Welcome to CR12 !
Little history of Four-Color Theorem proofs

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by Pascal Ochem (LIRMM, Montpellier) and Michael Rao (me, LIP)

Recall :

- How a computer can help in mathematics or computer science ?
- Presentation of the history/proof of some computer aided theorems (TODAY : 4 color theorem)
- Presentation of classic tools for combinatorial exploration : reduction to SAT or LP, backtracking, transfer matrix methods...
- Concrete examples of different approaches to accelerate explorations : reduction of the search space, heuristics, speed up code...
- Lot of practise !
- Evaluation by homework and projects.
- Note : no "Formal proof assistants" like Coq here !

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Next week : Pascal Ochem will talk about backtracking (and others things)

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Disadvantages : know how to program, sometimes it's more difficult to find "errors" the the proofs, and the "power" of the computer is not infinite...

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The Four-Color Theorem

Every map can be coloured with (at most) 4 colors, in such a way that no two adjacent regions have the same color.

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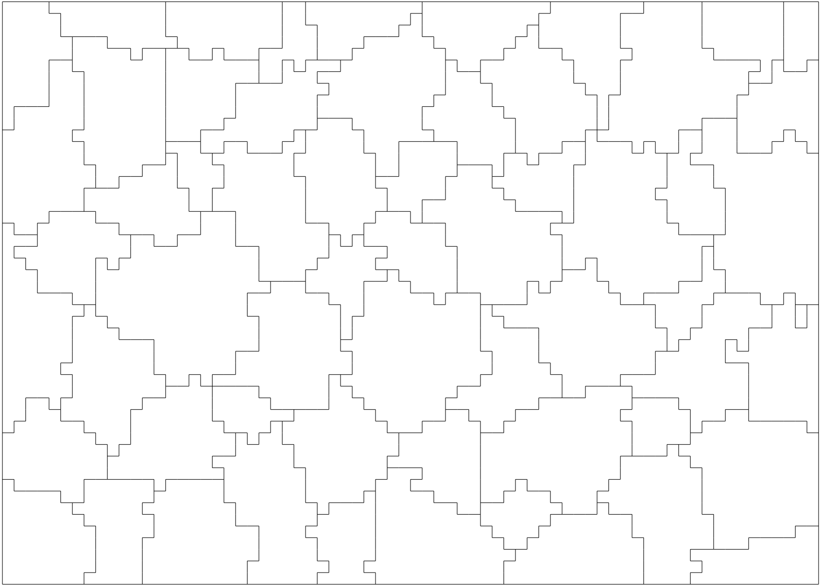

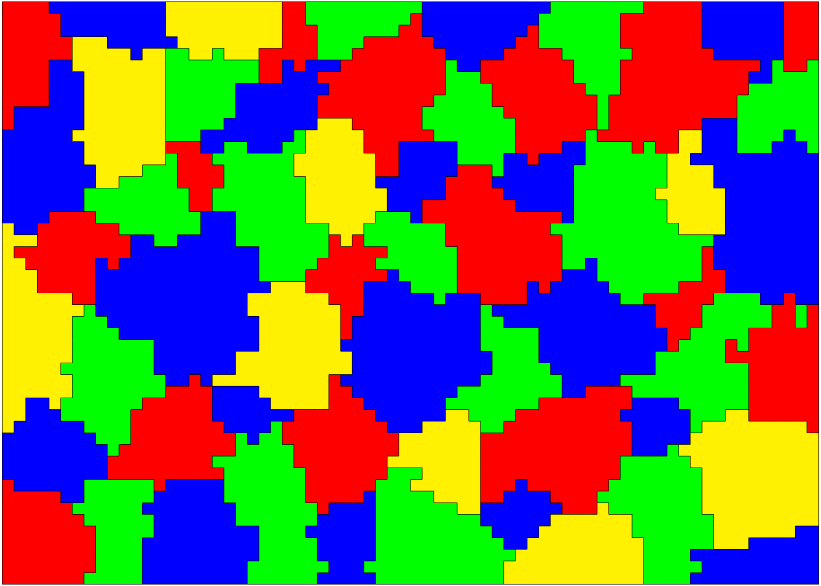

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In "graph theory" :

