantitative equations. This includes most examples from the literature on quantitative algebras, e.g, the finite powerset monad with the Hausdorff metric and the probability distributions monad with the Kantorovich metric [MPP16], among others (see, e.g., [MV20, MSV21, MSV22]).

5. We prove that all the results above, stated for the category **FRel** of fuzzy relation spaces, can be restricted to any chosen category **GMet** of generalised metric spaces (including, for example, the familiar category **Met** of metric spaces).

1.2 Organisation of the Document

The rest of the document is organised as follows.

In Section 2 we provide the necessary technical background regarding Universal Algebra, Category Theory and some basic notions regarding fuzzy relations. This section is relatively lengthy as we have included all notions required to make this article self-contained.

In Section 3 we formally introduce our theory of quantitative algebras with all its key definitions: quantitative algebras, equations and quantitative equations, their semantics, quantitative theories, and so on. In Section 3.1 we enumerate our key results, which will be proved in the subsequent sections.

Our main results (Item 1–Item 5) are formally stated and proved in Sections Section 4, Section 5, Section 6, Section 7 and Section 8, respectively.

After having developed all our technical results, in Section 9 we give a formal comparison of our theory of quantitative algebras and the original one of Mardare, Panangaden and Plotkin [MPP16].

Finally, we conclude in Section 10 suggesting possible lines of future work.

2 Technical Background

In this section we provide the mathematical background needed to formally state our results and to verify the proofs. In Section 2.1 we cover material from Universal Algebra. In Section 2.2 from Category theory. And in Section 2.3 we give the necessary definitions regarding fuzzy relations and generalised metric spaces.

2.1 Universal Algebra

We recall in this section some basic definitions of Universal Algebra. We refer the reader to [BS81] and [Wec92] as standard references, the latter is specifically intended for computer scientists.

A *signature* Σ is a (possibly infinite) set of function symbols $op \in \Sigma$ each having a finite arity $ar(op) \in \mathbb{N}$. Operations of arity 0 are referred to as constants.

Definition 2.1 (Σ-algebra)**.** Given a signature $\Sigma = \{op_i\}_{i \in I}$, a Σ-*algebra* $\mathbb{A}$ is a pair of the form $\mathbb{A} = (A, \{op^A\}_{op \in \Sigma})$, where:

1. A is a (possibly empty) set,
2. $\{op^A\}_{op \in \Sigma}$ is a collection of interpretations of the operations containing, for each function symbol in Σ, a function of type: $op^A : A^{ar(op)} \to A$.

Remark 2.2. Some authors (see, e.g., the literature on universal algebra [BS81, Wec92] referenced above) consider only algebras having nonempty carriers $A \neq \emptyset$. In contrast, in this paper we admit algebras with empty carriers, following standard references in category theory such as [Mac88]. While there is no substantial difference between such approaches, the latter is more general and it allows us to state results in a more uniform way. For example, by including the algebra with empty carrier we have that all categories of algebras have an initial object (i.e., the free object generated by $\emptyset$, see Definition 2.18).

Definition 2.3 (Homomorphism)**.** Given two Σ-algebras $(A, \{op^A\}_{op \in \Sigma})$ and $(B, \{op^B\}_{op \in \Sigma})$, a function $f : A \to B$ is a *homomorphism* if:

$$f(op^A(a_1, \dots, a_n)) = op^B(f(a_1), \dots, f(a_n))$$

for all $a_1, \dots, a_n \in A$ and n-ary $op \in \Sigma$.

We denote with $\mathbf{Alg}(\Sigma)$ the collection (a proper class) of all Σ-algebras.

Definition 2.4 (Terms over Σ)**.** Given a signature Σ and a set A, we denote with $\mathrm{Terms}_\Sigma(A)$ the collection of all Σ-terms built from A, i.e., the set inductively defined as follows:

$$a \in \mathrm{Terms}_\Sigma(A) \qquad t_1, \dots, t_n \in \mathrm{Terms}_\Sigma(A) \Longrightarrow op(t_1, \dots, t_n) \in \mathrm{Terms}_\Sigma(A)$$

for all $a \in A$ and n-ary $op \in \Sigma$.

The following definition follows the notational approach of [Wec92], denoting equations by "$\forall A.s = t$", where "$\forall A.$" explicitly indicates the set A of variables involved.

Definition 2.5 (Equations)**.** Given a signature Σ, a Σ-*equation* is a triple (A, s, t) where A is a set and $s, t \in \mathrm{Terms}_\Sigma(A)$. We write such triple as:

$$\forall A.s = t$$

and we denote with $\mathrm{Eq}(\Sigma)$ the class of all Σ-equations.

In the rest of the paper, the signature Σ will often be clear from the context and we will just talk about "equations" rather than Σ-equations. We use the letters ϕ, ψ to range over equations, and Φ, Ψ to range over classes of equations.

Definition 2.6 (Interpretation)**.** Given a Σ-algebra $\mathbb{A} = (A, \{op^A\}_{op \in \Sigma})$ and a set B, an *interpretation of* B *in* $\mathbb{A}$ is a function $\tau : B \to A$. The interpretation τ extends to a function of type $[\![\_]\!]^A_\tau : \mathrm{Terms}_\Sigma(B) \to A$ defined inductively as:

$$[\![b]\!]^A_\tau = \tau(b) \qquad [\![op(t_1, \dots, t_n)]\!]^A_\tau = op^A([\![t_1]\!]^A_\tau, \dots, [\![t_n]\!]^A_\tau).$$

The following definition gives semantics to equations and motivates the "$\forall(\_).$" syntactical notation we adopted, which hints at the universal quantification over interpretations of the variables.

Definition 2.7 (Semantics of Equations)**.** Given a Σ-algebra $\mathbb{A}$ and an equation ϕ of the form $\forall B.s = t$, we say that $\mathbb{A}$ *satisfies* ϕ (written $\mathbb{A} \models \phi$) if, for all interpretations $\tau : B \to A$, it holds that $[\![s]\!]^A_\tau = [\![t]\!]^A_\tau$.

Definition 2.8 (Equational Theory of a Class of Models)**.** Let $\mathcal{K} \subseteq \mathbf{Alg}(\Sigma)$ be a class of Σ-algebras. The *equational theory of* $\mathcal{K}$ is defined as the class of equations satisfied by all Σ-algebras in $\mathcal{K}$, formally:

$$\mathbf{Th}_\Sigma(\mathcal{K}) = \{\phi \in \mathrm{Eq}(\Sigma) \mid \forall \mathbb{A} \in \mathcal{K}, \mathbb{A} \models \phi\}.$$

Definition 2.9 (Models and Equationally defined classes)**.** Let $\Phi \subseteq \mathrm{Eq}(\Sigma)$ be a class of Σ-equations. The *models* of Φ are the Σ-algebras that satisfy all equations in Φ, formally:

$$\mathbf{Mod}_\Sigma(\Phi) = \{\mathbb{A} \in \mathbf{Alg}(\Sigma) \mid \forall \phi \in \Phi, \mathbb{A} \models \phi\}$$

A class $\mathcal{K} \subseteq \mathrm{Alg}(\Sigma)$ of Σ-algebras is said to be *equationally defined by* Φ if $\mathcal{K} = \mathbf{Mod}_\Sigma(\Phi)$.

Definition 2.10 (Model Theoretic Entailment Relation)**.** We define a binary relation[7] $\Vvdash_{\mathbf{Set}} \subseteq \mathcal{P}(\mathrm{Eq}(\Sigma)) \times \mathrm{Eq}(\Sigma)$ that describes how an equation ϕ can be a consequence of a class of equations $\Phi \subseteq \mathrm{Eq}(\Sigma)$. It is defined by

$$\Phi \Vvdash_{\mathbf{Set}} \phi \qquad \Longleftrightarrow \qquad \phi \in \mathbf{Th}_\Sigma(\mathbf{Mod}_\Sigma(\Phi)).$$

Therefore, the meaning of $\Phi \Vvdash_{\mathbf{Set}} \phi$ is that any Σ-algebra that satisfies Φ (i.e., all the equations in Φ) necessarily also satisfies the equation ϕ.

A fundamental result of Birkhoff establishes that $\Vvdash_{\mathbf{Set}}$ coincides with the derivability relation $\Phi \vdash_{\mathbf{Set}} \phi$ of the deductive system of "equational logic" (the relation $\vdash_{\mathbf{Set}}$ is inductively defined, see, e.g., [Wec92, §3.2.4, Definition 8]). Thus, this celebrated result is a logical axiomatisation of the entailment relation $\Vvdash_{\mathbf{Set}}$.

2.2 Category Theory

We assume basic knowledge of category theory. In this section we recall only some main definitions and results used in the rest of the paper. We refer to [Mac88] and [Awo10] as standard references.

For a given signature Σ, with some abuse of notation, we denote with $\mathbf{Alg}(\Sigma)$ both the class of Σ-algebras and the category having as objects Σ-algebras and as arrows the homomorphisms of Σ-algebras. Similarly, $\mathbf{Mod}_\Sigma(\Phi)$ denotes both the class of Σ-algebras satisfying Φ and the full subcategory of $\mathbf{Alg}(\Sigma)$ whose objects are in $\mathbf{Mod}_\Sigma(\Phi)$. In other words, $\mathbf{Mod}_\Sigma(\Phi)$ is the category having as objects Σ-algebras $\mathbb{A}$ such that $\mathbb{A} \models \phi$, for all $\phi \in \Phi$, and as morphisms all their homomorphisms of Σ-algebras.

There is a forgetful functor:

$$U_{\mathbf{Mod}_\Sigma(\Phi)\to\mathbf{Set}} : \mathbf{Mod}_\Sigma(\Phi) \to \mathbf{Set}$$

[7] $\mathcal{P}(\mathrm{Eq}(\Sigma))$ denotes the collection of all classes of Σ-equations, i.e. all subclasses of $\mathrm{Eq}(\Sigma)$. Therefore, it is a conglomerate in the sense of [AHS06, 2.3], and so is $\Vvdash_{\mathbf{Set}}$.

mapping an algebra in $\mathbf{Mod}_\Sigma(\Phi)$ to its carrier, and acting as identity on morphisms:

$$U_{\mathbf{Mod}_\Sigma(\Phi)\to\mathbf{Set}}(A, \{op^A\}_{op\in\Sigma}) = A$$
$$U_{\mathbf{Mod}_\Sigma(\Phi)\to\mathbf{Set}}(f) = f$$

We often just write U when no confusion arises.

2.2.1 Monads and Adjunctions

Definition 2.11 (Monad)**.** Given a category $\mathcal{C}$, a *monad* on $\mathcal{C}$ is a triple (M, η, μ) composed of a functor $M: \mathcal{C} \to \mathcal{C}$ together with two natural transformations: a *unit* $\eta\colon \mathrm{id}_\mathcal{C} \Rightarrow M$, where $\mathrm{id}_\mathcal{C}$ is the identity functor on $\mathcal{C}$, and a *multiplication* $\mu\colon M^2 \Rightarrow M$, satisfying $\mu \circ \eta_M = \mu \circ M\eta = \mathrm{id}_M$ and $\mu \circ M\mu = \mu \circ \mu_M$.

We often denote a monad (M, η, μ) simply with its underlying functor M.

Example 2.12. For any class $\Phi \subseteq \mathrm{Eq}(\Sigma)$ of Σ-equations, we have an associated monad $(T^{\mathbf{Set}}_{\Sigma,\Phi}, \eta, \mu)$ on **Set**, defined as follows:

- The functor $T^{\mathbf{Set}}_{\Sigma,\Phi}$ maps a set A to the set $\mathrm{Terms}_\Sigma(A)/_\equiv$ of terms over A quotiented by the relation $\equiv$ defined as follows, for all $s, t \in \mathrm{Terms}_\Sigma(A)$:

$$s \equiv t \iff \Phi \Vdash_{\mathbf{Set}} \forall A.s = t$$

  and maps a function $f : A \to B$ to the homomorphism

$$T^{\mathbf{Set}}_{\Sigma,\Phi}(f) : \mathrm{Terms}_\Sigma(A)/_\equiv \to \mathrm{Terms}_\Sigma(B)/_\equiv$$

  specified[8] as follows:

$$T^{\mathbf{Set}}_{\Sigma,\Phi}(f)([a]_\equiv) = [f(a)]_\equiv$$
$$T^{\mathbf{Set}}_{\Sigma,\Phi}(f)([op(t_1, ..., t_n)]_\equiv) = op^{F(A)}(T^{\mathbf{Set}}_{\Sigma,\Phi}(f)([t_1]_\equiv), ..., T^{\mathbf{Set}}_{\Sigma,\Phi}(f)([t_n]_\equiv))$$

  where $op^{F(A)}$ is defined as $op^{F(A)}([t_1]_\equiv, ..., [t_n]_\equiv) = [op(t_1, ..., t_n)]_\equiv$ (the reason why we denote the interpretation of operations by $op^{F(A)}$ will become clear in Example 2.19).

- For each set A, the unit $\eta_A : A \to \mathrm{Terms}_\Sigma(A)/_\equiv$ is defined as:

$$a \mapsto [a]_\equiv.$$

[8] It can be verified that this is a good definition, i.e., it does not depend on the choice of representatives for the equivalence classes.

- For each set A, the multiplication

$$\mu_A : \mathrm{Terms}_\Sigma(\mathrm{Terms}_\Sigma(A)/_\equiv)/_\equiv \to \mathrm{Terms}_\Sigma(A)/_\equiv$$

  is defined by the following "flattening" operation:

$$[s([t_1]_\equiv, \ldots, [t_n]_\equiv)]_\equiv \mapsto [s\{t_1/[t_1]_\equiv, \ldots, t_n/[t_n]_\equiv\}]_\equiv$$

  where $s([t_1]_\equiv, \ldots, [t_n]_\equiv)$ denotes that $[t_1]_\equiv, \ldots, [t_n]_\equiv$ are all and only the elements of $\mathrm{Terms}_\Sigma(A)/_\equiv$ appearing in the term s, and the term $s\{t_1/[t_1]_\equiv, \ldots, t_n/[t_n]_\equiv\}$ denotes the simultaneous substitution in s of each of these equivalence classes with one representative.

It can be shown that, indeed, the above definitions do not depend on specific choices of representatives of the $\equiv$-equivalence classes.

A monad M has an associated category of M-algebras.

Definition 2.13 (Eilenberg–Moore algebras for a monad)**.** Let (M, η, μ) be a monad on $\mathcal{C}$. An *algebra* for M (or *M-algebra*) is a pair (A, α) where $A \in \mathcal{C}$ is an object and $\alpha : M(A) \to A$ is a morphism such that (1) $\alpha \circ \eta_A = \mathrm{id}_A$ and (2) $\alpha \circ M\alpha = \alpha \circ \mu_A$ hold. An *M-algebra morphism* between two M-algebras (A, α) and (A', α') is a morphism $f : A \to A'$ in $\mathcal{C}$ such that $f \circ \alpha = \alpha' \circ M(f)$. The category of M-algebras and their morphisms, denoted by $\mathbf{EM}(M)$, is called the Eilenberg–Moore category for M. There is a forgetful functor $\mathbf{EM}(M) \to \mathcal{C}$ that forgets the algebra structures.

Definition 2.14 (Monad morphisms)**.** Let (M, η, μ) and (M', η', μ') be two monads on $\mathcal{C}$. A *monad morphism* from M to M' is a natural transformation $\lambda : M \Rightarrow M'$ such that (1) $\lambda \circ \eta^M = \eta^{M'}$ and (2) $\lambda \circ \mu^M = \mu^{M'} \circ \lambda M' \circ M\lambda$. It is a *monad isomorphism* whenever each component $\lambda_X : MX \to M'X$ is an isomorphism in $\mathcal{C}$.

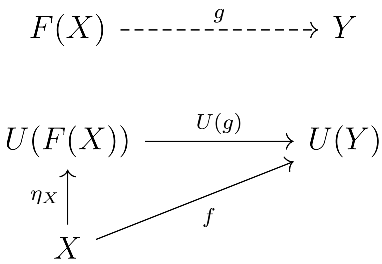
The following result is well known, e.g., it is a simple corollary of [BW05, Theorem 6.3].

Proposition 2.15. *Let (M, η, μ) and (M', η', μ') be two monads on a category $\mathcal{C}$. There is a monad isomorphism $M \cong M'$ if and only if there is an isomorphism of categories $\mathbf{EM}(M) \cong \mathbf{EM}(M')$ that commutes with the forgetful functors to $\mathcal{C}$.*

Monads can be defined as arising from adjunctions.[9]

[9] See, e.g., [Awo10, Chapter 9] for several equivalent definitions.

Definition 2.16 (Adjunction)**.** Let $U : \mathcal{D} \to \mathcal{C}, F : \mathcal{C} \to \mathcal{D}$ be functors. F is a left adjoint of U (notation: $F \dashv U$) if there is a natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow U \circ F$ such that for any $\mathcal{C}$-object X, any $\mathcal{D}$-object Y and $\mathcal{C}$-morphism $f : X \to U(Y)$ there is an unique $\mathcal{D}$-morphism $g : F(X) \to Y$ such that $f = U(g) \circ \eta_X$. Diagrammatically:



The natural transformation η is called the unit of the adjunction. Given an adjunction $F \dashv U$, we also have a natural transformation $\varepsilon : F \circ U \Rightarrow \mathrm{id}_{\mathcal{D}}$, which is called the counit of the adjunction and which satisfies the following identities:[10]

$$U\varepsilon \circ \eta_U = \mathrm{id}_U \qquad \varepsilon_F \circ F\eta = \mathrm{id}_F.$$

Proposition 2.17. *[Awo10, Proposition 10.3] Every adjunction $F : \mathcal{C} \to \mathcal{D} \dashv U : \mathcal{D} \to \mathcal{C}$ defines a monad (M, η, μ) where:*

- *M is the functor $U \circ F$*
- *the unit $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow M$ of the monad is the unit of the adjunction*
- *the multiplication $\mu : M^2 \Rightarrow M$ is given by*

$$\mu_X = U(\varepsilon_{F(X)})$$

where $\varepsilon : F \circ U \to \mathrm{id}_{\mathcal{D}}$ is the counit of the adjunction.

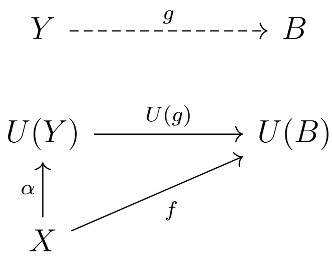
As we discuss in the following section, the monad of quotiented terms from Example 2.12 arises from the adjunction between the forgetful functor $U : \mathbf{Mod}_{\Sigma}(\Phi) \to \mathbf{Set}$ and the functor mapping sets to free objects in $\mathbf{Mod}_{\Sigma}(\Phi)$.

2.2.2 Free Objects

Definition 2.18 (Free object)**.** Let $U : \mathcal{D} \to \mathcal{C}$ be a functor, $X \in \mathcal{C}$, $Y \in \mathcal{D}$ and $\alpha : X \to U(Y)$. We say that Y is a U-free object generated by X with respect to α if the following **UMP** (Universal Mapping Property) holds: for

[10] See, e.g., [Awo10, Proposition 10.1].

every $B \in \mathcal{D}$ and every $\mathcal{C}$-morphism $f : X \to U(B)$, there exists a unique $\mathcal{D}$-morphism $g : Y \to B$ such that $f = U(g) \circ \alpha$, as indicated in the following diagram.



We say that the category $\mathcal{D}$ has U-free objects if for every $X \in \mathcal{C}$ there exist:

1. an object $D_X \in \mathcal{D}$, and
2. a function $\alpha_X : X \to U(D_X)$

such that D_X is a U-free object generated by X with respect to α_X.

If the functor U and the map α_X are clear from the context, we just refer to "the free object in $\mathcal{D}$ generated by X" instead of "U-free object generated by X with respect to α_X".

Free objects, when they exist, are unique up to isomorphism.

Example 2.19. It is a standard result in universal algebra that the forgetful functor $U : \mathbf{Mod}_\Sigma(\Phi) \to \mathbf{Set}$ has U-free objects. Concretely, for any set A, take the algebra $F(A) = (\text{Terms}_\Sigma(A)/_\equiv, \{op^{F(A)}\}_{op\in\Sigma})$, where $\text{Terms}_\Sigma(A)/_\equiv$ and $op^{F(A)}$ are defined as in Example 2.12. Then $F(A)$ is the U-free object generated by A with respect to the function $\alpha : A \to F(A)$ that sends every $a \in A$ to $\alpha(a) = [a]_\equiv$.

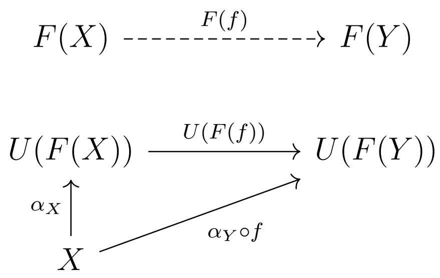
The following proposition states that if $U : \mathcal{D} \to \mathcal{C}$ is a functor such that $\mathcal{D}$ has U-free objects, then there is a functor F, called the free functor, which assigns to objects of $\mathcal{C}$ the U-free object they generate, and which gives an adjunction $F \dashv U$.

Proposition 2.20. *[Mac88, §IV.1, Theorem 2.(ii)] Let $U : \mathcal{D} \to \mathcal{C}$ be a functor such that free U-objects exist in $\mathcal{D}$, i.e., such that for every $X \in \mathcal{C}$ there exist an object $D_X \in \mathcal{D}$ and a function $\alpha_X : X \to U(D_X)$ such that D_X is the U-free object generated by X with respect to α_X. Then $U : \mathcal{D} \to \mathcal{C}$ has a left adjoint $F : \mathcal{C} \to \mathcal{D}$*

$$F \dashv U$$

with F the functor mapping an object X to the U-free object D_X and mapping a morphism $f : X \to Y$ to the unique $\mathcal{D}$-morphism $F(f) : F(X) \to F(Y)$

that makes the following diagram commute:



From this adjunction, we can then build the monad of U-free objects, as explained in Proposition 2.17.

Example 2.21. In Example 2.19 we have identified U-free objects, for $U : \mathbf{Mod}_\Sigma(\Phi) \to \mathbf{Set}$. As explained in Proposition 2.20, we can then define a functor F such that we obtain an adjunction $F \dashv U$. By Proposition 2.17, we have a monad with underlying functor $U \circ F$, which is exactly the monad of quotiented terms $T^{\mathbf{Set}}_{\Sigma,\Phi}$ from Example 2.12.

2.2.3 Strict Monadicity

Proposition 2.22 (Existence of the comparison functor). *Let $F : \mathcal{C} \to \mathcal{D} \dashv U : \mathcal{D} \to \mathcal{C}$ be an adjunction, and let UF be the induced monad. Then there exists a functor*

$$K : \mathcal{D} \to \mathbf{EM}(UF)$$

called the (canonical) comparison functor.[11]

There are interesting cases in which this comparison functor is an isomorphism. In such cases, we say that the functor is strictly monadic.

Definition 2.23 (Strictly Monadic Adjunction, Strictly Monadic Functor). Let $F : \mathcal{C} \to \mathcal{D} \dashv U : \mathcal{D} \to \mathcal{C}$ be an adjunction. We say that the adjunction is *strictly monadic* if the comparison functor is an isomorphism. Given a functor $U : \mathcal{D} \to \mathcal{C}$, we say that U is strictly monadic if it has a left adjoint F such that the adjunction is strictly monadic.

The next theorem, due to Beck, gives useful equivalent characterisations of strict monadicity relying on coequalizers.

