Mid–Term Exam

In solving the following exercises you can use the statements of the main theorems proved during the course such as: Büchi's theorem about the decidability of the MSO theory of $(\mathbb{N}, <)$ and Gödel's theorem about the undecidability of the FO theory of $(\mathbb{N}, +, \times)$. On the other hand you have to be very precise and clear about how you invoke these theorems to solve the exercises.

Exercise 1. Let $\mathbb{N}$ be the set of natural numbers. We have proved during the course, using the finite–automata method, that the First order (FO) theory of $(\mathbb{N}, \leq, +)$, known as *Presburger arithmetic*, is decidable.

Let $Q \subseteq \mathbb{N}$ be the predicate (i.e., unary relation) on natural numbers which holds on the powers of 2:

$$Q = \{1, 2, 4, 8, 16, 32, \ldots\} \quad \text{or, in other words,} \quad Q(n) \quad \Leftrightarrow \quad n = 2^m \text{ for some } m \in \mathbb{N}$$

To do: Is the FO–theory of $(\mathbb{N}, \leq, +, Q)$ decidable?

Exercise 2. For a given $k \in \mathbb{N}$, let $\times_k : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be multiplication on natural numbers defined only on numbers less than k. Formally,

$$n \times_k m = \begin{cases} n \times m & \text{if both } n < k \text{ and } m < k \\ 0 & \text{otherwise} \end{cases}$$

where $\times$ is the ordinary (full) multiplication on natural numbers.

To do: Is the FO–theory of $(\mathbb{N}, \leq, +, \{\times_k\}_{k \in \mathbb{N}})$ decidable?

Exercise 3. Is it possible to write a formula $\phi(x)$ of Presburger arithmetic such that:

$$\phi(n) \text{ holds} \quad \Leftrightarrow \quad n = 2^{2k} - 2^k, \text{ for some positive } k \in \mathbb{N}$$

In other words, $\phi(n)$ holds if n, when written in binary (using standard notation), is of the form $n = 1^m 0^m$.

Hint: to prove that it is possible, just write down the formula ϕ. To prove that it is impossible typically we use the *pumping lemma*.

Exercise 4. Let $\Sigma = \{a, b, c\}$. Let $L_1, L_2 \subseteq \Sigma^\omega$ be defined as:

- $L_1 = \{b, c\}^*.\{a\}.\{b, c\}^\omega = \big\{w \in \{a, b, c\}^\omega \mid w \text{ has exactly one position labeled by } a\big\}$, and
- $L_2 = \Sigma^*.\{a\}.\{a\}.\Sigma.\{b\}.\Sigma^\omega$

To do:

1. Write down a deterministic Büchi automaton that accepts L_1.
2. Is the language L_2 definable by a *deterministic* Büchi automaton?

Exercise 5. Let Σ be the alphabet $\{0,1,2\}$. Given an infinite word $w \in \Sigma^\omega$, we define $\infty(w) \subseteq \Sigma$ as the set of elements of the alphabet that occur *infinitely many times in* w. For example:

1. if $w = (12)^\omega = 1212121212\ldots$ then $\infty(w) = \{1,2\}$,
2. if $w = 0.(12)^\omega = 01212121212\ldots$ then $\infty(w) = \{1,2\}$,
3. if $w = 0^\omega = 0000000\ldots$ then $\infty(w) = \{0\}$,

Let $L \subseteq \Sigma^\omega$ be the set

$$\{w \mid \text{ the greatest number in } \infty(w) \text{ is even}\}$$

So for example:

- if $w = (012)^\omega$ then $w \in L$ because $\infty(w) = \{0,1,2\}$ and $\max\{0,1,2\} = 2$ is even.
- if $w = 2.(01)^\omega$ then $w \notin L$ because $\infty(w) = \{0,1\}$ and $\max\{0,1\} = 1$ is not even.

To do:

- Is L definable by a Büchi automaton?
- Is L definable by a deterministic Büchi automaton?