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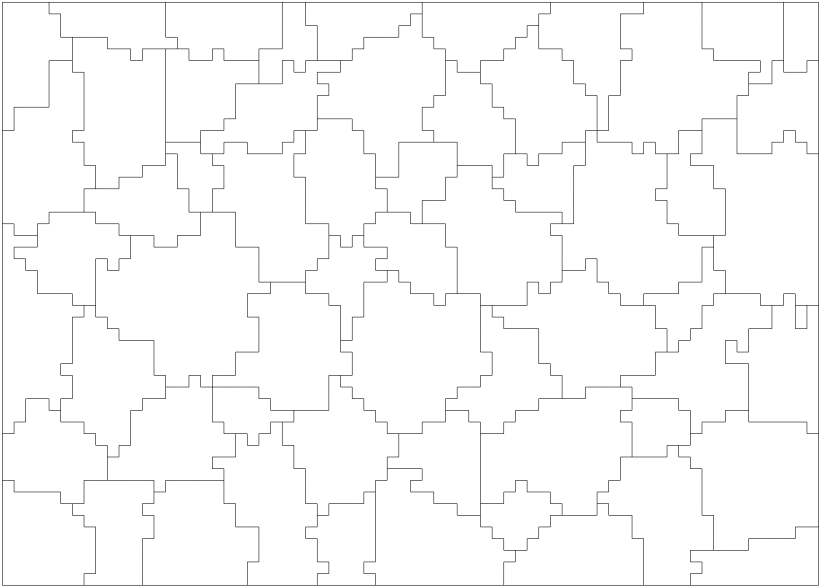