Definition 2.24 (Coequalizer). Let A, B be two objects and $f, g : A \to B$ be two morphisms in $\mathcal{C}$. A *coequalizer* of f and g is a morphism $e : B \to C$ satisfying $e \circ f = e \circ g$ with the following universal property: for any other morphism $o : B \to O$ satisfying $o \circ f = o \circ g$, there is a unique morphism $! : D \to O$ such that $! \circ e = o$.

[11] See, e.g., [Mac88, §VI.3, Theorem 1] for the construction of the comparison functor.

Proposition 2.25. *[Beck's monadicity theorem]*
Let $F : \mathcal{C} \to \mathcal{D} \dashv U : \mathcal{D} \to \mathcal{C}$ be an adjunction. The following are equivalent:

1. *U is strictly monadic*
2. *$U : \mathcal{D} \to \mathcal{C}$ strictly creates coequalizers for all $\mathcal{D}$-arrows f, g such that $U(f), U(g)$ has an absolute coequalizer (in $\mathcal{C}$).*
3. *$U : \mathcal{D} \to \mathcal{C}$ strictly creates coequalizers for all $\mathcal{D}$-arrows f, g such that $U(f), U(g)$ has a split coequalizer (in $\mathcal{C}$).*

where:

- *an absolute coequalizer (in $\mathcal{C}$) of $\mathcal{C}$-arrows $f, g : A \to B$ is a $\mathcal{C}$-arrow $e : B \to C$ such that for all functors F, $F(e)$ is a coequalizer of $F(f), F(g)$.*
- *a split coequalizer (in $\mathcal{C}$) of $\mathcal{C}$-arrows $f, g : A \to B$ is a $\mathcal{C}$-arrow $e : B \to C$ such that $e \circ f = e \circ g$ and such that there exist arrows $s : C \to B$ and $t : B \to A$ such that $e \circ s = \mathrm{id}_C$, $f \circ t = \mathrm{id}_B$ and $g \circ t = s \circ e$.*
- *$U : \mathcal{D} \to \mathcal{C}$ strictly creates coequalizers for the $\mathcal{D}$-arrows f, g if for any coequalizer $e : U(B) \to C$ of $U(f), U(g)$ (in $\mathcal{C}$), there are unique D and $u : B \to D$ (in $\mathcal{D}$) such that $U(D) = C$, $U(u) = e$ and u is a coequalizer of f, g.*

For a proof of Proposition 2.25, see, e.g., [Mac88, §VI.7, Theorem 1].[12]

The following result is well known and its proof, which indeed relies on the characterisations of strict monadicity given by Beck's theorem (Proposition 2.25), can be found in [Mac88, §VI.8, Theorem 1].

Proposition 2.26. *For any signature Σ and class Φ of equations over Σ, the functor $U : \mathbf{Mod}_\Sigma(\Phi) \to \mathbf{Set}$ is strictly monadic.*

As recalled in Example 2.21, the monad $T^{\mathbf{Set}}_{\Sigma,\Phi}$ arises from the adjunction $F \dashv U$, where $U : \mathbf{Mod}_\Sigma(\Phi) \to \mathbf{Set}$ and F is the functor mapping sets to U-free objects. Hence, Proposition 2.26 allows us to conclude that the category $\mathbf{EM}(T^{\mathbf{Set}}_{\Sigma,\Phi})$ of Eilenberg-Moore algebras for $T^{\mathbf{Set}}_{\Sigma,\Phi}$ is isomorphic to the category $\mathbf{Mod}_\Sigma(\Phi)$ of models of Φ.

Definition 2.27 (**Set** presentation)**.** A *presentation* of a monad (M, η, μ) on **Set** is a class of equations $\Phi \subseteq \mathrm{Eq}(\Sigma)$ along with a monad isomorphism $T^{\mathbf{Set}}_{\Sigma,\Phi} \cong M$.

[12]Our Definition 2.23 of *strict monadicity* coincides with that used in [Mac88, p. 143], where however it is just called *monadicity*. We chose to use the adjective *strict* as it has become standard terminology in recent literature, where *monadicity* has a different meaning (see e.g. [Rie17, p. 167]).

Thanks to Proposition 2.15 and Proposition 2.26, when a monad M on **Set** is presented by $\Phi \subseteq \mathrm{Eq}(\Sigma)$, there is an isomorphism $\mathbf{EM}(M) \cong \mathbf{Mod}_{\Sigma}(\Phi)$, hence, we can view M-algebras as the models of Φ.

2.3 Fuzzy Relations and Generalised Metric Spaces

We define here fuzzy relation spaces, which are sets equipped with a $[0,1]$-valued function (see, e.g., [Zad71]).

Definition 2.28. A *fuzzy relation* on a set A is a map $d : A \times A \to [0,1]$. The pair (A, d_A) is called a fuzzy relation space (often, we directly call (A, d_A) a fuzzy relation as well). A morphism between two fuzzy relation spaces (A, d_A) and (B, d_B) is a map $f : A \to B$ which is *nonexpansive*, namely,

$$\forall a, a' \in A, d_B(f(a), f(a')) \leq d_A(a, a').$$

We denote by **FRel** the category of fuzzy relation spaces and nonexpansive maps.

Note that **FRel** has the empty fuzzy relation space $(\emptyset, d_\emptyset)$ as initial object, where $d_\emptyset : \emptyset \times \emptyset \to [0,1]$ is the only map of this type (as $\emptyset \times \emptyset \cong \emptyset$ is initial in **Set**).

We denote with $U_{\mathbf{FRel}\to\mathbf{Set}} : \mathbf{FRel} \to \mathbf{Set}$ the forgetful functor defined as expected, and with U when no confusion arises.

We denote with $D_{\mathbf{Set}\to\mathbf{FRel}} : \mathbf{Set} \to \mathbf{FRel}$ (or just with D if clear from the context) the *discrete functor* mapping a set $A \in \mathbf{Set}$ to the discrete fuzzy relation $(A, d_\perp^A)$ defined as:

$$\forall a, a' \in A, \quad d_\perp^A(a, a') = 1$$

and acting as identity on morphisms $f : A \to B \in \mathbf{Set}$. We indeed note that $D(f) : (A, d_\perp^A) \to (B, d_\perp^B)$ is always nonexpansive, given the definition of $d_\perp^A$.

Proposition 2.29. *The functor D is left adjoint to U, that is, $D \dashv U$.*

We will also be interested in full subcategories of **FRel** obtained by restricting to fuzzy relations that satisfy certain constraints, expressed by means of universally quantified logical implications (Horn formulas) in the language of first-order logic.

Definition 2.30 ($\mathscr{L}$-implications)**.** Let $\mathscr{L}$ be the language of first-order logic with the equality binary predicate $(\_ = \_)$ and with, for each $\epsilon \in [0,1]$, the binary predicate $(d(\_,\_) \leq \epsilon)$. We call $\mathscr{L}$*-implications* all closed formulas H of this language that have the following shape:

$$H = \qquad \forall x_1, \ldots, x_n, \Big(\big(\bigwedge_{1 \leq i \leq k} G_i \big) \Rightarrow F \Big)$$

where the subformulas G_i and F are atomic, i.e., G_i and F are either of the form $(x = x')$ or $(d(x, x') \leq \epsilon)$, for some $\epsilon \in [0, 1]$ and for $x, x' \in \{x_1, ..., x_n\}$.

Such formulas are interpreted on fuzzy relations (A, d_A) as in standard first-order logic, with equality $(\_ = \_)$ being the identity relation on A, and the binary predicate $d(\_,\_) \leq \epsilon$ holding true whenever d_A assigns distance less than or equal to ϵ.

Definition 2.31 (Semantics of $\mathscr{L}$-implications)**.** Given a fuzzy relation (A, d_A) and an $\mathscr{L}$-implication H of the form described in Definition 2.30, we say that (A, d_A) satisfies H (notation: $(A, d_A) \models^{\mathscr{L}} H$) if for all functions $\iota : \{x_1, ..., x_n\} \to A$,

$$\text{if } (A, d_A) \models^{\mathscr{L}}_{\iota} G_i \text{ for all } 1 \leq i \leq k, \text{ then } (A, d_A) \models^{\mathscr{L}}_{\iota} F$$

where $(A, d_A) \models^{\mathscr{L}}_{\iota} x = x'$ holds if $\iota(x) = \iota(x')$, and $(A, d_A) \models^{\mathscr{L}}_{\iota} d(x, x') \leq \epsilon$ holds if $d_A(\iota(x), \iota(x')) \leq \epsilon$.

Given a (possibly infinite) set $\mathcal{H}$ of $\mathscr{L}$-implications, we say that (A, d_A) satisfies $\mathcal{H}$, written $(A, d_A) \models^{\mathscr{L}} \mathcal{H}$, if for all $H \in \mathcal{H}$, (A, d_A) satisfies H.

Consider, for example, the following useful $\mathscr{L}$-implications H:

$$\forall x, \qquad x = x \implies d(x, x) \leq 0 \tag{1}$$

$$\forall x, y, \qquad d(x, y) \leq 0 \implies x = y \tag{2}$$

$$\forall x, y, \qquad d(x, y) \leq \epsilon \implies d(y, x) \leq \epsilon \tag{3}$$

$$\forall x, y, z, \qquad d(x, y) \leq \epsilon \ \wedge \ d(y, z) \leq \delta \implies d(x, z) \leq \gamma \quad (\text{where } \gamma = \epsilon + \delta) \tag{4}$$

and the set $\mathcal{H}_{\mathbf{Met}}$ consisting of all instances (for all values of ϵ, δ and $\gamma = \epsilon + \delta$) of these $\mathscr{L}$-implications:

$$\mathcal{H}_{\mathbf{Met}} = \{\textit{Equation } (1), \textit{Equation } (2), \textit{Equation } (3), \textit{Equation } (4)\}$$

It is easy to see that $(A, d_A) \models^{\mathscr{L}} \mathcal{H}_{\mathbf{Met}}$ if and only if the fuzzy relation d_A is a metric. Indeed (1) expresses that each point is at distance zero from itself, (2) states that points at distance zero must be equal, (3) expresses symmetry ($d_A(a, b) = d_A(b, a)$) of the fuzzy relation and (4) expresses the triangular inequality property. Similarly, the subset:

$$\mathcal{H}_{\mathbf{PseudoMet}} = \{Equation\ (1), Equation\ (3), Equation\ (4)\}$$

is satisfied exactly by the fuzzy relations (A, d_A) such that d_A is a pseudometric [BvBR98]. In the literature, many other generalisations of metrics are defined as fuzzy relations satisfying a list of axioms expressible with $\mathscr{L}$-implications. Important examples include: quasimetrics [Wil31a], ultrametrics [BvBR98], semimetrics [Wil31b], dislocated metrics [HS00] also called diffuse metrics in [CKPR21], rectangular metrics [Bra00] and b-metrics [Cze93].

Definition 2.32 (**GMet** categories)**.** Given a collection $\mathcal{H}$ of $\mathscr{L}$-implications, we denote with $\mathbf{GMet}_{\mathcal{H}}$ (or just **GMet** if $\mathcal{H}$ is clear from the context or abstracted away) the full subcategory of **FRel** whose objects are fuzzy relations (A, d_A) such that $(A, d_A) \models^{\mathscr{L}} \mathcal{H}$ and whose morphisms are all the nonexpansive maps between them. We call objects of **GMet** *generalised metric spaces.*

Note that, in accordance with the above definition, we have that $\mathbf{FRel} = \mathbf{GMet}_{\emptyset}$, i.e., **FRel** is the special case of $\mathcal{H}$ being empty. Given its importance, we reserve the symbol **Met** for the category of metric spaces and nonexpansive maps, i.e., $\mathbf{Met} = \mathbf{GMet}_{\mathcal{H}_{\mathbf{Met}}}$.

Given any **GMet** category, we denote with $U_{\mathbf{GMet}\to\mathbf{Set}} : \mathbf{GMet} \to \mathbf{Set}$ the forgetful functor defined as the restriction of $U_{\mathbf{FRel}\to\mathbf{Set}}$ to **GMet**, which we simply denote by U when no confusion arises.

Remark 2.33. The terminology "generalised metric space" has appeared in the literature with different meanings. For instance, in [BvBR98], generalised metric spaces are fuzzy relations satisfying reflexivity (1) and triangular inequality (4). Our definition is thereby a further generalisation, which also covers as special cases the spaces considered in [BvBR98].

3 Presentation of the Framework and Results

In this section we present our framework of Universal Quantitative Algebra. We will introduce it following the same pattern as in the background Section 2.1 on Universal Algebra. We begin with the central notion of this section, the concept of quantitative algebra.

Definition 3.1 (Quantitative Algebra)**.** Given a signature Σ, an **FRel** *quantitative Σ-algebra* $\mathbb{A}$, or just a *quantitative algebra* for short, is a triple $\mathbb{A} = (A, d_A, \{op^A\}_{op\in\Sigma})$ where:

- (A, d_A) is an **FRel** space, i.e., A is a set and

$$d_A : A^2 \to [0, 1]$$

  is an arbitrary map,

- $(A, \{op^A\}_{op\in\Sigma})$ is a Σ-algebra, i.e.,

$$op^A : A^{ar(op)} \to A$$

  is an interpretation of all the operation symbols in Σ.

Remark 3.2. Note that, in contrast with the definition in [MPP16] (and with much subsequent literature [BMPP21, MV20, MPP17, MPP21]), under our definition the distance d_A is not required to satisfy the axioms of metric spaces, as it can be an arbitrary fuzzy relation, and the interpretations op^A of the operations in Σ are not required to be nonexpansive and can be arbitrary set-theoretical functions. See Section 9 for a more detailed comparison.

Definition 3.3 (Homomorphisms)**.** Given a signature Σ and quantitative algebras $\mathbb{A}$ and $\mathbb{B}$,

$$\mathbb{A} = (A, d_A, \{op^A\}_{op\in\Sigma}) \quad \mathbb{B} = (B, d_B, \{op^B\}_{op\in\Sigma})$$

a homomorphism (of quantitative algebras) is a function $f : A \to B$ such that:

- $f : (A, d_A) \to (B, d_B)$ is nonexpansive, i.e.,

$$d_B(f(a_1), f(a_2)) \leq d_A(a_1, a_2)$$

  for all $a_1, a_2 \in A$, and

- f is a homomorphism between the Σ-algebras $\mathbb{A} = (A, \{op^A\}_{op\in\Sigma})$ and $\mathbb{B} = (B, \{op^B\}_{op\in\Sigma})$, i.e.,

$$f(op^A(a_1, \dots, a_n)) = op^B(f(a_1), \dots, f(a_n))$$

  for all $a_1, \dots, a_n \in A$ and $op \in \Sigma$.

We denote with $\mathbf{QAlg^{FRel}}(\Sigma)$, or often just $\mathbf{QAlg}(\Sigma)$, the category of **FRel** quantitative Σ-algebras and their homomorphisms.

We denote with $U_{\mathbf{QAlg}(\Sigma)\to\mathbf{FRel}}$ and $U_{\mathbf{QAlg}(\Sigma)\to\mathbf{Alg}(\Sigma)}$ the forgetful functors defined as expected, which make the following diagram (where, without ambiguity, we denote all functors involved just by U) commute:

$$\begin{array}{ccc} \mathbf{QAlg}(\Sigma) & \xrightarrow{U} & \mathbf{FRel} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathbf{Alg}(\Sigma) & \xrightarrow[U]{} & \mathbf{Set} \end{array}$$

Definition 3.4 (Equations and quantitative equations)**.** An **FRel** Σ-*equation*, or just an *equation* for short, is a judgment of the form:

$$\forall(A, d_A).s = t$$

where (A, d_A) is an **FRel** space and $s, t \in \text{Terms}_\Sigma(A)$. An **FRel** *quantitative* Σ-*equation*, or just a *quantitative equation* for short, is a judgment of the form:

$$\forall(A, d_A).s =_\epsilon t$$

where (A, d_A) is an **FRel** space, $s, t \in \text{Terms}_\Sigma(A)$ and $\epsilon \in [0, 1]$.

We use the letters ϕ, ψ to range over equations and quantitative equations, and we denote with $\text{QEq}(\Sigma)$, the proper class of all **FRel** Σ-equations and quantitative Σ-equations.

Definition 3.5 (Interpretations)**.** Given an **FRel** quantitative Σ-algebra $\mathbb{A} = (A, d_A, \{op^A\}_{op\in\Sigma})$ and an **FRel** space (B, d_B), an *interpretation* of (B, d_B) in $\mathbb{A}$ is a nonexpansive function $\tau : (B, d_B) \to (A, d_A)$. The interpretation τ extends uniquely to a (set-theoretic) function of type $[\![\_]\!]^A_\tau : \text{Terms}_\Sigma(B) \to A$ specified as in Definition 2.6.

In accordance with the above definition, all interpretations of quantitative algebras are nonexpansive. While this prevents any confusion, we will sometimes stress the fact that the interpretations are nonexpansive as this is often crucial in some statements and proofs.

Definition 3.6 (Semantics of Equations and Quantitative Equations)**.** Let $\mathbb{A} = (A, d_A, \{op^A\})$ be an **FRel** quantitative Σ-algebra. Let ϕ_1 and ϕ_2 be the following **FRel** Σ-equation and quantitative Σ-equation, respectively:

$$\phi_1 = \forall(B, d_B).s = t \qquad \phi_2 = \forall(B, d_B).s =_\epsilon t.$$

We say that $\mathbb{A}$ *satisfies* ϕ_1, written $\mathbb{A} \models \phi_1$, if for all nonexpansive interpretations $\tau : (B, d_B) \to (A, d_A)$ of (B, d_B) in $\mathbb{A}$, $[\![s]\!]^A_\tau = [\![t]\!]^A_\tau$ holds. Similarly, we say that $\mathbb{A}$ satisfies ϕ_2, written $\mathbb{A} \models \phi_2$, if for all nonexpansive interpretations $\tau : (B, d_B) \to (A, d_A)$ of (B, d_B) in $\mathbb{A}$, $d_A([\![s]\!]^A_\tau, [\![t]\!]^A_\tau) \le \epsilon$ holds.

Definition 3.7 (Quantitative Equational Theory of a Class of Models)**.** Let $\mathcal{K} \subseteq \mathbf{QAlg}(\Sigma)$ be a class of **FRel** quantitative Σ-algebras. The *quantitative equational theory* of $\mathcal{K}$ is defined as the class of **FRel** Σ-equations and quantitative Σ-equations satisfied by all quantitative algebras in $\mathcal{K}$, formally:

$$\mathbf{QTh}_\Sigma(\mathcal{K}) = \{\phi \in \mathrm{QEq}(\Sigma) \mid \forall \mathbb{A} \in \mathcal{K}.\ \mathbb{A} \models \phi\}.$$

Definition 3.8 (Models and Quantitative Equationally Defined Classes)**.** Let $\Phi \subseteq \mathrm{QEq}(\Sigma)$ be a class of **FRel** Σ-equations and quantitative Σ-equations. The *models* of Φ are the quantitative algebras that satisfy all equations and quantitative equations in Φ, formally:

$$\mathbf{QMod}_\Sigma(\Phi) = \{\mathbb{A} \in \mathbf{QAlg}(\Sigma) \mid \forall \phi \in \Phi.\ \mathbb{A} \models \phi\}.$$

A class $\mathcal{K} \subseteq \mathbf{QAlg}(\Sigma)$ of quantitative Σ-algebras is said to be a *quantitative equationally defined by* Φ if $\mathcal{K} = \mathbf{QMod}_\Sigma(\Phi)$.

Note that, accordingly, $\mathbf{QAlg}(\Sigma) = \mathbf{QMod}_\Sigma(\emptyset)$. With some abuse of notation, we also denote with $\mathbf{QMod}_\Sigma(\Phi)$ the full subcategory of $\mathbf{QAlg}(\Sigma)$ whose objects are in $\mathbf{QMod}_\Sigma(\Phi)$. In other words, $\mathbf{QMod}_\Sigma(\Phi)$ is the category having as objects quantitative Σ-algebras $\mathbb{A}$ such that $\mathbb{A} \models \phi$, for all $\phi \in \Phi$, and as morphisms all their homomorphisms of quantitative algebras.

We denote with $U_{\mathbf{QMod}_\Sigma(\Phi)\to\mathbf{FRel}}$ the forgetful functor defined as the restriction of $U_{\mathbf{QAlg}(\Sigma)\to\mathbf{FRel}}$. As usual, this is most often just denoted by U when no confusion arises.

Definition 3.9 (Model Theoretic Entailment Relation)**.** Let $\Phi \subseteq \mathrm{QEq}(\Sigma)$ be a class of **FRel** Σ-equations and quantitative Σ-equations. We define a binary (consequence) relation[13] $\Vdash_{\mathbf{FRel}} \subseteq \mathcal{P}(\mathrm{QEq}(\Sigma)) \times \mathrm{QEq}(\Sigma)$ (or just $\Vdash$ for short) as follows:

$$\Phi \Vdash_{\mathbf{FRel}} \phi \qquad \Longleftrightarrow \qquad \phi \in \mathbf{QTh}_\Sigma(\mathbf{QMod}_\Sigma(\Phi)).$$

Thus, the meaning of $\Phi \Vdash_{\mathbf{FRel}} \phi$ is that any **FRel** quantitative Σ-algebra that satisfies Φ (i.e., all the **FRel** equations and quantitative equations in Φ) necessarily also satisfies ϕ.

We can summarize the introduced notions, in relation with the corresponding ones from Universal Algebra, as follows:

[13] As in Definition 2.10, both $\mathcal{P}(\mathrm{QEq}(\Sigma))$ and $\Vdash_{\mathbf{FRel}}$ are collections of classes, i.e. conglomerates in the sense of [AHS06, 2.3].

Universal Algebra	Universal Quantitative Algebra
Σ-algebra $(A, \{op^A\}_{op\in\Sigma})$	Quantitative Σ-algebra $(A, d_A, \{op^A\}_{op\in\Sigma})$
Homomorphism of Σ-algebras	(Nonexpansive) homomorphism of quantitative Σ-algebras
Category $\mathbf{Alg}(\Sigma)$ of Σ-algebras	Category $\mathbf{QAlg}(\Sigma)$ of quantitative Σ-algebras
Σ-equation $\forall A.s = t$	Σ-equation $\forall(A, d_A).s = t$ and quantitative Σ-equation $\forall(A, d_A).s =_\epsilon t$
Interpretation of Σ-equations	(Nonexpansive) interpretation of Σ-equations and of quantitative Σ-equations
Category $\mathbf{Mod}_\Sigma(\Phi)$ of models of $\Phi \subseteq \mathrm{Eq}(\Sigma)$	Category $\mathbf{QMod}_\Sigma(\Phi)$ of models of $\Phi \subseteq \mathrm{QEq}(\Sigma)$
Equational theory $\mathbf{Th}_\Sigma(\mathcal{K})$	Quantitative equational theory $\mathbf{QTh}_\Sigma(\mathcal{K})$
Equationally defined class of Σ-algebras $\mathcal{K} = \mathbf{Mod}_\Sigma(\Phi)$	Quantitative equationally defined class of quantitative Σ-algebras $\mathcal{K} = \mathbf{QMod}_\Sigma(\Phi)$
Entailment relation $\Vdash_{\mathbf{Set}}$	Entailment relation $\Vdash_{\mathbf{FRel}}$

3.1 Summary of Contributions

We now give an overview of the main results that we will prove in the following sections.

(I) The entailment relation $\Vdash_{\mathbf{FRel}}$ can be axiomatised by means of a deductive system analogous to the deductive system of Birkhoff's equational logic. More formally, there is an inductively defined relation $\vdash_{\mathbf{FRel}} \subseteq \mathcal{P}(\mathrm{QEq}(\Sigma)) \times \mathrm{QEq}(\Sigma)$, specified as the smallest relation containing a given set of pairs and closed under a given set of deductive rules (see Section 4 for details), which is sound and complete with respect to $\Vdash_{\mathbf{FRel}}$, i.e., for all $\Phi \subseteq \mathrm{QEq}(\Sigma)$ and $\phi \in \mathrm{QEq}(\Sigma)$,

$$\Phi \vdash_{\mathbf{FRel}} \phi \iff \Phi \Vdash_{\mathbf{FRel}} \phi.$$

The soundness of the deductive system (i.e., the implication $\Phi \vdash_{\mathbf{FRel}} \phi \Rightarrow \Phi \Vdash_{\mathbf{FRel}} \phi$) is proved in Section 4. The completeness (i.e., the

implication $\Phi \Vdash_{\mathbf{FRel}} \phi \Rightarrow \Phi \vdash_{\mathbf{FRel}} \phi$) is proved in Section 5.4, as a consequence of our second result (Item II) below.
We recall, from the introduction, that our deductive system $\vdash_{\mathbf{FRel}}$ has significant differences with the one presented in the seminal paper [MPP16]. A detailed discussion is available in Section 9.

(II) For every signature Σ and collection $\Phi \subseteq \mathrm{QEq}(\Sigma)$ of Σ-equations and quantitative Σ-equations, the category $\mathbf{QMod}_\Sigma(\Phi)$ has U-free objects. The U-free object $F(A, d_A)$ generated by $(A, d_A) \in \mathbf{FRel}$ can be identified (up to isomorphism of quantitative algebras) as follows:

$$F(A, d_A) = (\mathrm{Terms}_\Sigma(A)/_\equiv, \Delta^{F(A,d_A)}, \{op^{F(A,d_A)}\}_{op\in\Sigma})$$

where:

(a) the equivalence relation $\equiv \subseteq \mathrm{Terms}_\Sigma(A) \times \mathrm{Terms}_\Sigma(A)$ is defined as:

$$s \equiv t \iff \Phi \vdash_{\mathbf{FRel}} \forall(A, d_A).s = t$$

(b) the fuzzy relation $\Delta^{F(A,d_A)} : (\mathrm{Terms}_\Sigma(A)/_\equiv)^2 \to [0, 1]$ is defined as:

$$\Delta^{F(A,d_A)}([s]_\equiv, [t]_\equiv) \leq \epsilon \iff \Phi \vdash_{\mathbf{FRel}} \forall(A, d_A).s =_\epsilon t$$

(c) The interpretation $op^{F(A,d_A)} : (\mathrm{Terms}_\Sigma(A)/_\equiv)^n \to (\mathrm{Terms}_\Sigma(A)/_\equiv)$ of any n-ary operation $op \in \Sigma$, is defined as:

$$op^{F(A,d_A)}([s_1]_\equiv, \dots, [s_n]_\equiv) = [op(s_1, \dots, s_n)]_\equiv \ .$$

It can be shown that the definitions of $\Delta^{F(A,d_A)}$ and $op^{F(A,d_A)}$ are well specified regardless of the choice of representatives s, t for the classes $[s]_\equiv, [t]_\equiv$, and that indeed the quantitative algebra $F(A, d_A)$ belongs to $\mathbf{QMod}_\Sigma(\Phi)$.

These results are formally stated and proved in Section 5, and they give us the analogous of the result mentioned in Example 2.19 for Universal Algebra.

(III) As a corollary of the two results above and of Proposition 2.20, there is a functor $F : \mathbf{FRel} \to \mathbf{QMod}_\Sigma(\Phi)$ which associates to each $\mathbf{FRel}$ space (A, d_A) the corresponding free object $F(A, d_A)$. The functor F is a left adjoint of U:

$$\mathbf{QMod}_\Sigma(\Phi) \underset{F}{\overset{U}{\rightleftarrows}} \mathbf{FRel} \qquad (F \dashv U)$$

where we just wrote U for $U_{\mathbf{QMod}_\Sigma(\Phi)\to\mathbf{FRel}}$ to improve readability.

This adjunction gives us a monad $T^{\mathbf{FRel}}_{\Sigma,\Phi}$ on **FRel**, which is defined similarly to the **Set** monad $T^{\mathbf{Set}}_{\Sigma,\Phi}$ of quotiented terms discussed in Example 2.12 and Example 2.21. In Section 6 we concretely identify this adjunction and monad, and we prove that the functor $U : \mathbf{QMod}_\Sigma(\Phi) \to \mathbf{FRel}$ is strictly monadic, i.e., there is an isomorphism of categories:

$$\mathbf{EM}(T^{\mathbf{FRel}}_{\Sigma,\Phi}) \cong \mathbf{QMod}_\Sigma(\Phi)$$

where $\mathbf{EM}(T^{\mathbf{FRel}}_{\Sigma,\Phi})$ is the category of Eilenberg–Moore algebras for the monad $T^{\mathbf{FRel}}_{\Sigma,\Phi}$.

(IV) We identify two relevant collections of **FRel** monads and of classes of equations and quantitative equations $\Phi \subseteq \mathrm{QEq}(\Sigma)$, respectively. On one side, we consider monads M in **FRel** that are *monad liftings* of a monad N in **Set** having an equational presentation $\Psi \subseteq \mathrm{Eq}(\Sigma)$. On the other, classes of **FRel** equations and quantitative equations Φ that are *quantitative extensions* of Ψ. In Section 7, after having defined the above notions and that of *quantitative equational presentation* of a monad on **FRel**, we establish (Theorem 7.5) the following correspondence:

(1) If M is a monad lifting of N, then M is presented by a quantitative extension of Ψ.

(2) If Φ is a quantitative extension of Ψ, then Φ is a quantitative equational presentation of an **FRel** monad that lifts N.

(V) All the results in Item I–Item IV above, stated and proved for the category **FRel**, can be specialised and hold true for generalised metric spaces, i.e., for all full subcategories **GMet** of **FRel** defined as in Section 2.3. We show this in Section 8.

For example, it is possible to consider the category $\mathbf{QAlg}^{\mathbf{Met}}(\Sigma)$ of quantitative algebras whose underlying fuzzy relation space is a metric space $(A, d_A) \in \mathbf{Met}$. Accordingly, it is possible to define the entailment relation $\Vdash_{\mathbf{Met}}$ restricted to $\mathbf{QAlg}^{\mathbf{Met}}(\Sigma)$ and have a sound and complete proof system $\vdash_{\mathbf{Met}}$. Furthermore, free quantitative algebras generated by metric spaces exist in $\mathbf{QMod}_{\Sigma}^{\mathbf{Met}}(\Phi)$ and the forgetful functor $U : \mathbf{QMod}_{\Sigma}^{\mathbf{Met}}(\Phi) \to \mathbf{Met}$ is strictly monadic. Finally, we also obtain the analogous of points Item IV1 and Item IV2 above, relating **Met** monads liftings and quantitative extensions.

Finally, our contributions are compared to those of the seminal work of Mardare, Panangaden and Plotkin [MPP16] in Section 9.

4 The Deductive System

In this section we introduce a deductive system which can be used to derive judgments of the form: $\Phi \vdash_{\mathbf{FRel}} \phi$, for $\Phi \in \mathcal{P}(\mathrm{QEq}(\Sigma))$ and $\phi \in \mathrm{QEq}(\Sigma)$. Thus, formally, we define by induction a relation $\vdash_{\mathbf{FRel}} \subseteq \mathcal{P}(\mathrm{QEq}(\Sigma)) \times \mathrm{QEq}(\Sigma)$. As standard, we often use $\vdash_{\mathbf{FRel}}$ in infix notation, i.e., we write $\Phi \vdash_{\mathbf{FRel}} \phi$ for $(\Phi, \phi) \in \vdash_{\mathbf{FRel}}$.

In the rest of this section, we will often just write $\vdash$ instead of $\vdash_{\mathbf{FRel}}$ to improve readability.

Definition 4.1. The relation $\vdash_{\mathbf{FRel}} \subseteq \mathcal{P}(\mathrm{QEq}(\Sigma)) \times \mathrm{QEq}(\Sigma)$ is defined as the smallest relation satisfying the following properties:

1. Closure under the INIT rule: given any $\Phi \in \mathcal{P}(\mathrm{QEq}(\Sigma))$ and $\phi \in \mathrm{QEq}(\Sigma)$, if $\phi \in \Phi$ then $\Phi \vdash \phi$ holds. That is:

$$\frac{}{\Phi \vdash \phi}\ \text{INIT (proviso: } \phi \in \Phi\text{)}$$

2. Closure under the CUT rule: given any $\Phi, \Phi' \in \mathcal{P}(\mathrm{QEq}(\Sigma))$ and $\psi \in \mathrm{QEq}(\Sigma)$, if we have that $\Phi \vdash \phi$ holds for all $\phi \in \Phi'$ and that $\Phi \cup \Phi' \vdash \psi$ holds, then $\Phi \vdash \psi$ holds. That is:

$$\frac{\{\Phi \vdash \phi\}_{\phi \in \Phi'} \qquad \Phi, \Phi' \vdash \psi}{\Phi \vdash \psi}\ \text{CUT}$$

3. Closure under the WEAKENING rule: given any $\Phi, \Phi' \in \mathcal{P}(\mathrm{QEq}(\Sigma))$ and $\phi \in \mathrm{QEq}(\Sigma)$, if $\Phi \vdash \phi$ holds, then $\Phi \cup \Phi' \vdash \phi$ holds. That is:

$$\frac{\Phi \vdash \phi}{\Phi \cup \Phi' \vdash \phi}\ \text{WEAKENING}$$

4. The relation $\vdash$ contains all the pairs $\Phi \vdash \phi$ listed below (a)–(j). These pairs are “axiom schemes”, meaning that pairs are obtained from axiom schemes by instantiating the involved fuzzy relation (A, d_A), terms $s, t \in \mathrm{Terms}_\Sigma(A)$, substitutions σ, *etc.*, to concrete ones.

   (a) (REFL of $=$):
   $$\emptyset \vdash \forall (A, d_A). s = s$$

   (b) (SYMM of $=$):
   $$\forall (A, d_A). s = t \vdash \forall (A, d_A). t = s$$

   (c) (TRANS of $=$):
   $$\forall (A, d_A). s = t, \forall (A, d_A). t = u \vdash \forall (A, d_A). s = u$$

   (d) (CONG of $=$): for all $op \in \Sigma$ of arity n,
   $$\forall (A, d_A). s_1 = t_1, \ldots, \forall (A, d_A). s_n = t_n \vdash \forall (A, d_A). op(s_1, \ldots, s_n) = op(t_1, \ldots, t_n)$$

   (e) (SUBSTITUTION for $=$ and $=_\epsilon$):
   Given

   - (A, d_A) and (B, d_B) **FRel** spaces,
   - $\sigma : A \to \mathrm{Terms}_\Sigma(B)$, a substitution,

   we have the following two similar axiom schemes: one allowing substitution on conclusions that are equations ($=$) and the other on conclusions that are quantitative equations ($=_\epsilon$):
   $$\Psi_\sigma, \forall (A, d_A). s = t \vdash \forall (B, d_B). \sigma(s) = \sigma(t)$$
   and
   $$\Psi_\sigma, \forall (A, d_A). s =_\epsilon t \vdash \forall (B, d_B). \sigma(s) =_\epsilon \sigma(t)$$
   where in both axiom schemes, the set Ψ_σ is defined as:
   $$\Psi_\sigma = \left\{ \forall (B, d_B). \sigma(a_i) =_{\epsilon_{i,j}} \sigma(a_j) \mid a_i, a_j \in A,\ \epsilon_{i,j} := d_A(a_i, a_j) \right\}$$
   and where the function $\sigma : A \to \mathrm{Terms}_\Sigma(B)$ is extended to a function of type $\sigma : \mathrm{Terms}_\Sigma(A) \to \mathrm{Terms}_\Sigma(B)$ (which we denote with the same symbol, abusing notation) as expected by induction on terms, by letting $\sigma(op(s_1, ..., s_n)) = op(\sigma(s_1), ...\sigma(s_n))$.

(f) (USE VARIABLES): For a metric space (A, d_A) and $a, a' \in A$ and $\epsilon = d_A(a, a')$:

$$\emptyset \vdash \forall(A, d_A).a =_\epsilon a'$$

(g) (UP-CLOSURE): for all $\epsilon \leq \delta$:

$$\forall(A, d_A).s =_\epsilon t \vdash \forall(A, d_A).s =_\delta t$$

(h) (1-MAX):

$$\emptyset \vdash \forall(A, d_A).s =_1 t$$

(i) (ORDER COMPLETENESS): For an index set I,

$$\left\{\forall(A, d_A).s =_{\epsilon_i} t\right\}_{i \in I} \vdash \forall(A, d_A).s =_{\inf\{\epsilon_i\}_{i \in I}} t$$

(j) (Left and Right CONGRUENCE) of $=$ with respect to $=_\epsilon$:

$$\forall(A, d_A).s = t, \forall(A, d_A).t =_\epsilon u \vdash \forall(A, d_A).s =_\epsilon u$$

and

$$\forall(A, d_A).s = t, \forall(A, d_A).u =_\epsilon s \vdash \forall(A, d_A).u =_\epsilon t$$

The first basic result regarding the deductive system is the soundness theorem.

Theorem 4.2 (Soundness)**.** *The inclusion* $\vdash_{\mathbf{FRel}} \subseteq \Vdash_{\mathbf{FRel}}$ *holds.*

Proof. To improve notation we just write $\vdash$ for $\vdash_{\mathbf{FRel}}$, as already done above, and also $\Vdash$ for $\Vdash_{\mathbf{FRel}}$.

Assume $\Phi \vdash \phi$. We prove that $\Phi \Vdash \phi$ holds by induction on the derivation tree used to derive $\Phi \vdash \phi$.

Most cases are straightforward, including the occurrences of INIT, CUT and WEAKENING rules and most axiom schemes, so we only detail some of those. The only non-obvious case is the (SUBSTITUTION) axiom scheme (e), which we will prove in full detail.

For an instance of an easy to prove axiom scheme, consider the (USE VARIABLES) axiom scheme (f). We need to show that for an arbitrary quantitative algebra $\mathbb{C} = (C, d_C, \{op^C\}_{op \in \Sigma})$ we have

$$\mathbb{C} \models \forall(A, d_A).a =_\epsilon a'$$

for any **FRel** space (A, d_A) and $a, a' \in A$ with $\epsilon = d_A(a, a')$. This means that, for any nonexpansive interpretation $\tau : (A, d_A) \to (C, d_C)$, we need to prove that:

$$d_C(\llbracket a \rrbracket^{\mathbb{C}}_\tau, \llbracket a' \rrbracket^{\mathbb{C}}_\tau) \leq \epsilon$$

which by the definition of semantics is equivalent to

$$d_C(\tau(a), \tau(a')) \leq \epsilon.$$

This holds since τ is nonexpansive, so we have proven that the axiom scheme is sound.

For another simple example, consider the (Left CONGRUENCE) axiom scheme (j). We need to show that a quantitative algebra $\mathbb{C} = (C, d_C, \{op^C\}_{op\in\Sigma})$ satisfying all the left-side premises of the axiom, i.e.,

1. $\mathbb{C} \models \forall(A, d_A).s = t$
2. $\mathbb{C} \models \forall(A, d_A).t =_\epsilon u$

necessarily also satisfies the right-side quantitative equation, i.e.,

$$\mathbb{C} \models \forall(A, d_A).s =_\epsilon u$$

Take a nonexpansive interpretation $\tau : (A, d_A) \to (C, d_C)$. Then by the two premises we obtain:

1. $[\![s]\!]^{\mathbb{C}}_\tau = [\![t]\!]^{\mathbb{C}}_\tau$
2. $d_C([\![t]\!]^{\mathbb{C}}_\tau, [\![u]\!]^{\mathbb{C}}_\tau) \leq \epsilon$

from which we immediately derive the desired conclusion:

$$d_C([\![s]\!]^{\mathbb{C}}_\tau, [\![u]\!]^{\mathbb{C}}_\tau) \leq \epsilon.$$

We now proceed with the proof of soundness of the (SUBSTITUTION) axiom scheme (e), which is the non-obvious case, as anticipated above.

Let $\sigma : A \to \text{Terms}_\Sigma(B)$ be an arbitrary substitution. We need to show that a quantitative algebra $\mathbb{C} = (C, d_C, \{op^C\}_{op\in\Sigma})$ satisfying all the left-side premises of (e), i.e.,

1. $\mathbb{C} \models \forall(A, d_A).s =_\epsilon t$
2. $\mathbb{C} \models \big\{\forall(B, d_B).\sigma(a_i) =_{\epsilon_{i,j}} \sigma(a_j) \mid a_i, a_j \in A,\ \epsilon_{i,j} := d_A(a_i, a_j)\big\}$

necessarily also satisfies the right-side equation or quantitative equation (we just consider the quantitative equation case as the two are similar), i.e.,

$$\mathbb{C} \models \forall(B, d_B).\sigma(s) =_\epsilon \sigma(t).$$

Towards this end, take an arbitrary nonexpansive interpretation:

$$\tau : (B, d_B) \to (C, d_C).$$

From the interpretation τ and the substitution σ, we define a new interpretation $\hat{\sigma}$ as follows:

$$\hat{\sigma} : (A, d_A) \to (C, d_C) \qquad \hat{\sigma}(a) := [\![\sigma(a)]\!]^{\mathbb{C}}_{\tau}.$$

Before proceeding further, we need to show that $\hat{\sigma}$ is nonexpansive. So, take any $a, a' \in A$ and assume $d_A(a, a') = \delta$. Since (from the second hypothesis):

$$\mathbb{C} \models \forall (B, d_B).\sigma(a) =_{\delta} \sigma(a')$$

it holds (taking the interpretation τ) that:

$$d_C([\![\sigma(a)]\!]^{\mathbb{C}}_{\tau}, [\![\sigma(a')]\!]^{\mathbb{C}}_{\tau}) \leq \delta$$

By definition of $\hat{\sigma}$, this means that:

$$d_C(\hat{\sigma}(a), \hat{\sigma}(a')) \leq \delta,$$

which concludes the proof that $\hat{\sigma}$ is nonexpansive.

From the first hypothesis (i.e., $\mathbb{C} \models \forall (A, d_A).s =_{\epsilon} t$) and taking as interpretation $\hat{\sigma}$, we know that:

$$d_C\big([\![s]\!]^{\mathbb{C}}_{\hat{\sigma}}, [\![t]\!]^{\mathbb{C}}_{\hat{\sigma}}\big) \leq \epsilon.$$

It is now sufficient to observe that:

$$[\![\sigma(s)]\!]^{\mathbb{C}}_{\tau} = [\![s]\!]^{\mathbb{C}}_{\hat{\sigma}} \qquad [\![\sigma(t)]\!]^{\mathbb{C}}_{\tau} = [\![t]\!]^{\mathbb{C}}_{\hat{\sigma}}$$

from which we derive that:

$$d_C\big([\![\sigma(s)]\!]^{\mathbb{C}}_{\tau}, [\![\sigma(t)]\!]^{\mathbb{C}}_{\tau}\big) \leq \epsilon.$$

Since τ was chosen as an arbitrary nonexpansive interpretation, we can derive the desired:

$$\mathbb{C} \models \forall (A, d_A).\sigma(s) =_{\epsilon} \sigma(t).$$

□

The above proof of soundness is rather direct and simple. Proving the opposite direction, i.e., the completeness theorem, in contrast requires more work. Indeed we will use the proof system $\vdash_{\mathbf{FRel}}$ to construct free objects in $\mathbf{QMod}_{\Sigma}(\Phi)$, prove some results about such free objects and finally derive the completeness result.

Our results regarding free objects in $\mathbf{QMod}_{\Sigma}(\Phi)$ are presented in Section 5. The completeness Theorem 5.17 is also established in that section, as a corollary.

5 Free Quantitative Algebras

In the following, we fix a signature Σ and a class of **FRel** (quantitative) Σ-equations $\Phi \subseteq \mathrm{QEq}(\Sigma)$. Let us denote by U the forgetful functor:

$$U : \mathbf{QMod}_{\Sigma}(\Phi) \to \mathbf{FRel}$$

which acts on objects as:

$$(A, d_A, \{op^A\}_{op \in \Sigma}) \mapsto (A, d_A)$$

and as identity on morphisms.

We are going to prove the following statement using (the soundness of) the deductive system $\vdash_{\mathbf{FRel}}$ as main tool. Recall that free objects are defined as in Definition 2.18.

Theorem 5.1. *For every $(A, d_A) \in \mathbf{FRel}$ the U-free object generated by (A, d_A) exists in $\mathbf{QMod}_{\Sigma}(\Phi)$.*

The proof of this statement occupies the rest of this section.

Let us fix an arbitrary $(A, d_A) \in \mathbf{FRel}$. We are going to explicitly construct the U-free object in $\mathbf{QMod}_{\Sigma}(\Phi)$, denoted by $F(A, d_A)$, with respect to a nonexpansive map

$$\alpha : (A, d_A) \to U(F(A, d_A))$$

defined later on in Lemma 5.13.

First, consider the case when $(A, d_A) = (\emptyset, d_\emptyset)$ and Σ does not contain any constant, i.e., the case when $\mathrm{Terms}_{\Sigma}(A) = \emptyset$. In this specific case it is easy to verify, using the fact that $(\emptyset, d_\emptyset)$ is initial in **FRel**, that the empty quantitative algebra (operations $op^\emptyset$ are uniquely determined by initiality of $\emptyset$ in **Set**):

$$(\emptyset, d_\emptyset, \{op^\emptyset\}_{op \in \Sigma})$$

is the U-free object generated by $(\emptyset, d_\emptyset)$ relative to the (unique of this type) nonexpansive map: $\alpha : (\emptyset, d_\emptyset) \to (\emptyset, d_\emptyset)$.

Now, for the other cases, assume that either $(A, d_A) \neq (\emptyset, d_\emptyset)$ or that Σ contains some constants, i.e., the case when $\mathrm{Terms}_{\Sigma}(A) \neq \emptyset$.

We are going to proceed as follows:

1. (Section 5.1) Formally define the quantitative algebra $F(A, d_A) \in \mathbf{QAlg}^{\mathbf{FRel}}(\Sigma)$.

2. (Section 5.2) Prove that, indeed, $F(A, d_A)$ belongs to $\mathbf{QMod}_{\Sigma}(\Phi)$. In other words, we show that $F(A, d_A)$ satisfies all the **FRel** equations and quantitative equations in Φ.

3. (Section 5.3) Finally, define the map

$$\alpha : (A, d_A) \to U(F(A, d_A))$$

and show that $F(A, d_A)$ satisfies the universal property (from Definition 2.18) defining the (unique, up to isomorphism) free algebra generated by (A, d_A).

5.1 Definition of $F(A, d_A)$

We recall that, by the soundness (Theorem 4.2) of the deductive system, whenever $\Phi \vdash_{\mathbf{FRel}} \phi$ holds, also $\Phi \Vdash_{\mathbf{FRel}} \phi$ holds. We denote the relation $\vdash_{\mathbf{FRel}}$ just with $\vdash$ to improve readability.

We start by defining a binary relation $(\equiv)$:

$$\equiv \subseteq \text{Terms}_\Sigma(A) \times \text{Terms}_\Sigma(A)$$

and a fuzzy relation (d):

$$d : \text{Terms}_\Sigma(A) \times \text{Terms}_\Sigma(A) \to [0, 1]$$

on the set of terms $\text{Terms}_\Sigma(A)$ built from A, as follows.

Definition 5.2. We define $\equiv$ as follows, for all $s, t \in \text{Terms}_\Sigma(A)$:

$$s \equiv t \quad \Leftrightarrow \quad \Phi \vdash \forall(A, d_A).s = t.$$

We define d as follows, for all $s, t \in \text{Terms}_\Sigma(A)$:

$$d(s, t) = \inf_\epsilon \left\{\Phi \vdash \forall(A, d_A).s =_\epsilon t\right\}.$$

Lemma 5.3. *The relation $\equiv$ is an equivalence relation.*

Proof. This is due to the presence in the deductive system of the axiom schemes: (REFL of =) (a), (SYMM of =) (b) and (TRANS of =) (c). □

Lemma 5.4. *The relation $\equiv$ is a congruence relation:*

$$s_1 \equiv t_1, \ldots, s_n \equiv t_n \quad \Rightarrow \quad op(s_1, \ldots, s_n) \equiv op(t_1, \ldots, t_n)$$

for all $op \in \Sigma$ and $s_1, t_1, \ldots, s_n, t_n \in \text{Terms}_\Sigma(A)$.

Proof. This is due to the presence in the system of the (CONG of =) axiom scheme (d). □

Lemma 5.5. *The function d is a fuzzy relation on* $\mathrm{Terms}_\Sigma(A)$ *and satisfies:*

$$d(s,t) = \epsilon \qquad \Longrightarrow \qquad \Phi \vdash \forall(A, d_A).s =_\epsilon t.$$

Proof. The fact that d is a fuzzy relation (i.e., a function of type $(\mathrm{Terms}_\Sigma(A))^2 \to [0,1]$) follows from the (1-MAX) axiom scheme (h) (d is defined on all pairs of terms, with $d(s,t) \leq 1$) and the fact that all ϵ's are positive (hence $d(s,t) \geq 0$).

The property:

$$d(s,t) = \epsilon \qquad \Longrightarrow \qquad \Phi \vdash \forall(A, d_A).s =_\epsilon t$$

follows from the presence of the (ORDER COMPLETENESS) axiom scheme (i). □

The following technical lemma relates the proof system ($\vdash$) with the definition of the fuzzy relation d.

Lemma 5.6. $\Phi \vdash \forall(A, d_A).s =_\epsilon t \iff d(s,t) \leq \epsilon$.

Proof. The ($\Rightarrow$) direction follows immediately from the definition of d as an infimum.

For the ($\Leftarrow$) direction, assume $d(s,t) \leq \epsilon$. Let $d(s,t) = \delta$ with $\delta \leq \epsilon$.

As we already established in Lemma 5.5,

$$d(s,t) = \delta \qquad \Longrightarrow \qquad \Phi \vdash \forall(A, d_A).s =_\delta t$$

holds and therefore we deduce that:

$$\Phi \vdash \forall(A, d_A).s =_\delta t$$

holds. From this, using the (UP-CLOSURE) axiom scheme (g), we can derive (within the deductive system $\vdash$, using other rules such as the CUT rule):

$$\Phi \vdash \forall(A, d_A).s =_\epsilon t$$

as desired. □

The following technical lemma shows that the fuzzy relation d is compatible with the equivalence relation $\equiv$.

Lemma 5.7. *The equivalence relation $\equiv$ is a left and right congruence with respect to the fuzzy relation d, in the following sense: for all $s, t, u \in$* $\mathrm{Terms}_\Sigma(A)$*:*

$$s \equiv t \quad \text{and} \quad d(t,u) \leq \epsilon \quad \Longrightarrow \quad d(s,u) \leq \epsilon$$

and

$$s \equiv t \quad \text{and} \quad d(u,s) \leq \epsilon \quad \Longrightarrow \quad d(u,t) \leq \epsilon.$$

Proof. We just consider the first implication (left-congruence) as the other case is similar. Assume $s \equiv t$ and $d(t,u) \leq \epsilon$. By definition of $\equiv$ and by Lemma 5.6 this means that:

$$\Phi \vdash \forall(A, d_A).s = t \qquad \Phi \vdash \forall(A, d_A).t =_\epsilon u$$

Using the (Left CONGRUENCE) axiom scheme (j) we obtain that

$$\Phi \vdash \forall(A, d_A).s =_\epsilon u,$$

and from this we derive, by definition of d, that

$$d(s,u) \leq \epsilon. \qquad \square$$

Since we have established that $\equiv$ is an equivalence relation, the quotient $\text{Terms}_\Sigma(A)/_\equiv$, consisting of $\equiv$-equivalence classes, is well-defined. Furthermore, the equivalence $\equiv$ is a left and right congruence for the fuzzy relation d. This implies that the following is a good definition, regardless of the choice of representatives.

Definition 5.8. The fuzzy relation $\Delta : (\text{Terms}_\Sigma(A)/_\equiv \times \text{Terms}_\Sigma(A)/_\equiv) \to [0,1]$ is defined as:

$$\Delta([s]_\equiv, [t]_\equiv) = d(s,t).$$

Moreover, since we have already established in Lemma 5.4 that $\equiv$ is a congruence on $\text{Terms}_\Sigma(A)$, the interpretation $op^{F(A,d_A)}$ of each operation $op \in \Sigma$ specified as:

$$op^{F(A,d_A)}\big([s_1]_\equiv, \dots, [s_n]_\equiv\big) = [op(s_1, \dots, s_n)]_\equiv$$

is well-defined and does not depend on a specific choice of representatives for the equivalence classes.

We can collect the results of this subsection as follows:

Corollary 5.9. *The structure* $(\text{Terms}_\Sigma(A)/_\equiv, \Delta, \{op^{F(A,d_A)}\}_{op\in\Sigma})$ *is a quantitative* Σ*-algebra.*

The quantitative algebra identified above is our definition of $F(A, d_A)$.

Definition 5.10. The quantitative algebra $F(A, d_A)$ is defined as:

$$F(A, d_A) = (\text{Terms}_\Sigma(A)/_\equiv, \Delta, \{op^{F(A,d_A)}\}_{op\in\Sigma}).$$

Sometimes, to improve notation, we will explicitly identify some relevant parameters involved in the construction of the quantitative algebra (such as Φ, (A, d_A) and Σ). For example we will write:

$$F(B, d_B) = (\text{Terms}_\Sigma(B)/_\equiv, \Delta^{F(B,d_B)}, \{op^{F(B,d_B)}\}_{op\in\Sigma})$$

to highlight that the fuzzy relation $\Delta^{F(B,d_B)}$ is the one associated with (B, d_B). Note that also the relation $\equiv$ is parametric with respect to the generating fuzzy relation space.

5.2 Proof that $F(A, d_A) \in \mathbf{QMod}_\Sigma(\Phi)$

We show in Lemma 5.12 below that the quantitative algebra $F(A, d_A)$ constructed from (A, d_A), as in the previous Section 5.1, satisfies all **FRel** equations and quantitative equations in Φ.

The proof exploits the following property of $F(A, d_A)$.

Lemma 5.11. *Assume* $\text{Terms}_\Sigma(A) \neq \emptyset$. *Let* $\tau : X \to \text{Terms}_\Sigma(A)/_\equiv$ *be a function. Let* $c : \text{Terms}_\Sigma(A)/_\equiv \to \text{Terms}_\Sigma(A)$ *be a choice function, i.e., such that* $c([s]_\equiv) \in [s]_\equiv$*, thus choosing an element* $s \in \text{Terms}_\Sigma(A)$ *for each* $\equiv$*-equivalence class* $[s]_\equiv \in \text{Terms}_\Sigma(A)/_\equiv$*. Note that one such* c *exists by the axiom of choice. Define* $\sigma_\tau : X \to \text{Terms}_\Sigma(A)$ *as:* $\sigma_\tau(x) = c(\tau(x))$*. Then, for all* $s \in \text{Terms}_\Sigma(A)$*, it holds that:*

$$[\![s]\!]^{F(A,d_A)}_\tau = [\sigma_\tau(s)]_\equiv$$

where $\sigma_\tau(s)$ *is the term obtained by applying the substitution* σ_τ *to* s*.*

Proof. The proof is by induction on s. For $s = x$, the result is immediate by definition of σ_τ, i.e.,

$$[\![x]\!]^{F(A,d_A)}_\tau = \tau(x) = [\sigma_\tau(x)]_\equiv.$$

For $s = op(s_1, ...s_n)$, we have

$$\begin{aligned}
[\![s]\!]^{F(A,d_A)}_\tau &= op^{F(A,d_A)}([\![s_1]\!]^{F(A,d_A)}_\tau, ..., [\![s_n]\!]^{F(A,d_A)}_\tau) && \text{(by definition of } [\![_]\!]) \\
&= op^{F(A,d_A)}([\sigma_\tau(s_1)]_\equiv, ..., [\sigma_\tau(s_n)]_\equiv) && \text{(by inductive hypothesis)} \\
&= [op(\sigma_\tau(s_1), ..., \sigma_\tau(s_n)]_\equiv && \text{(by definition of } op^{F(A,d_A)}) \\
&= [\sigma_\tau(op(s_1, ..., s_n))]_\equiv && \text{(by definition of } \sigma_\tau \text{ on terms).}
\end{aligned}$$

□

Lemma 5.12. *It holds that:* $F(A, d_A) \in \mathbf{QMod}_\Sigma(\Phi)$*.*

Proof. Let

$$F(A, d_A) = \big(\text{Terms}_\Sigma(A)/_\equiv, \Delta, \{op^{F(A,d_A)}\}_{op \in \Sigma}\big)$$

as specified in Definition 5.10 of the previous Section 5.1.

We need to show that if $\phi \in \Phi$ with:

$$\phi = \quad \forall (X, d_X).s = t \qquad \text{equation}$$

or

$$\phi = \quad \forall (X, d_X).s =_\epsilon t \qquad \text{quantitative equation}$$

for some **FRel** space (X, d_X) and terms $s, t \in \text{Terms}_\Sigma(X)$, then

$$F(A, d_A) \models \phi \ .$$

Therefore, we need to show that for every nonexpansive interpretation $\tau : (X, d_X) \to (\text{Terms}_\Sigma(A)/_\equiv, \Delta)$, it holds that:

$$[\![s]\!]_\tau^{F(A,d_A)} = [\![t]\!]_\tau^{F(A,d_A)} \qquad \text{equation}$$

or

$$\Delta\Big([\![s]\!]_\tau^{F(A,d_A)}, [\![t]\!]_\tau^{F(A,d_A)}\Big) \leq \epsilon \qquad \text{quantitative equation}$$

or equivalently, by applying Lemma 5.11 where we take σ_τ defined as in the lemma, we need to show that:

$$[\sigma_\tau(s)]_\equiv \ = \ [\sigma_\tau(t)]_\equiv \qquad \text{equation}$$

or

$$\Delta\Big([\sigma_\tau(s)]_\equiv \ , \ [\sigma_\tau(t)]_\equiv\Big) \leq \epsilon \qquad \text{quantitative equation}$$

which in turn, by definition of the equivalence relation $\equiv$ and of Δ (which is defined in Definition 5.8 in terms of the fuzzy relation d), means that:

$$\sigma_\tau(s) \equiv \sigma_\tau(t) \qquad \text{equation}$$

or

$$d\big(\sigma_\tau(s), \sigma_\tau(t)\big) \leq \epsilon \qquad \text{quantitative equation.}$$

Hence, by unfolding the definitions of $\equiv$ and d (Definition 5.2), we need to show that:

$$\Phi \vdash \forall (A, d_A).\ \sigma_\tau(s) = \sigma_\tau(t) \qquad \text{equation}$$

or

$$\Phi \vdash \forall (A, d_A).\ \sigma_\tau(s) =_\epsilon \sigma_\tau(t) \qquad \text{quantitative equation}$$

is derivable.

We just consider below the case when ϕ is a quantitative equation, the other case (ϕ is an equation) is identical, just using the appropriate version of the (SUBSTITUTION) axiom scheme (e). The assumption that τ is nonexpansive means that for all $x_i, x_j \in X$:

$$d_X(x_i, x_j) \leq \epsilon_{ij} \quad \Longrightarrow \quad \Delta(\llbracket x_i \rrbracket_\tau^{F(A,d_A)}, \llbracket x_j \rrbracket_\tau^{F(A,d_A)}) \leq \epsilon_{ij}$$

which by Lemma 5.11 is equivalent to

$$d_X(x_i, x_j) \leq \epsilon_{ij} \quad \Longrightarrow \quad \Delta([\sigma_\tau(x_i)]_\equiv, [\sigma_\tau(x_j)]_\equiv) \leq \epsilon_{ij}$$

By definition of Δ, d and Lemma 5.6, this gives us:

$$d_X(x_i, x_j) \leq \epsilon_{\epsilon_{ij}} \quad \Longrightarrow \quad \Phi \vdash \forall(A, d_A).\sigma_\tau(x_i) =_{\epsilon_{ij}} \sigma_\tau(x_j). \tag{5}$$

We know that:

(I) Since $\forall(X, d_X).s =_\epsilon t$ belongs to Φ, by the INIT rule we have:

$$\Phi \vdash \forall(X, d_X).s =_\epsilon t$$

(II) By (5) we have all of the following judgments:

$$\big\{\Phi \vdash \forall(A, d_A).\sigma_\tau(x_i) =_{\epsilon_{ij}} \sigma_\tau(x_j) \mid x_i, x_j \in X, \quad \epsilon_{ij} := d_X(x_i, x_j)\big\}$$

Note that, using σ_τ as substitution, (I) and (II) above constitute the premises of the (SUBSTITUTION) axiom scheme (quantitative equation instance of (e)), which we restate here for convenience[14]:

$$\Psi \vdash \forall(A, d_A).\sigma_\tau(s) =_\epsilon \sigma_\tau(t)$$

where

$$\Psi = \big\{\forall(A, d_A).\sigma_\tau(x_i) =_{\epsilon_{ij}} \sigma_\tau(x_j) \mid x_i, x_j \in X, \ \epsilon_{ij} := d_X(x_i, x_j)\big\} \cup \{\forall(X, d_X).s =_\epsilon t\}.$$

From which we can derive the desired:

$$\Phi \vdash \forall(A, d_A).\sigma_\tau(s) =_\epsilon \sigma_\tau(t)$$

as follows:

$$\dfrac{(\text{I}) \qquad (\text{II}) \qquad \dfrac{}{\Psi \vdash \forall(A, d_A).\sigma_\tau(s) =_\epsilon \sigma_\tau(t)}\ \text{SUBST}}{\Phi \vdash \forall(A, d_A).\sigma_\tau(s) =_\epsilon \sigma_\tau(t)}\ \text{CUT}$$

□

[14] Compared to the version presented in the list of axioms of the proof system, apply the renaming $A \to X$ and $B \to A$.

5.3 Proof of freeness of $F(A, d_A)$

We now show that the quantitative algebra

$$F(A, d_A) = (\text{Terms}_\Sigma(A)/_\equiv, \Delta, \{op^{F(A,d_A)}\}_{op\in\Sigma}).$$

constructed as specified in Definition 5.10 is indeed the free object in $\mathbf{QMod}_\Sigma(\Phi)$ generated by (A, d_A). We first observe that the map $(a \mapsto [a]_\equiv)$ is nonexpansive.

Lemma 5.13. *The map* $\alpha : (A, d_A) \to (\text{Terms}_\Sigma(A)/_\equiv, \Delta)$ *defined as:*

$$\alpha(a) = [a]_\equiv$$

for all $a \in A$*, is nonexpansive.*

Proof. If $(A, d_A) = (\emptyset, d_\emptyset)$ the function α is trivially nonexpansive. Otherwise, fix arbitrary $a_1, a_2 \in A$ and assume $d_A(a_1, a_2) = \epsilon$. We need to prove that:

$$\Delta([a_1]_\equiv, [a_2]_\equiv) \leq \epsilon$$

This follows from the definition of Δ, Lemma 5.6, and the presence in the proof system of the (USE VARIABLES) axiom scheme (f). □

Remark 5.14. We note that the map α is generally not an isometry (i.e., distance preserving) nor an injection, although it is in many interesting cases. For example, consider the case when the generating fuzzy relation space (A, d_A) is defined as follows:

$$A = \{a_1, a_2\} \quad d_A(a_1, a_2) = d_A(a_2, a_1) = \frac{1}{2} \quad d_A(a_1, a_1) = d_A(a_2, a_2) = 0$$

i.e., it is a metric space consisting of two points at distance $\frac{1}{2}$, the signature Σ is empty, and the set $\Phi = \{\forall(A, d_A).a_1 = a_2\}$ consists of just one **FRel** equation. In this case it is easy to check that $\text{Terms}_\Sigma(A)/_\equiv$ has a single element (i.e., all terms in $\text{Terms}_\Sigma(A)$ are $\equiv$-equivalent), and thus the map α is neither an injection nor an isometry.

Recall that $U : \mathbf{QMod}_\Sigma(\Phi) \to \mathbf{FRel}$ is the forgetful functor which maps a quantitative algebra to its underlying fuzzy relation space. We are now ready to proceed with the proof that $F(A, d_A)$ is the U-free object generated by (A, d_A) relative to the map $\alpha : (A, d_A) \to (\text{Terms}_\Sigma(A)/_\equiv, \Delta)$ defined earlier, in the sense of Definition 2.18.

Theorem 5.15. *Let* $U : \mathbf{QMod}_\Sigma(\Phi) \to \mathbf{FRel}$*. The quantitative algebra* $F(A, d_A)$ *is the* U*-free object generated by* (A, d_A) *relative to the map* $\alpha : (A, d_A) \to (\mathrm{Terms}_\Sigma(A)/_\equiv, \Delta)$

$$a \overset{\alpha}{\mapsto} [a]_\equiv.$$

Proof. We need to show that for every quantitative algebra $\mathbb{B} \in \mathbf{QMod}_\Sigma(\Phi)$

$$\mathbb{B} = (B, d_B, \{op^{\mathbb{B}}\}_{op \in \Sigma})$$

and nonexpansive map

$$f : (A, d_A) \to (B, d_B)$$

there is a unique homomorphism of quantitative algebras

$$\hat{f} : F(A, d_A) \to \mathbb{B}$$

which extends f, i.e., which satisfies $f = U(\hat{f}) \circ \alpha$.

In the remainder of the proof we will generally omit the explicit use of the forgetful functors on morphisms, i.e., we will often write the same symbol to denote a function f seen as a set function, or as a morphism of metric spaces, or as a morphism of quantitative algebras.

We proceed as follows. First (Existence) we exhibit a nonexpansive homomorphism $\hat{f} : F(A, d_A) \to \mathbb{B}$, then (Extension) we show that $\hat{f}$ extends f, and lastly (Uniqueness) we show that $\hat{f}$ is the unique such homomorphism.

Existence. Recall that, by definition, we have

$$F(A, d_A) = (\mathrm{Terms}_\Sigma(A)/_\equiv, \Delta, \{op^{F(A,d_A)}\}_{op \in \Sigma}).$$

For all terms $s \in \mathrm{Terms}_\Sigma(A)$, define $\hat{f}$ as

$$\hat{f}([s]_\equiv) := [\![s]\!]^{\mathbb{B}}_f$$

Before moving on, we need to establish that this is well-defined, i.e., that it does not depend on any choice of representative s for the class $[s]_\equiv$.

To see this, observe that if $s \equiv t$ then, by definition of the relation $(\equiv)$, it holds that $\Phi \vdash \forall (A, d_A).s = t$. By the soundness (Theorem 4.2) of the deductive system, we have that

$$\forall (A, d_A).s = t \;\in\; \mathbf{QTh}_\Sigma(\mathbf{QMod}_\Sigma(\Phi))$$

Since $\mathbb{B} \in \mathbf{QMod}_\Sigma(\Phi)$ by hypothesis, this means that

$$\mathbb{B} \models \forall (A, d_A).s = t$$

and, in particular taking $f : (A, d_A) \to (B, d_B)$ as (nonexpansive) interpretation, it holds that $[\![s]\!]^{\mathbb{B}}_f = [\![t]\!]^{\mathbb{B}}_f$. Hence $\hat{f}$ is well-defined as a function.

It remains to show that:

1. $\hat{f}$ is a homomorphism and,

2. $\hat{f}$ is nonexpansive.

The first follows from the interpretation $op^{F(A,d_A)}$ of the operations in $F(A, d_A)$ as follows:

$$\begin{aligned}\hat{f}(op^{F(A,d_A)}([s_1]_\equiv, ..., [s_n]_\equiv)) &= \hat{f}([op(s_1, ..., s_n)]_\equiv) \\ &= [\![op(s_1, ..., s_n)]\!]^{\mathbb{B}}_f \\ &= op^{\mathbb{B}}([\![s_1]\!]^{\mathbb{B}}_f, ..., [\![s_n]\!]^{\mathbb{B}}_f) \\ &= op^{\mathbb{B}}(\hat{f}([s_1]_\equiv), ..., \hat{f}([s_n]_\equiv)).\end{aligned}$$

Regarding the second point (nonexpansiveness), take two arbitrary $[s]_\equiv, [t]_\equiv \in \text{Terms}_\Sigma(A)/_\equiv$ and let $\Delta([s]_\equiv, [t]_\equiv) = \epsilon$ be their distance in $F(A, d_A)$. We need to show that

$$d_B\big(\hat{f}([s]_\equiv), \hat{f}([t]_\equiv)\big) \leq \epsilon.$$

As established in Lemma 5.6, the hypothesis $\Delta([s]_\equiv, [t]_\equiv) = \epsilon$ implies that:

$$\Phi \vdash \forall(A, d_A).s =_\epsilon t$$

From the soundness of the deductive system (Theorem 4.2), we therefore know that:

$$\forall(A, d_A).s =_\epsilon t \quad \in \quad \mathbf{QTh}_\Sigma(\mathbf{QMod}_\Sigma(\Phi))$$

and since $\mathbb{B} \in \mathbf{QMod}_\Sigma(\Phi)$ by hypothesis, we deduce that:

$$\mathbb{B} \models \forall(A, d_A).s =_\epsilon t$$

Taking as nonexpansive interpretation $f : (A, d_A) \to (B, d_B)$ we therefore obtain that: $d_B([\![s]\!]^{\mathbb{B}}_f, [\![t]\!]^{\mathbb{B}}_f) \leq \epsilon$. By the definition of $\hat{f}$ we have

$$\hat{f}([s]_\equiv) = [\![s]\!]^{\mathbb{B}}_f \qquad \text{and } \hat{f}([t]_\equiv) = [\![t]\!]^{\mathbb{B}}_f$$

so we conclude as desired that

$$d_B(\hat{f}([s]_\equiv), \hat{f}([t]_\equiv)) \leq \epsilon.$$

Extension. We need to show that $f = \hat{f} \circ \alpha$. For any $a \in A$, we have

$$\begin{aligned}\big(\hat{f} \circ \alpha\big)(a) &= \hat{f}(\alpha(a)) & \\ &= \hat{f}([a]_\equiv) & &\text{(definition of } \alpha) \\ &= [\![a]\!]^{\mathbb{B}}_f & &\text{(definition of } \hat{f}) \\ &= f(a) & &\text{(definition of } [\![\_]\!]^{\mathbb{B}}_f)\end{aligned}$$

Uniqueness. Let $g : F(A, d_A) \to \mathbb{B}$ be another nonexpansive homomorphism extending $f : (A, d_A) \to (B, d_B)$, i.e., $f = g \circ \alpha$. We now prove that for all $s \in \mathrm{Terms}_\Sigma(A)$ it holds that: $g([s]_\equiv) = \hat{f}([s]_\equiv)$.

Again, this follows from the interpretation of the operations in $F(A, d_A)$. Formally, the proof goes by induction on the structure of s as follows:

The base case ($s = a$) is immediate, as both g and $\hat{f}$ extend f, i.e.,

$$g([a]_\equiv) = g \circ \alpha(a) = f(a) = \hat{f} \circ \alpha(a) = \hat{f}([a]_\equiv)$$

For the inductive case we use the fact that g and $\hat{f}$ are homomorphisms, together with the inductive hypothesis:

$$\begin{aligned} g([op(s_1, ..., s_n)]_\equiv) &= g(op^{F(A,d_A)}([s_1]_\equiv, ..., [s_n]_\equiv)) \\ &= op^{\mathbb{B}}(g([s_1]_\equiv), ..., g([s_n]_\equiv)) \\ &= op^{\mathbb{B}}(\hat{f}([s_1]_\equiv), ..., \hat{f}([s_n]_\equiv)) \\ &= \hat{f}(op^{F(A,d_A)}([s_1]_\equiv, ..., [s_n]_\equiv)) \\ &= \hat{f}([op(s_1, ..., s_n)]_\equiv). \end{aligned}$$

□

5.4 Completeness of the Deductive System

By exploiting the existence of free objects, we can now establish the completeness of the deductive system $\vdash_{\mathbf{FRel}}$, Theorem 5.17 below.

The proof relies on the following property of $F(A, d_A)$, which, as we have seen, is the U-free object (here U is $U : \mathbf{QMod}_\Sigma(\Phi) \to \mathbf{FRel}$) generated by the **FRel** space (A, d_A) relative to the nonexpansive map $\alpha : (A, d_A) \to F(A, d_A)$.

Lemma 5.16. *For all* $s, t \in \mathrm{Terms}_\Sigma(A)$*,*

$$\textit{if } [\![s]\!]_\alpha^{F(A,d_A)} = [\![t]\!]_\alpha^{F(A,d_A)} \textit{ then } \Phi \vdash_{\mathbf{FRel}} \forall(A, d_A).s = t$$

and

$$\textit{if } \Delta([\![s]\!]_\alpha^{F(A,d_A)}, [\![t]\!]_\alpha^{F(A,d_A)}) \leq \epsilon \textit{ then } \Phi \vdash_{\mathbf{FRel}} \forall(A, d_A).s =_\epsilon t.$$

Proof. First, consider the case when $\mathrm{Terms}_\Sigma(A) = \emptyset$, i.e., when $A = \emptyset$ and Σ does not contain any constant. In this case the statement of the lemma trivially holds as the universal quantification on terms is empty.

Now assume $\mathrm{Terms}_\Sigma(A) \neq \emptyset$.

For the equation case, suppose that the equality $[\![s]\!]_\alpha^{F(A,d_A)} = [\![t]\!]_\alpha^{F(A,d_A)}$ holds. Note that by Lemma 5.11, where we instantiate τ with α, we have $[\![s]\!]_\alpha^{F(A,d_A)} = [\sigma_\alpha(s)]_\equiv$, where $\sigma_\alpha : A \to \mathrm{Terms}_\Sigma(A)$ is a choice function for α

as required in the lemma. Since σ_α maps elements of A to terms in their $\equiv$-equivalence class, by the definition of $\equiv$ (Definition 5.2) and by the (CONG of $=$) axiom scheme (d) we derive that $\Phi \vdash \forall(A, d_A).\sigma_\alpha(s) = s$. Thus by the definition of $\equiv$ we have:

$$[\![s]\!]_\alpha^{F(A,d_A)} = [\sigma_\alpha(s)]_\equiv = [s]_\equiv.$$

Analogously, we derive that $[\![t]\!]_\alpha^{F(A,d_A)} = [t]_\equiv$. Hence, $[s]_\equiv = [t]_\equiv$, which by definition of $\equiv$ means $\Phi \vdash \forall(A, d_A).s = t$.

Similarly, for the quantitative equation case, we have that the inequality $\Delta([\![s]\!]_\alpha^{F(A,d_A)}, [\![t]\!]_\alpha^{F(A,d_A)}) \leq \epsilon$ implies, by Lemma 5.11 as above, $\Delta([s]_\equiv, [t]_\equiv) \leq \epsilon$. Then, by the definition of Δ as d (Definition 5.8 and Definition 5.2) and by Lemma 5.6, we conclude $\Phi \vdash \forall(A, d_A).s =_\epsilon t$. □

We can now prove the completeness theorem.

Theorem 5.17 (Completeness of the deductive system)**.** *Fix a signature* Σ *and a class* $\Phi \subseteq \mathrm{QEq}(\Sigma)$ *of equations and quantitative equations. For all* $\phi \in \mathrm{QEq}(\Sigma)$*:*

$$\text{If } \Phi \Vdash_{\mathbf{FRel}} \phi \text{ then } \Phi \vdash_{\mathbf{FRel}} \phi.$$

Proof. Let us consider first the case of ϕ being an equation of the form $\phi := \forall(A, d_A).s = t$, for some **FRel** space (A, d_A) and terms $s, t \in \mathrm{Terms}_\Sigma(A)$.

Note that since $s, t \in \mathrm{Terms}_\Sigma(A)$, it must be the case that $\mathrm{Terms}_\Sigma(A) \neq \emptyset$. By the free algebra Theorem 5.15 we know that

$$F(A, d_A) = (\mathrm{Terms}_\Sigma(A)/_\equiv, \Delta, \{op^{F(A,d_A)}\}_{op \in \Sigma})$$

is the $U_{\mathbf{QMod}_\Sigma(\Phi) \to \mathbf{FRel}}$-free object generated by (A, d_A) relative to the nonexpansive map $\alpha : (A, d_A) \to (\mathrm{Terms}_\Sigma(A)/_\equiv, \Delta)$ defined as: $\alpha(a) = [a]_\equiv$.

By definition of the entailment relation ($\Vdash_{\mathbf{FRel}}$) the hypothesis $\Phi \Vdash_{\mathbf{FRel}} \phi$ implies that for all $\mathbb{B} \in \mathbf{QMod}_\Sigma(\Phi)$ and for all nonexpansive interpretations $\tau : (A, d_A) \to (B, d_B)$ it holds $[\![s]\!]_\tau^{\mathbb{B}} = [\![t]\!]_\tau^{\mathbb{B}}$. Hence, since $F(A, d_A) \in \mathbf{QMod}_\Sigma(\Phi)$ and α is nonexpansive, we have $[\![s]\!]_\alpha^{F(A,d_A)} = [\![t]\!]_\alpha^{F(A,d_A)}$. Then by Lemma 5.16 we conclude that $\Phi \vdash_{\mathbf{FRel}} \forall(A, d_A).s = t$.

Analogously, for quantitative equations, if ϕ is of the form $\forall(A, d_A).s =_\epsilon t$ then we derive from $\Phi \Vdash_{\mathbf{FRel}} \phi$ that $\Delta([\![s]\!]_\alpha^{F(A,d_A)}, [\![t]\!]_\alpha^{F(A,d_A)}) \leq \epsilon$. By Lemma 5.16 we conclude $\Phi \vdash_{\mathbf{FRel}} \forall(A, d_A).s =_\epsilon t$. □

Corollary 5.18 (Soundness and Completeness)**.** *Fix a signature* Σ *and a class* $\Phi \subseteq \mathrm{QEq}(\Sigma)$*. For all equations and quantitative equations* $\phi \in \mathrm{QEq}(\Sigma)$*:*

$$\Phi \Vdash_{\mathbf{FRel}} \phi \iff \Phi \vdash_{\mathbf{FRel}} \phi.$$

6 The Free-Forgetful Adjunction and Strict Monadicity

By relying on the construction of free objects shown in Section 5, we identify in this section the free-forgetful adjunction arising from it, together with the associated monad. We then proceed to prove the strict monadicity of the adjunction.

Recall from Proposition 2.20 in Section 2.2 that, given a functor $U : \mathcal{D} \to \mathcal{C}$ such that $\mathcal{D}$ has U-free objects, there is a functor $F : \mathcal{C} \to \mathcal{D}$ which assigns to each element of $\mathcal{C}$ its corresponding U-free object, and which gives an adjunction $F \dashv U$ and a monad with functor $U \circ F$.

We have seen in Section 5 that the forgetful functor $U : \mathbf{QMod}_\Sigma(\Phi) \to \mathbf{FRel}$ has U-free objects, identified (up to isomorphism) as the quantitative algebras of quotiented terms. Hence, using the recipe from Proposition 2.20 we obtain the adjunction $F \dashv U$, where F is the functor mapping each **FRel** space (A, d_A) to the free quantitative algebra of quotiented terms $F(A, d_A) = (\mathrm{Terms}_\Sigma(A)/_{\equiv_A}, \Delta^{F(A,d_A)}, \{op^{F(A,d_A)}\}_{op\in\Sigma})$. For a nonexpansive function $f : (A, d_A) \to (B, d_B)$, the functor F gives the nonexpansive homomorphism of quantitative algebras $F(f) : F(A, d_A) \to F(B, d_B)$ which is the unique homomorphic extension of f. This means that once the forgetful functor U is applied, i.e., when seen as a (nonexpansive) function on terms, it can be defined by induction on $t \in \mathrm{Terms}_\Sigma(A)$ as follows:

$$UF(f)([a]_{\equiv_A}) = [f(a)]_{\equiv_B}$$

and

$$UF(f)([op(t_1, ..., t_n)]_{\equiv_A}) = op^{F(B,d_B)}(UF(f)([t_1]_{\equiv_A})), ..., UF(f)([t_n]_{\equiv_A})).$$

We denote the obtained **FRel** monad on UF (by Proposition 2.17) by $T^{\mathbf{FRel}}_{\Sigma,\Phi}$, and we can describe it concretely as follows.

- The functor $T^{\mathbf{FRel}}_{\Sigma,\Phi} = U \circ F$ maps an object (A, d_A) to

  $$T^{\mathbf{FRel}}_{\Sigma,\Phi}(A, d_A) = (\mathrm{Terms}_\Sigma(A)/_{\equiv_A}, \Delta^{F(A,d_A)})$$

  and a morphism $f : (A, d_A) \to (B, d_B)$ to

  $$T^{\mathbf{FRel}}_{\Sigma,\Phi}(f) : (\mathrm{Terms}_\Sigma(A)/_{\equiv_A}, \Delta^{F(A,d_A)}) \to (\mathrm{Terms}_\Sigma(B)/_{\equiv_B}, \Delta^{F(B,d_B)})$$

  where $T^{\mathbf{FRel}}_{\Sigma,\Phi}(f)([t]_{\equiv_A}) = UF(f)([t]_{\equiv_A})$ is the nonexpansive homomorphism which can be specified by induction on terms t as above.

- The unit η is given by the unit of the adjunction, i.e., for every $(A, d_A) \in$ **FRel** we have that $\eta_{(A,d_A)}$ is the function $\alpha_{(A,d_A)}$ (see Lemma 5.13) given together with the free object $F(A, d_A)$ in Section 5. Concretely, this is the function assigning to $a \in A$ the equivalence class $[a]_{\equiv_A}$.

- The multiplication μ of the monad is given as $\mu_{(A,d_A)} = U(\varepsilon_{F(A,d_A)})$, where ε is the counit of the adjunction. We now show that, for $[t]_{\equiv_{F(A,d_A)}} \in \text{Terms}_\Sigma(\text{Terms}_\Sigma(A)/_{\equiv_A})_{\equiv_{F(A,d_A)}}$, the multiplication behaves as the function substituting each occurrence of an equivalence class of terms in t with a representative of the class, thus "flattening" the term as follows:

$$\mu_{(A,d_A)}\Big(\big[t\big([t_1]_{\equiv_A}, \dots, [t_n]_{\equiv_A}\big)\big]_{\equiv_{F(A,d_A)}}\Big) = [t(t_1, \dots, t_n)]_{\equiv_A}.$$

  This can be seen as follows where, to improve readability, we just write $\equiv$ for $\equiv_{F(A,d_A)}$. For $t \in \text{Terms}_\Sigma(\text{Terms}_\Sigma(A)/_{\equiv_A})$ of the form $t = [t']_{\equiv_A}$, for some $[t']_{\equiv_A} \in \text{Terms}_\Sigma(A)/_{\equiv_A}$, we have:

$$\begin{aligned}
\mu_{(A,d_A)}([t]_\equiv) &= \mu_{(A,d_A)}([[t']_{\equiv_A}]_\equiv) \\
&= U(\varepsilon_{F(A,d_A)})([[t']_{\equiv_A}]_\equiv) \\
&= U(\varepsilon_{F(A,d_A)}) \circ \eta_{UF(A,d_A)}([t']_{\equiv_A}) \\
&= \text{id}_{UF(A,d_A)}([t']_{\equiv_A}) \\
&= [t']_{\equiv_A}
\end{aligned}$$

  where the second to last equation follows from the properties of the unit-counit triangle identities of the adjunction (see Section 2.2). For $t \in \text{Terms}_\Sigma(\text{Terms}_\Sigma(A)/_{\equiv_A})$ of the form $t = op(t_1, \dots, t_n)$, for some $t_1, \dots, t_n \in \text{Terms}_\Sigma(\text{Terms}_\Sigma(A)/_{\equiv_A})$, we have:

$$\begin{aligned}
\mu_{(A,d_A)}([t]_\equiv) &= \mu_{(A,d_A)}\Big([op(t_1, \dots t_n)]_\equiv\Big) \\
&= U(\varepsilon_{F(A,d_A)})\Big([op(t_1, \dots t_n)]_\equiv\Big) \\
&= U(\varepsilon_{F(A,d_A)})\Big(op^{F\left(U(F(A,d_A))\right)}([t_1]_\equiv, \dots, [t_n]_\equiv)\Big) \\
&= op^{F(A,d_A)}\Big(U(\varepsilon_{F(A,d_A)})([t_1]_\equiv), \dots, U(\varepsilon_{F(A,d_A)})([t_n]_\equiv)\Big) \\
&= op^{F(A,d_A)}\Big(\mu_{(A,d_A)}([t_1]_\equiv), \dots, \mu_{(A,d_A)}([t_n]_\equiv)\Big)
\end{aligned}$$

  where we exploit the fact that $\varepsilon_{F(A,d_A)}$ is a homomorphism.

We now proceed to prove that the forgetful functor $U : \mathbf{QMod}_\Sigma(\Phi) \to$ **FRel** is strictly monadic (see Definition 2.23), i.e., that there is an isomorphism of categories:

$$\mathbf{EM}(T^{\mathbf{FRel}}_{\Sigma,\Phi}) \cong \mathbf{QMod}_\Sigma(\Phi)$$

where $\mathbf{EM}(T^{\mathbf{FRel}}_{\Sigma,\Phi})$ is the *Eilenberg–Moore* category of M (see Definition 2.13).

We will prove in Theorem 6.3 that $U : \mathbf{QMod}_{\Sigma}(\Phi) \to \mathbf{FRel}$ satisfies the condition (3) of Beck's theorem (Proposition 2.25), from which we conclude by Beck's theorem that it is strictly monadic. In order to do so, we first consider the special case when $\Phi = \emptyset$. In this case we recall that $\mathbf{QMod}_{\Sigma}(\emptyset) = \mathbf{QAlg}(\Sigma)$ consists of the category of all quantitative Σ-algebras.

Theorem 6.1. *The forgetful functor $U_{\emptyset} : \mathbf{QAlg}(\Sigma) \to \mathbf{FRel}$ is strictly monadic.*

Sketch. It is enough to prove that the forgetful functor $U_{\emptyset} : \mathbf{QAlg}(\Sigma) \to \mathbf{FRel}$ strictly creates coequalizers for all $\mathbf{QAlg}(\Sigma)$-arrows f, g such that $U_{\emptyset}(f), U_{\emptyset}(g)$ has an absolute coequalizer (in $\mathbf{FRel}$). From this we derive, by Beck's theorem (Proposition 2.25), that $U_{\emptyset}$ is strictly monadic. The argument follows the structure of the proof of [Mac88, §VI.8, Theorem 1], i.e., of the analogous result for $\mathbf{Set}$. A fully detailed proof is available in [MSV23]. □

We are now going to use the above result, which deals with the special case $\Phi = \emptyset$, to prove Theorem 6.3 in its full generality, showing that the functor $U : \mathbf{QMod}_{\Sigma}(\Phi) \to \mathbf{FRel}$ is strictly monadic for arbitrary Φ.

The proof is a generalisation of the analogous result in [Adá22, Theorem 2.17], which proves strict monadicity in the framework of [MPP16]. In particular, we prove that U satisfies the condition (3) of Beck's theorem, i.e., in contrast with Theorem 6.1, we use split coequalizers instead of absolute coequalizers. We do so since split coequalizers guarantee the existence of a right inverse of the coequalizer, which allows us to apply the following fact (Lemma 6.2): $\mathbf{QMod}_{\Sigma}(\Phi)$ is closed under the images of homomorphisms that have a right inverse in $\mathbf{FRel}$.

Recall that, given $f : A \to B$ and $g : B \to A$, we say that g is a right inverse of f if $f \circ g = id_A$. Note that if f has a right inverse g, then f is surjective.

Lemma 6.2. *Let $\mathbb{A} = (A, d_A, \{op^{\mathbb{A}}\}_{op\in\Sigma})$ be a quantitative algebra in $\mathbf{QMod}_{\Sigma}(\Phi)$, for some class of $\mathbf{FRel}$ equations and quantitative equations $\Phi \subseteq \mathrm{QEq}(\Sigma)$, and let $\mathbb{B} = (B, d_B, \{op^{\mathbb{B}}\}_{op\in\Sigma})$ be in $\mathbf{QAlg}(\Sigma)$. If there is a homomorphism of quantitative algebras $f : \mathbb{A} \to \mathbb{B}$ such that $U(f)$ has a nonexpansive right inverse $g : (B, d_B) \to (A, d_A)$ then $\mathbb{B}$ is in $\mathbf{QMod}_{\Sigma}(\Phi)$.*

Proof. We need to show that, under the hypothesis of the statement, for every $\phi \in \Phi$:

$$\mathbb{B} \models \phi$$

holds. We first consider the case of ϕ being an equation:

$$\phi = \quad \forall(X, d_X).s = t$$

for some **FRel** space (X, d_X) and terms $s, t \in \text{Terms}_\Sigma(X)$.

By definition (of $\mathbb{B} \models \phi$), we need to show that $[\![s]\!]^{\mathbb{B}}_\tau = [\![t]\!]^{\mathbb{B}}_\tau$, for all nonexpansive interpretations $\tau : (X, d_X) \to (B, d_B)$.

Let $\tau : (X, d_X) \to (B, d_B)$ be such an interpretation. Then $g \circ \tau : (X, d_X) \to (A, d_A)$ is an interpretation in $\mathbb{A}$ (which is nonexpansive as **FRel** is a category and thus the composition of nonexpansive functions is nonexpansive). We prove that for any term $r \in \text{Terms}_\Sigma(X)$ (and so, in particular, s and t) it holds that:

$$U(f)([\![r]\!]^{\mathbb{A}}_{g\circ\tau}) = [\![r]\!]^{\mathbb{B}}_\tau \tag{6}$$

The proof is by induction on r:

- if $r = x$ then we have $U(f)([\![x]\!]^{\mathbb{A}}_{g\circ\tau}) = U(f) \circ g \circ \tau(x) = \tau(x) = [\![x]\!]^{\mathbb{B}}_\tau$ since g is a right inverse of $U(f)$, i.e., $U(f) \circ g = \text{id}_B$.

- if $r = op(r_1, ..., r_n)$ then we have

$$\begin{aligned} U(f)([\![r]\!]^{\mathbb{A}}_{g\circ\tau}) &= U(f)(op^{\mathbb{A}}([\![r_1]\!]^{\mathbb{A}}_{g\circ\tau}, ..., [\![r_n]\!]^{\mathbb{A}}_{g\circ\tau})) \\ &= op^{\mathbb{B}}(U(f)([\![r_1]\!]^{\mathbb{A}}_{g\circ\tau}), ..., U(f)([\![r_n]\!]^{\mathbb{A}}_{g\circ\tau})) \\ &\qquad\qquad \text{(by } U(f) \text{ a homomorphism)} \\ &= op^{\mathbb{B}}([\![r_1]\!]^{\mathbb{B}}_\tau, ..., [\![r_n]\!]^{\mathbb{B}}_\tau)) \qquad \text{(by inductive hypothesis)} \\ &= [\![r]\!]^{\mathbb{B}}_\tau \end{aligned}$$

We now conclude the proof that $\mathbb{B}$ satisfies the equation under the interpretation τ. Since, by hypothesis, $\mathbb{A}$ is a model of Φ, we know that $\mathbb{A} \models \phi$ and therefore:

$$[\![s]\!]^{\mathbb{A}}_{g\circ\tau} = [\![t]\!]^{\mathbb{A}}_{g\circ\tau} \tag{7}$$

Then, we derive:

$$\begin{aligned} [\![s]\!]^{\mathbb{B}}_\tau &= U(f)([\![s]\!]^{\mathbb{A}}_{g\circ\tau}) && \text{(by (6))} \\ &= U(f)([\![t]\!]^{\mathbb{A}}_{g\circ\tau}) && \text{(by (7))} \\ &= [\![t]\!]^{\mathbb{B}}_\tau && \text{(by (6))} \end{aligned}$$

We now consider the case of quantitative equations $\phi \in \Phi$ of the form:

$$\phi = \quad \forall(X, d_X).s =_\epsilon t.$$

Since, by hypothesis $\mathbb{A}$ is a model of Φ, we know that $\mathbb{A} \models \phi$ and therefore:

$$d_A([\![s]\!]^{\mathbb{A}}_{g\circ\tau}, [\![t]\!]^{\mathbb{A}}_{g\circ\tau}) \leq \epsilon.$$

From which we derive:

$$\begin{aligned} d_B([\![s]\!]^{\mathbb{B}}_{\tau}, [\![t]\!]^{\mathbb{B}}_{\tau}) &= d_B(U(f)([\![s]\!]^{\mathbb{B}}_{g\circ\tau}), U(f)([\![t]\!]^{\mathbb{B}}_{g\circ\tau})) && \text{(by (6))} \\ &\leq d_A([\![s]\!]^{\mathbb{A}}_{g\circ\tau}, [\![t]\!]^{\mathbb{A}}_{g\circ\tau}) && \text{(by } U(f) \text{ nonexpansive)} \\ &\leq \epsilon && \text{(by } \mathbb{A} \text{ a model of } \Phi) \end{aligned}$$

□

Theorem 6.3 (Strict Monadicity of $U : \mathbf{QMod}_{\Sigma}(\Phi) \to \mathbf{FRel}$)**.** *The forgetful functor $U : \mathbf{QMod}_{\Sigma}(\Phi) \to \mathbf{FRel}$ is strictly monadic.*

Proof. From result (III) (see Section 6), we know that there is an adjunction $F \dashv U$. We now prove that the forgetful functor $U : \mathbf{QMod}_{\Sigma}(\Phi) \to \mathbf{FRel}$ strictly creates coequalizers for all $\mathbf{QMod}_{\Sigma}(\Phi)$-arrows f, g such that $U(f), U(g)$ has a split coequalizer (in $\mathbf{FRel}$). From this, it immediately follows by Beck's theorem (Proposition 2.25) that U is strictly monadic.

Let $f, g : \mathbb{A} \to \mathbb{B}$ be $\mathbf{QMod}_{\Sigma}(\Phi)$-arrows such that $U(f), U(g) : (A, d_A) \to (B, d_B)$ have a split coequalizer $e : (B, d_B) \to (C, d_C)$. We show that there exists a unique algebra $\mathbb{C}$ in $\mathbf{QMod}_{\Sigma}(\Phi)$ such that $U(\mathbb{C}) = (C, d_C)$, such that $e = U(u)$ for $u : \mathbb{B} \to \mathbb{C}$ an arrow in $\mathbf{QMod}_{\Sigma}(\Phi)$, and such that u is a coequalizer of f, g in $\mathbf{QMod}_{\Sigma}(\Phi)$.

Recall that $U_{\emptyset}$ is the forgetful functor $U_{\emptyset} : \mathbf{QAlg}(\Sigma) \to \mathbf{FRel}$ from Theorem 6.1. Since $U_{\emptyset}$ is strictly monadic (by Theorem 6.1), it satisfies condition (3) of Proposition 2.25. Since $\mathbf{QAlg}(\Sigma)$-arrows between objects in $\mathbf{QMod}_{\Sigma}(\Phi)$ coincide with $\mathbf{QMod}_{\Sigma}(\Phi)$-arrows, i.e., they are both defined as nonexpansive homomorphisms of quantitative Σ-algebras, the condition (3) of Proposition 2.25 implies that there is a unique algebra $\mathbb{C}$ in $\mathbf{QAlg}(\Sigma)$ such that $U_{\emptyset}(\mathbb{C}) = C$, such that $e = U_{\emptyset}(u)$ for $u : \mathbb{B} \to \mathbb{C}$ an arrow in $\mathbf{QAlg}(\Sigma)$, and such that u is a coequalizer of f, g in $\mathbf{QAlg}(\Sigma)$.

Now, $e = U_{\emptyset}(u)$ is a split coequalizer, so it has a right inverse $r : (C, d_C) \to (B, d_B)$. Hence, Lemma 6.2 applies and it says that $\mathbb{C}$ satisfies all the equations and quantitative equations satisfied by $\mathbb{B}$. In particular, $\mathbb{C}$ is a model of Φ because $\mathbb{B}$ is a model of Φ, i.e., $\mathbb{C}$ and u are an object and morphism in $\mathbf{QMod}_{\Sigma}(\Phi)$.

We conclude by noting that the uniqueness and universal property (being a coequalizer of f, g) of u that were true in $\mathbf{QAlg}(\Sigma)$ are also true in $\mathbf{QMod}_{\Sigma}(\Phi)$ because the latter is a full subcategory of the former. □

7 Lifting Presentations from Set to FRel

We recall, from Definition 2.27, that a **Set** monad M has an *equational presentation* if there exists a class of equations $\Phi \subseteq \mathrm{Eq}(\Sigma)$ over some signature Σ such that $T^{\mathbf{Set}}_{\Sigma,\Phi} \cong M$. A well known result, dating back to the seminal works of Lawvere [Law63] connecting the theory of monads with Universal Algebra, states that a **Set** monad M has an equational presentation if and only if it is *finitary* (see, e.g., [AR94, Chapter 3]). This result provides a useful correspondence between a logical notion (definability by equations) and a categorical one (finitary monad).

In the context of our theory of quantitative algebras, it is natural to give the following definition of **FRel** monads having a *quantitative equational presentation.*

Definition 7.1 (Quantitative Equational Presentation)**.** An **FRel** monad M has a quantitative equational presentation if there is a class of equations and quantitative equations $\Phi \subseteq \mathrm{QEq}(\Sigma)$, over some signature Σ, such that $T^{\mathbf{FRel}}_{\Sigma,\widehat{\Phi}} \cong M$.

The problem of characterising which **FRel** monads have a quantitative equational presentation in terms of categorical properties seems to be hard. For instance, Adámek provides in [Adá22, Example 4.1] an example of a class $\Phi \subseteq \mathrm{QEq}(\Sigma)$ such that $T^{\mathbf{FRel}}_{\Sigma,\Phi}$ is not finitary. This example is formulated in the context of the framework of Mardare, Panangaden and Plotkin [MPP16], but can be reformulated in our setting (see also Section 9 where we compare the framework of [MPP16] with ours).

In this section we establish (Theorem 7.5) a correspondence between **FRel** monads that are *liftings of* **Set** monads (Definition 7.2) having an equational presentation $\Psi \subseteq \mathrm{Eq}(\Sigma)$ (i.e., finitary **Set** monads) and *quantitative equational presentations* $\Phi \subseteq \mathrm{QEq}(\Sigma)$ that are *extensions* of $\Psi \subseteq \mathrm{Eq}(\Sigma)$ (Definition 7.4).

Before proceeding with the formal definitions, since we have to deal with both **Set** and **FRel** monads, and with both classes of equations in $\mathcal{P}(\mathrm{Eq}(\Sigma))$ and classes of **FRel** equations and quantitative equations in $\mathcal{P}(\mathrm{QEq}(\Sigma))$, in the the rest of this section we adopt the following notational convention:

1. We reserve the letters M and Φ for **Set** monads and classes of equations $\Phi \subseteq \mathrm{Eq}(\Sigma)$, respectively.

2. We use the letters $\widehat{M}$ and $\widehat{\Phi}$, with the "hat" notation, for **FRel** monads $\widehat{M}$ and classes of **FRel** equations and quantitative equations $\widehat{\Phi} \subseteq \mathrm{QEq}(\Sigma)$, respectively.

We first give the standard (see, e.g., [Bec69, p. 121]) definition of lifting of a monad.

Definition 7.2 (Monad Lifting)**.** An **FRel** monad $(\widehat{M}, \widehat{\eta}, \widehat{\mu})$ is a *lifting* of a **Set** monad (M, η, μ) if:

$$U\widehat{M} = MU \quad U\widehat{\eta} = \eta U \quad U\widehat{\mu} = \mu U,$$

where $U : \mathbf{FRel} \to \mathbf{Set}$ is the forgetful functor. More explicitly,

(i) the action of $\widehat{M}$ on objects is an assignment $(A, d_A) \mapsto (MA, \widehat{d}_A)$ that lifts every fuzzy relation d_A on A to a fuzzy relation $\widehat{d}_A$ on MA,

(ii) the actions of $\widehat{M}$ and M on morphisms coincide, set-theoretically,

(iii) the units η and $\widehat{\eta}$ coincide, set-theoretically. This means that for any $(A, d_A) \in \mathbf{FRel}$, the function $\eta_A : A \to MA$ is nonexpansive when seen as the map of type $\widehat{\eta}_{(A,d_A)} : (A, d_A) \to (MA, \widehat{d}_A)$.

(iv) the multiplications μ and $\widehat{\mu}$ coincide, set-theoretically. This means that for any $(A, d_A) \in \mathbf{FRel}$, the function $\mu_A : MMA \to MA$ is nonexpansive when seen as the map of type $\widehat{\mu}_{(A,d_A)} : (MMA, \widehat{\widehat{d}}_A) \to (MA, \widehat{d}_A)$.

Remark 7.3. Many monads of interest, on different **GMet** categories (like the usual category **Met** of metric spaces, see Section 2.3), are liftings of **Set** monads which have an equational presentation. Two important instances are the Hausdorff lifting of the finite powerset monad and the Kantorovich lifting of the finite distribution monad, on (pseudo-)metric spaces (see, e.g., [BBKK15, Examples 4.3 and 4.4]). There is also a combination of the two liftings: the Hausdorff–Kantorovich lifting of the convex sets of distributions monad [MV20] on (pseudo-)metric spaces. As a last example, we mention the formal ball monad on quasi-metric spaces [GL19] which is a lifting of a writer monad on **Set** (see also the quantitative writer monad of [BMPP21, §4.3.2]). It is a consequence of Theorem 7.5 (and its **GMet** variant Theorem 8.11) that all such liftings have quantitative equational presentations.

We now formally define when a class $\widehat{\Phi} \subseteq \mathrm{QEq}(\Sigma)$ is an extension of a class $\Phi \subseteq \mathrm{Eq}(\Sigma)$.

Definition 7.4 (Quantitative Extension)**.** Let Σ be a signature. A class $\widehat{\Phi} \subseteq \mathrm{QEq}(\Sigma)$ is a *quantitative extension* of a class $\Phi \subseteq \mathrm{Eq}(\Sigma)$ if

$$\begin{gathered}\text{for all } (A, d_A) \in \mathbf{FRel} \text{ and } s, t \in \mathrm{Terms}_\Sigma(A), \\ \Phi \Vdash_{\mathbf{Set}} \forall A.s = t \iff \widehat{\Phi} \Vdash_{\mathbf{FRel}} \forall (A, d_A).s = t.\end{gathered} \tag{8}$$

This guarantees that the equations entailed by $\widehat{\Phi}$ "coincide" with those of Φ, in the sense that $\forall A.s = t$ follows from Φ if and only if $\forall (A, d_A).s = t$ follows from $\widehat{\Phi}$, for all possible fuzzy relations d_A on A.

We are now ready to state our main result of this section.

Theorem 7.5. *Let* (M, η, μ) *be a monad on* **Set** *presented by* $\Phi \subseteq \mathrm{Eq}(\Sigma)$. *Then:*

(1) For any quantitative extension $\widehat{\Phi}$ *of* Φ*, there is a monad lifting* $\widehat{M}$ *of* M *presented by* $\widehat{\Phi}$*.*

(2) For any monad lifting $\widehat{M}$ *of* M*, there is a quantitative extension* $\widehat{\Phi}$ *of* Φ *presenting* $\widehat{M}$*.*

The goal of the rest of this section is to sketch the proof of the above theorem. A detailed proof is available in [MSV23].

For the statement in Item 1 of the theorem, we are a given a class $\widehat{\Phi} \subseteq \mathrm{QEq}(\Sigma)$ extending $\Phi \subseteq \mathrm{Eq}(\Sigma)$, and a **Set** monad M presented by Φ (with a given monad isomorphism $\rho : T^{\mathbf{Set}}_{\Sigma,\Phi} \cong M$). Our goal is to exhibit an **FRel** monad $\widehat{M}$ that lifts M and is presented by $\widehat{\Phi}$.

As a first step, we establish that, from the assumption that $\widehat{\Phi}$ extends Φ, it follows that $T^{\mathbf{FRel}}_{\Sigma,\widehat{\Phi}}$ is a monad lifting of $T^{\mathbf{Set}}_{\Sigma,\Phi}$. Hence, diagrammatically, the assumptions can be depicted as below (left) and our goal is to complete the diagram as in the (right):

$$\begin{array}{ccc} & & T^{\mathbf{FRel}}_{\Sigma,\widehat{\Phi}} \\ & & \big\downarrow U \\ M & \overset{\rho}{\cong} & T^{\mathbf{Set}}_{\Sigma,\Phi} \end{array} \qquad\qquad \begin{array}{ccc} \widehat{M} & \overset{\widehat{\rho}}{\cong} & T^{\mathbf{FRel}}_{\Sigma,\widehat{\Phi}} \\ U \big\downarrow & & \big\downarrow U \\ M & \overset{\rho}{\cong} & T^{\mathbf{Set}}_{\Sigma,\Phi} \end{array}$$

We thus need to define an **FRel** monad $\widehat{M}$ lifting M. We remark that, from Definition 7.2 of monad lifting, the unit, the multiplication and the action on morphisms on any such $\widehat{M}$ are fully determined (set-theoretically) by M. We therefore just need to specify the action of $\widehat{M}$ on objects $(A, d_A) \in \mathbf{FRel}$, respecting the constraint of Definition 7.2:

$$(A, d_A) \mapsto (MA, \widehat{d}_A).$$

Therefore, we only need to specify the fuzzy relation: $\widehat{d}_A : (MA)^2 \to [0, 1]$. To do this, we use the monad isomorphism $\rho : T^{\mathbf{Set}}_{\Sigma,\Phi} \cong M$ to get a bijection

$$\rho_A^{-1} : MA \to \mathrm{Terms}_\Sigma(A)/_{\equiv^A_\Phi}$$

between MA and the set $\text{Terms}_\Sigma(A)/_{\equiv^A_\Phi}$ underlying $T^{\mathbf{Set}}_{\Sigma,\Phi}$ and, as we have already established, also $T^{\mathbf{FRel}}_{\Sigma,\widehat{\Phi}}$. We can now define $\widehat{d}_A$ as follows:

$$\forall m, m' \in MA, \quad \widehat{d}_A(m, m') = \Delta^{F(A,d_A)}(\rho_A^{-1}(m), \rho_A^{-1}(m')),$$

where $\Delta^{F(A,d_A)}$ is the distance on quotiented terms obtained in Definition 5.8.

This completes the set-theoretic definition of the **FRel** monad $\widehat{M}$. The verification that all these definitions are valid in **FRel** (i.e., that the unit, the multiplication and the action of morphisms yield nonexpansive maps) is straightforward. The fact that $\widehat{M}$ is a lifting of M follows directly from its construction.

Finally, we define the components of the monad isomorphism $\widehat{\rho}$:

$$\widehat{\rho}_{(A,d_A)} : T^{\mathbf{FRel}}_{\Sigma,\widehat{\Phi}}(A, d_A) \to \widehat{M}(A, d_A)$$

to coincide with $\rho_A : T^{\mathbf{Set}}_{\Sigma,\Phi}A \to MA$, for every $(A, d_A) \in \mathbf{FRel}$. Checking that $\widehat{\rho}_{(A,d_A)}$ is indeed a map in **FRel** (i.e., it is nonexpansive) and that $\widehat{\rho}$ satisfies the constraints of a monad isomorphism is also straightforward.

For the statement in Item 2 of the theorem, we are given an **FRel** monad $\widehat{M}$ which is a lifting of a given **Set** monad M presented by some class of equations $\Phi \subseteq \text{Eq}(\Sigma)$ (with a given monad isomorphism $\rho : T^{\mathbf{Set}}_{\Sigma,\Phi} \cong M$). Our goal is to exhibit a class of **FRel** equations and quantitative equations $\widehat{\Phi} \subseteq \text{QEq}(\Sigma)$ such that: (i) $\widehat{\Phi}$ is a quantitative extension of Φ and (ii) there is a monad isomorphism $\widehat{\rho} : T^{\mathbf{FRel}}_{\Sigma,\widehat{\Phi}} \cong \widehat{M}$.

We define $\widehat{\Phi}$ to be the union of a class of **FRel** equations $\widehat{\Phi}_{\text{EQ}}$ and a class of **FRel** quantitative equations $\widehat{\Phi}_{\text{QEQ}}$

$$\widehat{\Phi} = \widehat{\Phi}_{\text{EQ}} \cup \widehat{\Phi}_{\text{QEQ}}$$

defined as follows.

The class $\widehat{\Phi}_{\text{EQ}}$ consists of all equations $\forall X.s = t$ entailed by Φ, transformed to **FRel** equations $\forall (X, d).s = t$, for all possible fuzzy relations d on X:

$$\widehat{\Phi}_{\text{EQ}} = \{\forall (X, d).s = t \mid \Phi \Vdash_{\mathbf{Set}} \forall X.s = t \text{ and } (X, d) \in \mathbf{FRel}\}\,. \tag{9}$$

The class $\widehat{\Phi}_{\text{QEQ}}$ contains quantitative equations of the form $\forall (X, d).s =_\epsilon t$, for all possible **FRel** spaces (X, d) and $s, t \in \text{Terms}_\Sigma(X)$. The $\epsilon \in [0, 1]$, expressing the distance between s and t, is obtained by:

(a) using the monad isomorphism $\rho : T^{\mathbf{Set}}_{\Sigma,\Phi} \cong M$ to get a bijection:

$$\rho_X : \text{Terms}_\Sigma(X)/_{\equiv_\Phi} \to MX$$

where the equivalence $\equiv_\Phi$ is defined as: $s \equiv_\Phi t \Leftrightarrow \Phi \Vdash_{\mathbf{Set}} \forall X.s = t$ (see Example 2.12),

(b) using the distance provided by the given **FRel** monad $\widehat{M}$, $\widehat{M}(X,d) = \big(MX, \widehat{d}\big)$, to obtain the required value for ϵ:

$$\epsilon = \widehat{d}\big(\rho_X\big([s]_{\equiv_\Phi}\big), \rho_X\big([t]_{\equiv_\Phi}\big)\big).$$

Thus, formally:

$$\widehat{\Phi}_{\mathrm{QEQ}} = \left\{ \forall (X,d).s =_\epsilon t \;\middle|\; \begin{array}{c} (X,d) \in \mathbf{FRel}, s,t \in \mathrm{Terms}_\Sigma(X), \\ \text{and } \epsilon = \widehat{d}\,(\rho_X([s]_{\equiv_\Phi}), \rho_X([t]_{\equiv_\Phi})) \end{array} \right\}. \tag{10}$$

The rest of the proof consists in verifying that the defined $\widehat{\Phi} = \widehat{\Phi}_{\mathrm{EQ}} \cup \widehat{\Phi}_{\mathrm{QEQ}}$ satisfies the desired properties: (i) $\widehat{\Phi}$ extends Φ and (ii) $\widehat{\rho}$, defined set-theoretically as ρ, is a monad isomorphism $\widehat{\rho} : T^{\mathbf{FRel}}_{\Sigma,\widehat{\Phi}} \cong \widehat{M}$.

8 From Fuzzy Relations to Generalised Metric Spaces

Most of the literature on quantitative algebras following the seminal paper [MPP16] (see, e.g., [MSV21, Adá22, BMPP21, BMPP18]) considers quantitative algebras whose carriers are metric spaces. Up to this point, our results have been stated for quantitative algebras (in the sense of Definition 3.1) whose carriers are arbitrary fuzzy relations.

In this section, we show that all the results proved so far also hold when, instead of **FRel**, we take as base category an arbitrary category **GMet** of generalised metric spaces (see Section 2.3), such as the category **Met** of metric spaces.

In what follows, we fix a category of generalised metric spaces **GMet** defined by a set $\mathcal{H}$ of $\mathscr{L}$-implications (see Definition 2.30) and a signature Σ. We denote by $\mathbf{QAlg}^{\mathbf{GMet}}(\Sigma)$ the full subcategory of $\mathbf{QAlg}^{\mathbf{FRel}}(\Sigma)$ comprising only quantitative algebras whose underlying fuzzy relations satisfy the $\mathscr{L}$-implications defining **GMet**:

$$\mathbf{QAlg}^{\mathbf{GMet}}(\Sigma) = \{(A, d_A, \{op^A\}_{op\in\Sigma}) \mid (A, d_A) \models^{\mathscr{L}} \mathcal{H}\} \subseteq \mathbf{QAlg}^{\mathbf{FRel}}(\Sigma). \tag{11}$$

Given a class of **FRel** equations and quantitative equations $\Phi \subseteq \mathrm{QEq}(\Sigma)$, we also denote by $\mathbf{QMod}^{\mathbf{GMet}}_\Sigma(\Phi)$ the full subcategory of $\mathbf{QMod}^{\mathbf{FRel}}_\Sigma(\Phi)$ comprising only quantitative algebras that belong to $\mathbf{QAlg}^{\mathbf{GMet}}(\Sigma)$:

$$\mathbf{QMod}^{\mathbf{GMet}}_\Sigma(\Phi) = \mathbf{QAlg}^{\mathbf{GMet}}(\Sigma) \cap \mathbf{QMod}^{\mathbf{FRel}}_\Sigma(\Phi). \tag{12}$$

Note that since we are taking full subcategories, homomorphisms of **GMet** quantitative Σ-algebras are still nonexpansive homomorphisms of the underlying Σ-algebras.

We first show that $\mathbf{QAlg}^{\mathbf{GMet}}(\Sigma)$ is a quantitative equationally definable class of quantitative Σ-algebras in the sense of Definition 3.8. In other words, we show (Corollary 8.6) that there is a class $\Phi_{\mathcal{H}} \subseteq \mathrm{QEq}(\Sigma)$ of **FRel** equations and quantitative equations such that

$$\mathbf{QAlg}^{\mathbf{GMet}}(\Sigma) = \mathbf{QMod}^{\mathbf{FRel}}_{\Sigma}(\Phi_{\mathcal{H}}).$$

We prove this fact by giving an explicit procedure to translate any $\mathscr{L}$-implication $H \in \mathcal{H}$ to an **FRel** equation or quantitative equation $\phi_H \in \mathrm{QEq}(\Sigma)$ having the following property:

For any quantitative algebra $(A, d_A, \{op^A\}_{op\in\Sigma}) \in \mathbf{QAlg}^{\mathbf{FRel}}(\Sigma)$,

$$(A, d_A) \models^{\mathscr{L}} H \iff \mathbb{A} \models \phi_H.$$

In fact, the terms in ϕ_H will not be built using any of the operations $op \in \Sigma$, so this translation is independent of the signature Σ.

Definition 8.1 (Translation)**.** Let

$$H = \forall x_1, \dots, x_n.\Big(\ (G_1 \wedge \cdots \wedge G_m \wedge G'_1 \cdots \wedge G'_k)\ \Rightarrow F\Big)$$

be an $\mathscr{L}$-implication, where:

- We denote with $X = \{x_1, \dots, x_n\}$ the set of variables occurring in H. Note that this set cannot be empty as the atomic formulas in H (not empty because F is one of them) are predicates ($x = y$ or $d(x, y) \leq \epsilon$) which must use variables.
- All atomic formulas G_i, $1 \leq i \leq m$ (possibly an empty set when $m = 0$), are of the form:
$$x = y$$
for some $x, y \in X$,
- All atomic formulas G'_j, $1 \leq j \leq k$ (possibly an empty set when $k = 0$), are of the form:
$$d(x, y) \leq \epsilon$$
for some $x, y \in X$ and $\epsilon \in [0, 1]$.

We are going to define a (quantitative) equation $\phi_H \in \mathrm{QEq}(\Sigma)$ constructed from H. We first use the premises $(G_1, \ldots, G_m, G'_1 \ldots, G'_k)$ of H to construct a fuzzy relation space (X_H, d_H). Let $\sim \subseteq X \times X$ be the smallest equivalence relation on X generated by (i.e., containing all) the pairs:

$$\big\{(x, x') \mid \text{there is a formula } G_i \text{ in } H \text{ of the form: } x = x'\big\}.$$

Hence, $\sim$ consists of exactly all pairs (x, y) of variables in X such that $x = y$ is logically implied by the conjunction of all formulas G_i. Let us denote with X_H the quotient $X/_\sim$, i.e., the set of all $\sim$-equivalence classes. Finally, let $d_H : X_H \times X_H \to [0, 1]$ be the following fuzzy relation on X_H

$$d_H([x]_\sim, [x']_\sim) = \min \left\{ \epsilon \in [0,1] \;\middle|\; \begin{array}{l} \text{there is a formula } G'_j \text{ in } H \\ \text{of the form: } d(y, y') \leq \epsilon, \\ \text{with } y \in [x]_\sim \text{ and } y' \in [x']_\sim \end{array} \right\},$$

with the convention $\min(\emptyset) = 1$. We have thereby defined the fuzzy relation space (X_H, d_H).

Now we use the conclusion F of H to construct ϕ_H which can be either an **FRel** equation or an **FRel** quantitative equation depending on F:

- If F is of the form $x = y$, for some $x, y \in X$, then:

$$\phi_H \text{ is defined as: } \quad \forall (X_H, d_H).[x]_\sim = [y]_\sim$$

- If F is of the form $d(x, y) \leq \epsilon$, for some $x, y \in X$ and $\epsilon \in [0, 1]$, then:

$$\phi_H \text{ is defined as: } \quad \forall (X_H, d_H).[x]_\sim =_\epsilon [y]_\sim.$$

We note that the two terms ($[x]_\sim$ and $[y]_\sim$) appearing in ϕ_H belong to $\mathrm{Terms}_\Sigma(X_H)$, because they both belong to X_H, for any specific choice of signature Σ. Hence the translation is well-defined for all Σ.

Before proving the main result regarding this translation (Lemma 8.4) we provide some illustrative examples.

Example 8.2. Consider the $\mathscr{L}$-implication H (a logically equivalent variant of (1) in Section 2.3):

$$\forall x_1.x_2.\; \big(x_1 = x_2 \;\Rightarrow\; d(x_1, x_2) \leq 0\big). \tag{13}$$

We are in the case where $X = \{x_1, x_2\}$, $n = 2$ (two variables), $m = 1$ (one atomic equation among the premises) and $k = 0$ (no atomic formula of the form $d(x, y) \leq \epsilon$ among the premises). Since the only premise (G_1)

of the formula is $x_1 = x_2$, we have that $x_1 \sim x_2$, and thus X_H consists of only one element $X_H = \{\ [x_1]_\sim\}$. By the definition of d_H we have that $d_H([x_1]_\sim, [x_1]_\sim) = 1$. Finally, since the conclusion is of the form $d(x_1, x_2) \leq 0$, we have that ϕ_H is defined as:

$$\forall\big(\{[x_1]_\sim\}, d_H\big).\ \ [x_1]_\sim =_0 [x_1]_\sim.$$

Now, we note that a nonexpansive interpretation $\tau : (X_H, d_H) \to (A, d_A)$ is simply a choice of an element $a = \tau([x_1]_\sim) \in A$, and ϕ_H holds under such an assignment if and only if $d_A(a, a) = 0$. Therefore, an algebra $\mathbb{A} \in \mathbf{QAlg}^{\mathbf{FRel}}(\Sigma)$ satisfies ϕ_H if and only if all elements of A have self-distance 0. This is indeed also the meaning of the $\mathscr{L}$-implication (13) and of the logically equivalent variant (1).

Example 8.3. Consider the $\mathscr{L}$-implication H (cf. (3) in Section 2.3):

$$\forall x_1, x_2.\ \big(d(x_1, x_2) \leq \epsilon\ \Rightarrow\ d(x_2, x_1) \leq \epsilon\big). \tag{14}$$

We are in the case where $X = \{x_1, x_2\}$, $n = 2$ (two variables), $m = 0$ (no atomic equations among the premises) and $k = 1$ (one atomic formula of the form $d(x, y) \leq \epsilon$ among the premises). In this case, the equivalence $\sim$ is the identity relation on X, hence $X_H = X$. By the definition of d_H, we have

$$d_H(x_1, x_1) = 1 \quad d_H(x_1, x_2) = \epsilon \quad d_H(x_2, x_1) = 1 \quad d_H(x_2, x_2) = 1.$$

Finally, since the conclusion is of the form $d(x_2, x_1) \leq \epsilon$, we have that ϕ_H is defined as:

$$\forall\big(\{x_1, x_2\}, d_H\big).\ \ x_2 =_\epsilon x_1.$$

One can check that a quantitative Σ-algebra satisfies ϕ_H if and only if the underlying fuzzy relation is symmetric, which is exactly what satisfaction of (14) means.

Lemma 8.4. *Let $\mathbb{A} = (A, d_A, \{op^A\}_{op\in\Sigma})$ be an* **FRel** *quantitative Σ-algebra, H be an $\mathscr{L}$-implication, and ϕ_H be the corresponding (quantitative) equation constructed in Definition 8.1. Then*

$$(A, d_A) \models^{\mathscr{L}} H \qquad \Longleftrightarrow \qquad \mathbb{A} \models \phi_H.$$

Proof. Let H be of the form described in Definition 8.1

$$\forall x_1, \ldots, x_n.\Big(\ (G_1 \wedge \cdots \wedge G_m \wedge G'_1 \cdots \wedge G'_k)\ \Rightarrow F\Big),$$

and let $X = \{x_1, \ldots, x_n\}$. We consider in parallel the two cases when the conclusion F is of the form

$$x = y \qquad \text{or} \qquad d(x, y) \leq \epsilon$$

for some $x, y \in X$. Let ϕ_H be defined as in Definition 8.1 and be of the form:

$$\forall (X_H, d_H).\ [x]_\sim = [y]_\sim \qquad \text{or} \qquad \forall (X_H, d_H).\ [x]_\sim =_\epsilon [y]_\sim.$$

We first show $(A, d_A) \models^{\mathscr{L}} H$ implies $\mathbb{A} \models \phi_H$.

Assume $(A, d_A) \models^{\mathscr{L}} H$ holds. This means that for all interpretations $\iota : X \to A$ of the variables X, if all premises G_i and G'_j hold under the interpretation ι then also F holds under ι. Our goal is to prove that $\mathbb{A} \models \phi_H$. This amount to showing that, for all interpretations of the fuzzy relation (X_H, d_H) in $\mathbb{A}$, i.e., for all nonexpansive maps $\tau : (X_H, d_H) \to (A, d_A)$, it holds that:

$$[\![[x]_\sim]\!]^{\mathbb{A}}_\tau = [\![[y]_\sim]\!]^{\mathbb{A}}_\tau \qquad \text{or} \qquad d_A\big([\![[x]_\sim]\!]^{\mathbb{A}}_\tau, [\![[y]_\sim]\!]^{\mathbb{A}}_\tau\big) \leq \epsilon$$

or equivalently, by definition of $[\![\_]\!]$, that:

$$\tau([x]_\sim) = \tau([y]_\sim) \qquad \text{or} \qquad d_A\big(\tau([x]_\sim), \tau([y]_\sim)\big) \leq \epsilon. \tag{15}$$

So let us fix an arbitrary such τ. Let $\iota_\tau : X \to A$ be the interpretation of the variables X defined as follows:

$$\iota_\tau(x) = \tau([x]_\sim).$$

We can show that all the premises G_i and G'_j of H hold under the interpretation ι_τ. Indeed consider G_i of the form

$$x = y$$

By definition of the equivalence relation $\sim$ from Definition 8.1, we know that $x \sim y$, or equivalently $[x]_\sim = [y]_\sim$, so we infer that $\iota_\tau(x) = \iota_\tau(y)$, and this means the premise G_i holds under ι. Now consider a premise G'_j of the form

$$d(x, y) \leq \epsilon.$$

By definition of d_H as a minimum from Definition 8.1, we know that the inequality $d_H([x]_\sim, [y]_\sim) \leq \epsilon$ holds. From the fact that τ is nonexpansive, we can therefore deduce that:

$$d_A\big(\tau([x]_\sim), \tau([y]_\sim)\big) \leq \epsilon,$$

which, by definition of ι_τ, is equivalent to

$$d_A\big(\iota_\tau(x), \iota_\tau(y)\big) \leq \epsilon,$$

which in turn precisely means that ι_τ satisfies the premise G'_j.

Hence, we have established that the interpretation ι_τ satisfies all the premises G_i and G'_j of H and therefore, by the assumption $(A, d_A) \models^{\mathscr{L}} H$, we know that it also satisfies the conclusion F, which means that:

$$\iota_\tau(x) = \iota_\tau(y) \qquad \text{or} \qquad d_A(\iota_\tau(x), \iota_\tau(y)) \leq \epsilon.$$

This in turn, by definition of $\iota_\tau(x)$, means that

$$\tau([x]_\sim) = \tau([y]_\sim) \qquad \text{or} \qquad d_A(\tau([x]_\sim), \tau([y]_\sim)) \leq \epsilon$$

This concludes the proof of (15). Since τ is arbitrary, we have established that $\mathbb{A} \models \phi_H$.

We now show that $\mathbb{A} \models \phi_H$ implies $(A, d_A) \models^{\mathscr{L}} H$. Assume $\mathbb{A} \models \phi_H$. This means that for every interpretation of (X_H, d_H) in $\mathbb{A}$, i.e., every nonexpansive map $\tau : (X_H, d_H) \to (A, d_A)$, (15) holds. Our goal is to show that $(A, d_A) \models^{\mathscr{L}} H$. This amount to showing that, for all interpretations $\iota : X \to A$ of the variables X, if all the premises G_i and G'_j of H are satisfied under the interpretation ι then also the conclusion F is satisfied by ι, i.e.:

$$\iota(x) = \iota(y) \qquad \text{or} \qquad d_A(\iota(x), \iota(y)) \leq \epsilon \tag{16}$$

So let us fix an arbitrary ι satisfying all of the premises G_i and G'_j of H. Let $\tau_\iota : (X_H, d_H) \to (A, d_A)$ be the interpretation of (X_H, d_H) in $\mathbb{A}$ defined as follows:

$$\tau_\iota([x]_\sim) = \iota(x).$$

Before moving further, we need to verify two facts:

1. that τ_ι is well-defined, i.e., that the definition does not depend on any specific choice of representative $x \in [x]_\sim$ of the equivalence class. Formally, we need to show that if $x_1 \sim x_2$, then $\iota(x_1) = \iota(x_2)$.

   Proof of fact Item 1 Assume $x_1 \sim x_2$. By definition of $\sim$, this means that the satisfaction of all the premises G_i in H implies satisfaction of the predicate $x_1 = x_2$. Since, by assumption, ι satisfies all premises in H, this means that $\iota(x_1) = \iota(x_2)$ as desired.

2. that (the well-defined) τ_ι is a nonexpansive map.

   Proof of fact Item 2: Assume $d_H([x_1]_\sim, [x_2]_\sim) \leq \epsilon$, for some $[x_1]_\sim, [x_2]_\sim \in X_H$ and $\epsilon \in [0, 1]$. We need to show that $d_A\big(\tau_\iota([x_1]_\sim), \tau_\iota([x_2]_\sim)\big) \leq \epsilon$ holds. Equivalently, by definition of τ_ι, we need to show that the inequality $d_A\big(\iota(x_1), \iota(x_2)\big) \leq \epsilon$ holds. Recall from Definition 8.1 that $d_H([x_1]_\sim, [x_2]_\sim)$ is defined as a minimum of all δ's such that there is some premise G'_j in H of the form $d(x_1, x_2) \leq \delta$. Therefore, since ι satisfies all the premises of H by assumption, we know that:

$$d_A\big(\iota(x_1), \iota(x_2)\big) \leq \delta$$

for all the δ's involved in the minimised expression. Hence

$$d_A\big(\iota(x_1), \iota(x_2)\big) \leq \epsilon$$

as desired.

Now that we have defined the interpretation τ_ι, we can apply the hypothesis ($\mathbb{A} \models \phi_H$) and obtain that the following holds:

$$\tau_\iota([x]_\sim) = \tau_\iota([y]_\sim) \qquad \text{or} \qquad d_A\big(\tau_\iota([x]_\sim), \tau_\iota([y]_\sim)\big) \leq \epsilon$$

By definition of τ_ι, this means that:

$$\iota(x) = \iota(y) \qquad \text{or} \qquad d_A\big(\iota(x), \iota(y)\big) \leq \epsilon$$

which means that the interpretation ι satisfies the conclusion F. Since ι is arbitrary, we have established that $(A, d_A) \models^{\mathscr{L}} H$. □

We can obtain a few useful corollaries from Lemma 8.4. The first extends the result of Lemma 8.4 from one $\mathscr{L}$-implication H to a set $\mathcal{H}$ of $\mathscr{L}$-implications. In what follows, we define the set $\Phi_{\mathcal{H}} \subseteq \mathrm{QEq}(\Sigma)$ as the set of (quantitative) equations:

$$\Phi_{\mathcal{H}} = \{\phi_H \mid H \in \mathcal{H} \text{ and } \phi_H \text{ is a (quantitative) equation translating } H\}$$

where the translation is the one specified in Definition 8.1.

Corollary 8.5. *Let $\mathbb{A} = (A, d_A, \{op^A\}_{op\in\Sigma})$ be an* **FRel** *quantitative Σ-algebra. Let $\mathcal{H}$ be a set of $\mathscr{L}$-implications. Then*

$$(A, d_A) \models^{\mathscr{L}} \mathcal{H} \qquad \Longleftrightarrow \qquad \mathbb{A} \in \mathbf{QMod}^{\mathbf{FRel}}_{\Sigma}(\Phi_{\mathcal{H}}).$$

Proof. We have

$$\begin{aligned} (A, d_A) \models^{\mathscr{L}} \mathcal{H} &\Leftrightarrow \forall H \in \mathcal{H}.\ (A, d_A) \models^{\mathscr{L}} H && \text{(by definition)} \\ &\Leftrightarrow \forall \phi_H \in \Phi_{\mathcal{H}}.\ \mathbb{A} \models \phi_H && \text{(by Lemma 8.4)} \\ &\Leftrightarrow \mathbb{A} \in \mathbf{QMod}^{\mathbf{FRel}}_{\Sigma}(\Phi_{\mathcal{H}}) && \text{(by definition).} \end{aligned}$$ □

Hence, the class of quantitative algebras $\mathbf{QAlg}^{\mathbf{GMet}}(\Sigma)$, which contains exactly those algebras satisfying the $\mathscr{L}$-implications in $\mathcal{H}$, is quantitative equationally definable.

Corollary 8.6. *For any signature* Σ *and any* **GMet** *category defined by a set* $\mathcal{H}$ *of* $\mathscr{L}$*-implications,*

$$\mathbf{QMod}^{\mathbf{FRel}}_{\Sigma}(\Phi_{\mathcal{H}}) = \mathbf{QAlg}^{\mathbf{GMet}}(\Sigma).$$

Proof. We have that for any quantitative algebra $\mathbb{A} \in \mathbf{QAlg}^{\mathbf{FRel}}(\Sigma)$,

$$\begin{aligned} \mathbb{A} \in \mathbf{QMod}^{\mathbf{FRel}}_{\Sigma}(\Phi_{\mathcal{H}}) &\Leftrightarrow (A, d_A) \models^{\mathscr{L}} \mathcal{H} && \text{(by Corollary 8.5)} \\ &\Leftrightarrow \mathbb{A} \in \mathbf{QAlg}^{\mathbf{GMet}}(\Sigma) && \text{(by (11))} \end{aligned}$$

Since both $\mathbf{QMod}^{\mathbf{FRel}}_{\Sigma}(\Phi_{\mathcal{H}})$ and $\mathbf{QAlg}^{\mathbf{GMet}}(\Sigma)$ are full subcategories of the category $\mathbf{QAlg}^{\mathbf{FRel}}(\Sigma)$ and we have shown that they have the same objects, they are the same categories. □

An important instance of this corollary is when Σ is empty. The category $\mathbf{QAlg}^{\mathbf{GMet}}(\Sigma)$ is then simply the category of fuzzy relations that satisfy $\mathcal{H}$, i.e., it is **GMet**. Thus, we have shown that **GMet** is a quantitative equationally definable class of fuzzy relations.

The next corollary is a further generalisation of the previous one, showing that for any class of equations and quantitative equations $\Phi \subseteq \mathrm{QEq}(\Sigma)$, $\mathbf{QMod}^{\mathbf{GMet}}_{\Sigma}(\Phi)$ is a quantitative equationally definable family of quantitative Σ-algebras. Namely, the full subcategories $\mathbf{QMod}^{\mathbf{FRel}}_{\Sigma}(\Phi_{\mathcal{H}} \cup \Phi)$ and $\mathbf{QMod}^{\mathbf{GMet}}_{\Sigma}(\Phi)$ of $\mathbf{QAlg}^{\mathbf{FRel}}(\Sigma)$ coincide.

Corollary 8.7. *For any signature* Σ*, for any* **GMet** *category defined by a set* $\mathcal{H}$ *of* $\mathscr{L}$*-implications, and for any class* $\Phi \subseteq \mathrm{QEq}(\Sigma)$ *of* **FRel** *equations and quantitative equations,*

$$\mathbf{QMod}^{\mathbf{FRel}}_{\Sigma}(\Phi_{\mathcal{H}} \cup \Phi) = \mathbf{QMod}^{\mathbf{GMet}}_{\Sigma}(\Phi).$$

Proof. For any quantitative algebra $\mathbb{A} \in \mathbf{QAlg}^{\mathbf{FRel}}(\Sigma)$, we have

$$\begin{aligned} \mathbb{A} \in \mathbf{QMod}^{\mathbf{FRel}}_{\Sigma}(\Phi_{\mathcal{H}} \cup \Phi) &\Leftrightarrow \mathbb{A} \in \mathbf{QMod}^{\mathbf{FRel}}_{\Sigma}(\Phi) \text{ and } \mathbb{A} \in \mathbf{QMod}^{\mathbf{FRel}}_{\Sigma}(\Phi_{\mathcal{H}}) \\ &\qquad \text{(by Corollary 8.6)} \\ &\Leftrightarrow \mathbb{A} \in \mathbf{QMod}^{\mathbf{FRel}}_{\Sigma}(\Phi) \text{ and } \mathbb{A} \in \mathbf{QAlg}^{\mathbf{GMet}}(\Sigma) \\ &\qquad \text{(by (12))} \\ &\Leftrightarrow \mathbb{A} \in \mathbf{QMod}^{\mathbf{GMet}}_{\Sigma}(\Phi). \end{aligned}$$

Hence, $\mathbf{QMod}^{\mathbf{FRel}}_{\Sigma}(\Phi_{\mathcal{H}} \cup \Phi)$ and $\mathbf{QMod}^{\mathbf{GMet}}_{\Sigma}(\Phi)$ have the same objects, and, since they are full subcategories of $\mathbf{QAlg}^{\mathbf{FRel}}(\Sigma)$, they also have the same morphisms. □

We can now show that all the results proved for **FRel** in Section 4, Section 5, Section 6 and Section 7 also hold when specialised for a category **GMet** defined by a $\mathcal{H}$ of $\mathscr{L}$-implications.

Starting from the relation $\vdash_{\mathbf{FRel}}$ for **FRel** defined in Definition 4.1, we define a relation $\vdash_{\mathbf{GMet}}$ for **GMet** as follows:

$$\Phi \vdash_{\mathbf{GMet}} \phi \iff \Phi_{\mathcal{H}} \cup \Phi \vdash_{\mathbf{FRel}} \phi.$$

Theorem 8.8 shows that the relation $\vdash_{\mathbf{GMet}}$ is sound and complete for the relation $\Vdash_{\mathbf{GMet}}$, which is the restriction of $\Vdash_{\mathbf{FRel}}$ to **GMet** defined as follows:

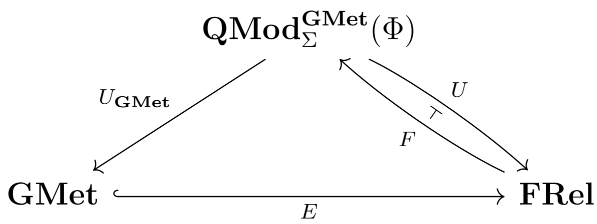
$$\Phi \Vdash_{\mathbf{GMet}} \phi \iff \forall \mathbb{A} \in \mathbf{QMod}_{\Sigma}^{\mathbf{GMet}}(\Phi),\ \mathbb{A} \models \phi.$$

Theorem 8.8 (Soundness and Completeness for **GMet**). $\Phi \vdash_{\mathbf{GMet}} \phi \iff \Phi \Vdash_{\mathbf{GMet}} \phi$.

Proof. We have

$$\begin{aligned}
\Phi \vdash_{\mathbf{GMet}} \phi &\Leftrightarrow \Phi_{\mathcal{H}} \cup \Phi \vdash_{\mathbf{FRel}} \phi && \text{(definition of } \vdash_{\mathbf{GMet}})\\
&\Leftrightarrow \Phi_{\mathcal{H}} \cup \Phi \Vdash_{\mathbf{FRel}} \phi && \text{(by Corollary 5.18)}\\
&\Leftrightarrow \forall \mathbb{A} \in \mathbf{QMod}_{\Sigma}^{\mathbf{FRel}}(\Phi_{\mathcal{H}} \cup \Phi),\ \mathbb{A} \models \phi && \text{(definition of } \Vdash_{\mathbf{FRel}})\\
&\Leftrightarrow \forall \mathbb{A} \in \mathbf{QMod}_{\Sigma}^{\mathbf{GMet}}(\Phi),\ \mathbb{A} \models \phi && \text{(by Corollary 8.7)}\\
&\Leftrightarrow \Phi \Vdash_{\mathbf{GMet}} \phi. && \text{(definition of } \Vdash_{\mathbf{GMet}})\ \square
\end{aligned}$$

Now fix a class of equations and quantitative equations $\Phi \subseteq \mathrm{QEq}(\Sigma)$. We have the following diagram:



(17)

where:

- E is the (full and faithful) functor embedding **GMet** into **FRel**;
- $U_{\mathbf{GMet}}$ is the forgetful functor of type $U_{\mathbf{GMet}} : \mathbf{QMod}_{\Sigma}^{\mathbf{GMet}}(\Phi) \to \mathbf{GMet}$;
- U is the forgetful functor of type $U : \mathbf{QMod}_{\Sigma}^{\mathbf{FRel}}(\Phi_{\mathcal{H}} \cup \Phi) \to \mathbf{FRel}$, which indeed also has type $U : \mathbf{QMod}_{\Sigma}^{\mathbf{GMet}}(\Phi) \to \mathbf{FRel}$ by Corollary 8.7;

- F is the left adjoint of the forgetful functor U, as given in Section 6.

We can then define the functor $F_{\mathbf{GMet}} : \mathbf{GMet} \to \mathbf{QMod}_{\Sigma}^{\mathbf{GMet}}(\Phi)$ as follows

$$F_{\mathbf{GMet}} = F \circ E.$$

Theorem 8.9. *The functor $F_{\mathbf{GMet}} : \mathbf{GMet} \to \mathbf{QMod}_{\Sigma}^{\mathbf{GMet}}(\Phi)$ is a left adjoint of $U_{\mathbf{GMet}} : \mathbf{QMod}_{\Sigma}^{\mathbf{GMet}}(\Phi) \to \mathbf{GMet}$.*

Proof. Let $\eta : \mathrm{id}_{\mathbf{FRel}} \Rightarrow U \circ F$ be the unit of the adjunction $F \dashv U$. For any **GMet** space (A, d_A), define

$$\eta'_{(A,d_A)} = \eta_{E(A,d_A)} : E(A, d_A) \to UFE(A, d_A)$$

Since, by the diagram (17) we have $UFE(A, d_A) = EU_{\mathbf{GMet}}FE(A, d_A) = EU_{\mathbf{GMet}}F_{\mathbf{GMet}}(A, d_A)$, and since E embeds a **GMet** space into an **FRel** space, we can see $\eta'_{(A,d_A)}$ as a function of type

$$\eta'_{(A,d_A)} : (A, d_A) \to U_{\mathbf{GMet}}F_{\mathbf{GMet}}(A, d_A)$$

Since η is a natural transformation and E acts like identity on morphisms, we also obtain a natural transformation

$$\eta' : \mathrm{id}_{\mathbf{GMet}} \Rightarrow U_{\mathbf{GMet}}F_{\mathbf{GMet}}.$$

Now take a **GMet** space (A, d_A), a quantitative algebra

$$\mathbb{B} = (B, d_B, \{op^{\mathbb{B}}\}_{op\in\Sigma}) \in \mathbf{QMod}_{\Sigma}^{\mathbf{GMet}}(\Phi)$$

and a nonexpansive map $f : (A, d_A) \to (B, d_B)$. Since $F \dashv U$ is an adjunction with unit η, by seeing f as the **FRel** morphism $E(f)$ we obtain that there is a unique quantitative algebra homomorphism $g : F(E(A, d_A)) \to \mathbb{B}$ such that $E(f) = U(g) \circ \eta_{E(A,d_A)}$.

By definition of the embedding E and of $\eta'_{(A,d_A)}$, this implies that there is a unique quantitative algebra homomorphism $g : F_{\mathbf{GMet}}(A, d_A) \to \mathbb{B}$ such that $f = U_{\mathbf{GMet}}(g) \circ \eta'_{(A,d_A)}$. Hence, $F_{\mathbf{GMet}}$ is a left adjoint of $U_{\mathbf{GMet}}$. □

Note that, by definition, the functor $F_{\mathbf{GMet}}$ acts as the functor F on **GMet** spaces, and thus the free $U_{\mathbf{GMet}}$-object generated by a generalised metric space (A, d_A) is the quantitative algebra of quotiented terms built as in Section 5. The monad on **GMet** obtained from the composite $U_{\mathbf{GMet}} \circ F_{\mathbf{GMet}}$ will be denoted $T_{\Sigma,\Phi}^{\mathbf{GMet}}$.

Moreover, we have strict monadicity of the functor $U_{\mathbf{GMet}}$.[15]

[15] This is direct a consequence of a more abstract result sometimes called “cancellability of monadicity”, see [Bou92, Proposition 5] or [AM23, Corollary 5.6]. We a give a direct proof to keep the document self contained.

Theorem 8.10. *The functor* $U_{\mathbf{GMet}} : \mathbf{QMod}_{\Sigma}^{\mathbf{GMet}}(\Phi) \to \mathbf{GMet}$ *is strictly monadic.*

Proof. By Beck's theorem, in order to show that $U_{\mathbf{GMet}}$ is strictly monadic it is enough to show that $U_{\mathbf{GMet}}$ strictly creates coequalizers for pairs of morphisms $f, g \in \mathbf{QMod}_{\Sigma}^{\mathbf{GMet}}(\Phi)$ such that $U_{\mathbf{GMet}}f$ and $U_{\mathbf{GMet}}g$ have an absolute coequalizer.

Let $f, g : \mathbb{A} \to \mathbb{B}$ be such a pair and

$$U_{\mathbf{GMet}}\mathbb{A} \underset{U_{\mathbf{GMet}}g}{\overset{U_{\mathbf{GMet}}f}{\rightrightarrows}} U_{\mathbf{GMet}}\mathbb{B} \xrightarrow{h} (C, d_C) \tag{18}$$

be an absolute coequalizer in $\mathbf{GMet}$. Applying E to (18), we obtain an absolute coequalizer in $\mathbf{FRel}$:

$$U\mathbb{A} \underset{Ug}{\overset{Uf}{\rightrightarrows}} U\mathbb{B} \xrightarrow{Eh} E(C, d_C) \ .$$

By Theorem 6.3 we know that U is strictly monadic, and thus strictly creates coequalizers for pairs of arrows f, g such that Uf and Ug have an absolute coequalizer. So, we derive that there exists a unique morphism $\hat{h} : \mathbb{B} \to \mathbb{C}$ in $\mathbf{QMod}_{\Sigma}^{\mathbf{GMet}}(\Phi)$ such that $U(\mathbb{C}) = E(C, d_C)$ and $U(\hat{h}) = Eh$ and $\hat{h}$ is a coequalizer for f, g.

Moreover, we also have that $\hat{h} : \mathbb{B} \to \mathbb{C}$ is unique in $\mathbf{QMod}_{\Sigma}^{\mathbf{GMet}}(\Phi)$ such that $U_{\mathbf{GMet}}(\mathbb{C}) = (C, d_C)$, $U_{\mathbf{GMet}}(\hat{h}) = h$ and $\hat{h}$ is a coequalizer of f and g. To see this, suppose that $u : \mathbb{B} \to \mathbb{C}'$ in $\mathbf{QMod}_{\Sigma}^{\mathbf{GMet}}(\Phi)$ is a coequalizer of f and g such that $U_{\mathbf{GMet}}(\mathbb{C}') = (C, d_C)$ and $U_{\mathbf{GMet}}(u) = h$. Then, applying E yields $U(\mathbb{C}') = E(C, d_C)$ and $U(u) = Eh$, and because U strictly creates coequalizers of f and g, we must have $\hat{h} = u$.

Hence, we have obtained that $\hat{h} : \mathbb{B} \to \mathbb{C}$ is unique in $\mathbf{QMod}_{\Sigma}^{\mathbf{GMet}}(\Phi)$ such that $U_{\mathbf{GMet}}(\mathbb{C}) = (C, d_C)$ and $U_{\mathbf{GMet}}(\hat{h}) = h$, and that it is a coequalizer of f, g. This allows us to conclude that $U_{\mathbf{GMet}}$ strictly creates coequalizers for pairs of arrows f, g such that $U_{\mathbf{GMet}}f$ and $U_{\mathbf{GMet}}g$ have an absolute coequalizer. □

Finally, we adapt the results of Section 7 to $\mathbf{GMet}$. The three central notions of quantitative equational presentations (Definition 7.1), monad liftings (Definition 7.2) and quantitative extensions (Definition 7.4) just need to be modified in a straightforward way by replacing all instances of $\mathbf{FRel}$ to $\mathbf{GMet}$.[16]

[16] Note that the notion of equation and quantitative equation remains as in Definition 3.4, i.e., (A, d_A) in $\forall(A, d_A).s = t$ and $\forall(A, d_A).s =_{\epsilon} t$ is an arbitrary $\mathbf{FRel}$ space.

First, a *quantitative equational presentation* of a monad M on **GMet** is a class of **FRel** equations and quantitative equations $\widehat{\Phi} \subseteq \mathrm{QEq}(\Sigma)$ along with a monad isomorphism $T^{\mathbf{GMet}}_{\Sigma,\widehat{\Phi}} \cong M$, where we recall that $T^{\mathbf{GMet}}_{\Sigma,\widehat{\Phi}}$ is the monad obtained from going around the triangle in (17). Second, a **GMet** monad $(\widehat{M}, \widehat{\eta}, \widehat{\mu})$ is a **GMet** lifting of a **Set** monad (M, η, μ) if

$$U\widehat{M} = MU \quad U\widehat{\eta} = \eta U \quad U\widehat{\mu} = \mu U,$$

where U is now the forgetful functor $U : \mathbf{GMet} \to \mathbf{Set}$. The explicit description of what it means to be a monad lifting (in Definition 7.2) is still valid after replacing fuzzy relations with generalised metric spaces. Third, a class of **FRel** equations and quantitative equations $\widehat{\Phi} \subseteq \mathrm{QEq}(\Sigma)$ is a **GMet** *quantitative extension* of $\Phi \subseteq \mathrm{Eq}(\Sigma)$ if:

$$\begin{aligned}&\text{for all } (A, d_A) \in \mathbf{GMet} \text{ and } s, t \in \mathrm{Terms}_\Sigma(A),\\ &\Phi \Vdash_{\mathbf{Set}} \forall A.s = t \iff \widehat{\Phi} \Vdash_{\mathbf{GMet}} \forall (A, d_A).s = t.\end{aligned} \tag{19}$$

This simple transformation can also be carried out in the proofs, yielding the following theorem.

Theorem 8.11. *Let (M, η, μ) be a monad on* **Set** *presented by $\Phi \subseteq \mathrm{Eq}(\Sigma)$. The following holds:*

(1) For any **GMet** *quantitative extension $\widehat{\Phi}$ of Φ, there is a* **GMet** *monad lifting $\widehat{M}$ of M presented by $\widehat{\Phi}$.*

(2) For any **GMet** *monad lifting $\widehat{M}$ of M, there is a* **GMet** *quantitative extension $\widehat{\Phi}$ of Φ that presents $\widehat{M}$.*

While the proof of Theorem 8.11 above is essentially identical to that of Theorem 7.5, the results that the two theorems state may present some subtle differences.

For instance, there are classes of equations and quantitative equations $\widehat{\Phi} \subseteq \mathrm{QEq}(\Sigma)$ such that $T^{\mathbf{GMet}}_{\Sigma,\widehat{\Phi}}$ is a **GMet** monad lifting (of some monad M on **Set**) but $T^{\mathbf{FRel}}_{\Sigma,\widehat{\Phi}}$ is not. For a concrete example, let $\Sigma = \emptyset$ and $\widehat{\Phi}$ be the class $\Phi_{\mathcal{H}_{\mathbf{Met}}}$ resulting from the translation (as in Definition 8.1) of the set of $\mathscr{L}$-implications $\mathcal{H}_{\mathbf{Met}}$ defining the category **Met** (see Definition 2.32). It is readily seen that the monad $T^{\mathbf{GMet}}_{\Sigma,\widehat{\Phi}}$ on **Met** is a lifting of the identity monad on **Set**. However, the monad $T^{\mathbf{FRel}}_{\Sigma,\widehat{\Phi}}$ on **FRel** is not a lifting of the any monad on **Set**. It sends (A, d_A) to (A, d_A) when d_A is a metric, but when e.g. $d_A(a, b) = 0$ for $a \neq b \in A$, the carrier set of $T^{\mathbf{FRel}}_{\Sigma,\widehat{\Phi}}(A, d_A)$ will be a quotient of A where a and b are identified. This means $T^{\mathbf{FRel}}_{\emptyset,\widehat{\Phi}}$ cannot lift a monad on **Set** because it sends two fuzzy relations with identical carrier set to fuzzy relations with different carriers.

9 Comparison with the Framework of Mardare, Panangaden and Plotkin

In Section 3 we have formally introduced our theory of quantitative algebras and in sections Section 4, Section 5, Section 6, Section 7 and Section 8 we have stated and proved the main results.

In this section we compare our theory with the original one presented in the seminal paper [MPP16]. For clarity purposes, in what follows we refer to our theory as "MSV theory" (for Mio, Sarkis and Vignudelli [MSV23]) and to that of [MPP16] as "MPP theory".

A first point of comparison is that the MPP theory deals with quantitative algebras over metric spaces formally specified as follows:[17]

Definition 9.1 (MPP Quantitative Algebra)**.** [MPP16, Definition 3.1] Given a signature Σ, a MPP quantitative Σ-algebra is a triple $(A, d_A, \{op^A\}_{op\in\Sigma})$ such that $(A, d_A) \in \mathbf{Met}$ is a metric space and such that all interpretations of operation symbols in the signature $op^A : A^n \to A$ (where $ar(op) = n$) are nonexpansive functions

$$op^A : (A^n, d_A^n) \to (A, d_A),$$

where d_A^n is the product metric defined as: $d_A^n((a_1, \dots, a_n), (a'_1, \dots, a'_n)) = \max\limits_{i=1\dots n} \{d_A(a_i, a'_i)\}$.

As a result, it only makes sense to compare the MPP theory to the MSV theory restricted to the category **Met** of metric spaces. This restriction is done by first seeing **Met** as the category of fuzzy relations satisfying the $\mathscr{L}$-implications in $\mathcal{H}_{\mathbf{Met}}$ as explained in Section 2.3, and then instantiating the results of Section 8 with **GMet** = **Met**.

Note that a MPP quantitative Σ-algebra is a MSV quantititative Σ-algebra $\mathbb{A} \in \mathbf{QAlg}^{\mathbf{Met}}(\Sigma)$ such that all the interpretations op^A are nonexpansive in the sense of Definition 9.1. It is straightforward to see that, for a given $op \in \Sigma$ of arity n, the interpretation op^A of a quantitative algebra $\mathbb{A} \in \mathbf{QAlg}^{\mathbf{Met}}(\Sigma)$ is nonexpansive if and only if $\mathbb{A}$ satisfies all the following quantitative equations ϕ_d^{op}, one for each fuzzy relation $d : X^2 \to [0, 1]$, where $X = \{x_1, \dots, x_n, x'_1, \dots, x'_n\}$

$$\forall(X, d).op(x_1, \dots, x_n) =_\epsilon op(x'_1, \dots, x'_n) \qquad\qquad \epsilon = \max_{i=1\dots n} \{d(x_i, x'_i)\}.$$

[17] A technical difference between Definition 9.1 and [MPP16, Definition 3.1] is that the metrics of the latter are actually extended metrics, i.e. $d_A(a, b)$ ranges in $[0, \infty]$ instead of $[0, 1]$.

In other words, op^A is nonexpansive in the sense of Definition 9.1 if and only if $\mathbb{A}$ belongs to $\mathbf{QMod}_\Sigma^{\mathbf{Met}}(\Phi_{NE}^{op})$, where Φ_{NE}^{op} is the class of quantitative equations

$$\Phi_{NE}^{op} = \bigcup_{d:X^2\to[0,1]} \{\phi_d^{op}\}.$$

Therefore the class of MPP quantitative Σ-algebras is a quantitative equationally definable class of MSV quantitative algebras in $\mathbf{QAlg}^{\mathbf{Met}}(\Sigma)$, and its theory is generated by

$$\Phi_{NE} = \bigcup_{op\in\Sigma} \Phi_{NE}^{op}.$$

Furthermore, it is an immediate consequence of Corollary 8.7 that the class of MPP quantitative algebras can be quantitative equationally defined in $\mathbf{QAlg}^{\mathbf{FRel}}(\Sigma)$ as the class $\mathbf{QMod}_\Sigma^{\mathbf{FRel}}(\Phi_{NE} \cup \Phi_{\mathcal{H}_{\mathbf{Met}}})$

Remark 9.2. Note that the MSV theory allows for other interesting properties of op^A to be defined by quantitative equations. For example, by using:

$$\epsilon = \alpha \cdot \max_{i=1\ldots n}\{d(x_i, x_i)\} \qquad \text{for some } \alpha > 0$$

in the definition of ϕ_d^{op} one expresses the property of being Lipschitz with constant α (nonexpansivness being the case $\alpha = 1$).

We now proceed to compare the logical expressiveness of the MPP and MSV frameworks.

First, we observe that in the MSV theory restricted to **Met**, equations of the form $\forall(X, d).s = t$ and quantitative equations $\forall(X, d).s =_0 t$ are semantically equivalent, and in fact mutually derivable in the deductive system $\vdash_{\mathbf{Met}}$. This just reflects the fact that metric spaces satisfy the property $x = y \Leftrightarrow d(x, y) = 0$ (c.f. the set $\mathcal{H}_{\mathbf{Met}}$ of $\mathscr{L}$-implications defining the category **Met** in Section 2.3). Hence in the MSV theory for **Met**, as far as expressiveness is concerned, we can just restrict our attention to quantitative equations.

Secondly, the basic logical judgment in the MPP framework is a form of implication (possibly with infinitely many premises), called *quantitative inference*, of the form[18]

$$\{s_i =_{\epsilon_i} t_i\}_{i\in I} \Rightarrow s =_\epsilon t,$$

[18]The notation used in [MPP16, Definition 2.1] is $\{s_i =_{\epsilon_i} t_i\}_{i\in I} \vdash s =_\epsilon t$, but it clashes with our use of the turnstile $\vdash$, so we write $\Rightarrow$ instead.

where $s_i, t_i, s, t \in \text{Terms}_\Sigma(X)$, for some set X, and $\epsilon_i, \epsilon \in [0,1]$. A MPP quantitative algebra $\mathbb{A} = (A, d_A, \{op^A\}_{op\in\Sigma})$ (or, alternatively, a MSV quantitative algebra in $\mathbf{QMod}_\Sigma^{\mathbf{Met}}(\Phi_{NE})$, as noted above) satisfies such a judgment J, written $\mathbb{A} \models_{\text{MPP}} J$, if for all set-theoretic interpretations $j : X \to A$ it holds that:

$$\text{if, for all } i \in I,\ d_A([\![s_i]\!]_j^{\mathbb{A}}, [\![t_i]\!]_j^{\mathbb{A}}) \leq \epsilon_i \text{ holds, then } d_A([\![s]\!]_j^{\mathbb{A}}, [\![t]\!]_j^{\mathbb{A}}) \leq \epsilon.$$

A judgment J is called a *basic quantitative inference* if all the terms s_i, t_i appearing on the left-side of the implication are variables in X, i.e., J is of the form:

$$\{x_i =_{\epsilon_i} x'_i\}_{i\in I} \Rightarrow s =_\epsilon t.$$

One can verify that for every basic quantitative inference J of the above shape and for any MSV quantitative algebra $\mathbb{A} = (A, d_A, \{op^A\}_{op\in\Sigma})$ in $\mathbf{QMod}_\Sigma^{\mathbf{GMet}}(\Phi_{NE})$ it holds that:

$$\mathbb{A} \models_{\text{MPP}} J \qquad \Longleftrightarrow \qquad \mathbb{A} \models \phi_J,$$

where ϕ_J is the formula $\forall(X, d_X).s =_\epsilon t$ with X being the set of variables appearing in the premises of J and $d_X : X^2 \to [0,1]$ defined as follows:

$$d_X(x, x') = \inf\{\epsilon_i \mid (x =_{\epsilon_i} x') \text{ is among the premises of } J\}$$

where the infimum of the empty set is 1. Note that, in the formula ϕ_J, the fuzzy relation (X, d_X) is constructed (in a similar fashion to the translation of Definition 8.1) to ensure that nonexpansive interpretations $\tau : (X, d_X) \to (A, d_A)$ correspond to set-theoretic interpretations $j : X \to A$ satisfying the premises of J (i.e. $\forall i \in I, d_X(j(x_i), j(x'_i)) \leq \epsilon_i$).

In the opposite direction, given any quantitative equation ϕ of the form $\forall(X, d_X).s =_\epsilon t$, it holds that

$$\mathbb{A} \models_{\text{MPP}} J_\phi \qquad \Longleftrightarrow \qquad \mathbb{A} \models \phi$$

where J_ϕ is

$$\{x =_{d_X(x,x')} x'\}_{x,x'\in X} \Rightarrow s =_\epsilon t.$$

We can therefore conclude that the expressive power of MPP basic inferences J and MSV quantitative equations ϕ is the same. This means that our MSV theory, restricted to **Met**, coincides with the MPP theory where only *basic* quantitative inferences are used. We note that this is a mild restriction, as most interesting results and application instances of the MPP framework only use basic quantitative inferences [MPP16, MPP17, BMPP18, BMPP21, MV20, MSV21, MSV22]

As a further point of comparison, we now discuss the proof systems. Note that our MSV proof system $\vdash_{\mathbf{Met}}$, which we have proved to be sound and complete, is not obtained by simply "restricting" the MPP proof system of [MPP16] to basic quantitative inferences (via the translation $J \mapsto \phi_J$). Indeed the MPP proof system is not "closed under basic quantitative inferences". The reason is the presence in the MPP proof system of the following *substitution rule*:

Rule (Subst) in [MPP16, Definition 2.1]

$$\frac{\left\{\ s_i =_{\epsilon_i} t_i\ \right\}_{i\in I} \Rightarrow s =_\epsilon t}{\left\{\ \sigma(s_i) =_{\epsilon_i} \sigma(t_i)\ \right\}_{i\in I} \Rightarrow \sigma(s) =_\epsilon \sigma(t)}\ \text{Substitution by } \sigma$$

where all terms have variables ranging over a set X and $\sigma : X \to \mathrm{Terms}_\Sigma(X)$ is a substitution, which is homomorphically extended to a function of type $\sigma : \mathrm{Terms}_\Sigma(X) \to \mathrm{Terms}_\Sigma(X)$.

Note that even in the case where the premise of the substitution rule is a basic quantitative inference (i.e., the terms s_i and t_i are variables for all $i \in I$), the conclusion of the rule is generally not a basic quantitative inference, because the substitution is also applied to the premises.

This highlights the novelty in the design of our MSV proof system $\vdash_{\mathbf{Met}}$ (the new substitution rule), which in turn also proves a novel result applicable to the MPP theory: a sound and complete proof system for basic quantitative inference exists (via the translation $\phi \mapsto J_\phi$).

To conclude, we now compare the expressiveness of the MPP theory and of the MSV theory in terms of which **Met** monads can be presented, respectively, by a class of basic quantitative inferences and by a class of quantitative equations. By exploiting the correspondence between monad liftings and quantitative extensions proved in Section 7, instantiated to the category **Met** via Theorem 8.11, we show the following result:

> *There exist monads on* **Met** *which can be presented by a class of quantitative equations in the MSV theory, but which cannot be presented by a class of basic quantitative inferences in the MPP theory.*

To see this, consider the (finite, non-empty) powerset monad $(\mathscr{P}, \eta, \mu)$ on **Set**, which is presented by the equations Φ of semilattices. Define the monad $(\widehat{\mathscr{P}}, \widehat{\eta}, \widehat{\mu})$ on **Met** where the functor $\widehat{\mathscr{P}} : \mathbf{Met} \to \mathbf{Met}$ is such that

$$\widehat{\mathscr{P}}(X,d) = (\mathscr{P}X, \widehat{d}) \quad \text{with} \quad \widehat{d}(S,S') = \begin{cases} 0 & S = S' \\ d(x,y) & S = \{x\} \text{ and } S' = \{y\} \\ 1 & \text{otherwise} \end{cases}$$

and where the unit $\widehat{\eta}$ and multiplication $\widehat{\mu}$ coincide, as **Set** functions, with the unit η and multiplication μ of the monad $\mathscr{P}$. The **Met** monad $\widehat{\mathscr{P}}$ is a monad lifting of the **Set** monad $\mathscr{P}$, and this implies by Theorem 8.11 that there is a **Met** quantitative extension $\widehat{\Phi}$ of the equations of semilattices Φ which is a presentation of $\widehat{\mathscr{P}}$. Hence, $\widehat{\mathscr{P}}$ has a presentation in the MSV theory.

In contrast, there is no class of basic quantitative inferences presenting the monad $\widehat{\mathscr{P}}$ in the MPP theory. This is a consequence of the fact that all monads which can be presented by a class of basic quantitative inferences in the MPP theory are enriched (see [ADV23, Theorem 8.10]), and that the monad $\widehat{\mathscr{P}}$ is not enriched. A proof of such properties of of the monad $\widehat{\mathscr{P}}$ is available in [MSV23].

10 Conclusions and Directions for Future Work

We have presented an extension of the theory of quantitative algebras of Mardare, Panangaden and Plotkin [MPP16]. In our theory the carriers of quantitative algebras are not restricted to be metric spaces and can be arbitrary fuzzy relations (or generalised metric spaces) and the interpretations of the algebraic operations are not required to be nonexpansive. We have established some key results, including the soundness and completeness of a novel proof system, the existence of free quantitative algebras, the strict monadicity of the associated Free-Forgetful adjunction, and the correspondence between monad liftings of a finitary monad and quantitative extensions of an equational presentation.

A first direction for future work is to adapt and generalise to our setting some theoretical results obtained for the framework of Mardare, Panangaden and Plotkin [MPP16]. Examples include: monad composition techniques [BMPP18] (see also [BMPP22]), fixed-points [MPP21], completion techniques [BMPP18], variety "HSP-style" theorems[19] [MPP17, Adá22].

[19] Jan Jurka, Stefan Milius and Henning Urbat already have preliminary results in this direction: talk titled "Varieties of Quantitative Algebras: A Categorical Perspective" given by Henning Urbat at the QUALOG 2023 workshop, the 25th of June 2023, Boston (USA).

A second direction, more oriented towards applications, consists in leveraging the additional flexibility provided by our theory. For example, in [DLHLP22] the authors investigate Curry's *combinatory logic* (an algebraic counterpart of the λ-calculus) under the lenses of quantitative algebras, and they point out the need of considering operations that are not nonexpansive and carriers that are *partial ultra-metrics*. As the latter is an example of **GMet** category, in the sense of Section 2.3, the research line of [DLHLP22] can be carried out within the framework presented in this work. Similarly, in [MSV22] the authors have investigated the Łukaszyk–Karmowski distance on *diffuse metric spaces* [HS00, CKPR21] of probability distributions. This is yet another type of **GMet** category that can be formalised within our framework. As a last example, in [GF23] the authors investigate "quantitative rewriting systems" and need to go beyond nonexpansive operations, by admitting (in what they call "graded rewriting systems") Lipschitz operations with constant $\alpha > 1$. As noted in Remark 9.2, it is possible in our theory to express, by means of quantitative equations, that operations are Lipschitz for any $\alpha > 1$.

A third direction for future work consists in exploring further generalisations of our framework. For example, our choice of considering fuzzy relations $d_A : A^2 \to [0, 1]$ has been made, somewhat arbitrarily, as a compromise between maximal generality and the convenience of dealing with a concrete notion of numeric distance. But it would be possible to work with distances $d_A : A^2 \to [0, \infty]$ (valued in the extended real line) as in [MPP16] or, even more generally, $d_A : A^2 \to Q$ where Q is an abstract quantale [PC96]. We expect that all our results can be easily adapted to such variants, but details needs to be carefully verified. In a similar direction, it could be interesting to follow the work of [FMS21] and move beyond "distances" and towards arbitrary relational structures.

Finally, a fourth direction is to use our deductive apparatus to reason quantitatively about program distances as, e.g., suggested in the preliminary examples given in [MSV21, §VI] in the context of process algebras. In particular, adapting the well-known framework of (equational) "up-to techniques" (see, e.g., [BPPR17]) to the quantitative setting (see, e.g., [BKP18]) appears to be a promising endeavour.

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