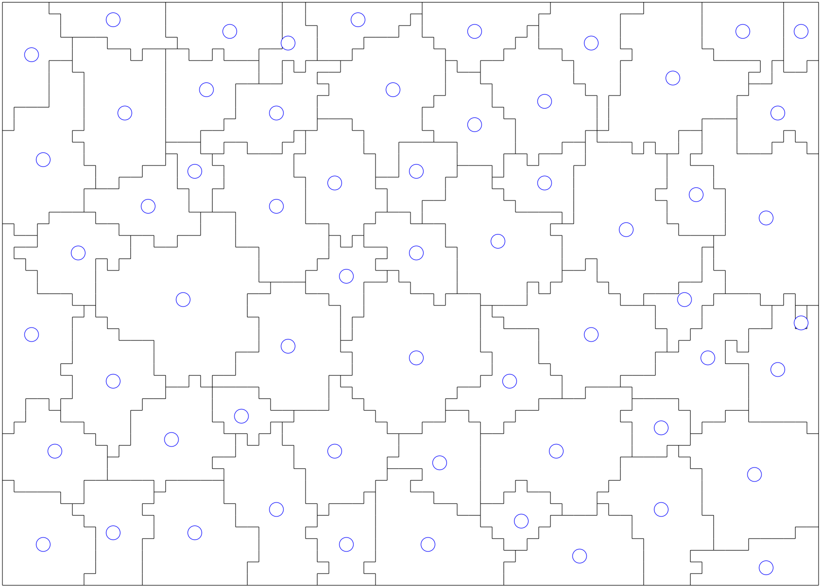

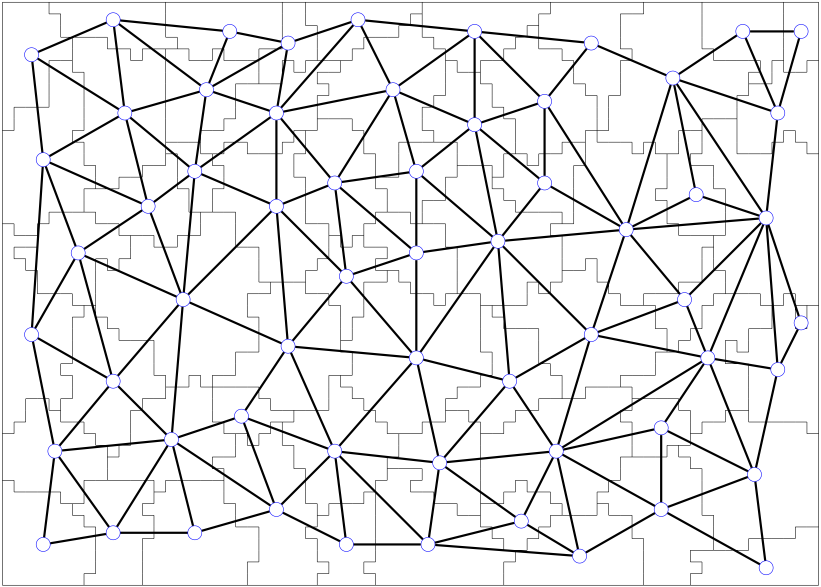

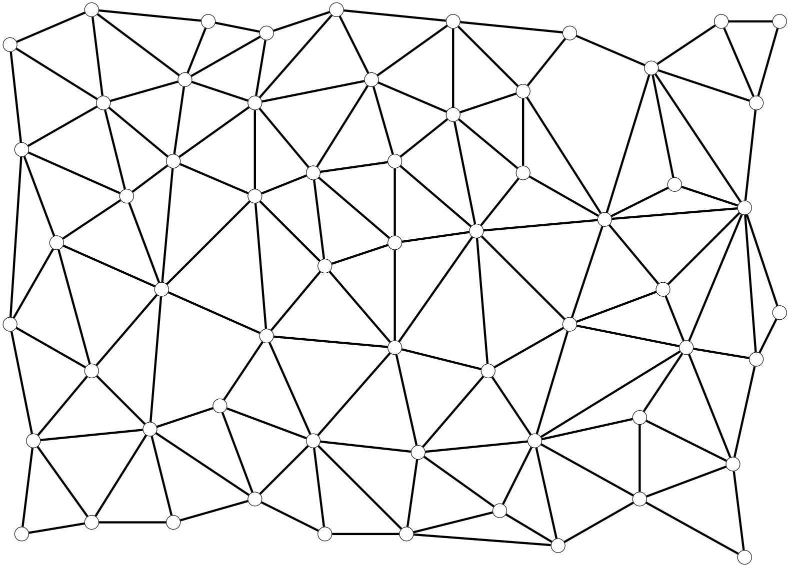

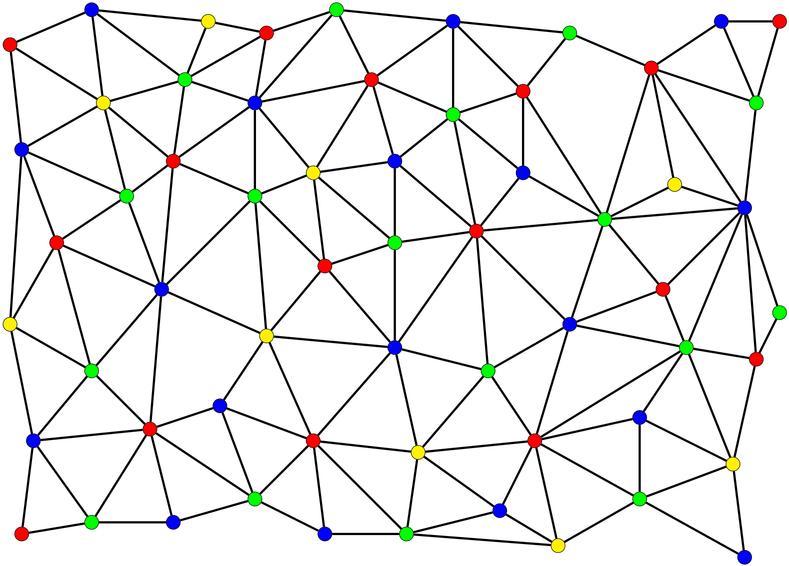

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Contradiction !

Euler formula and 6 colors

Every planar graph is 6-colorable :

Recursive algorithm to color with 6 colors

- Let v be a vertex of G with degree at most 5
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One can call it the "6-Color Theorem".

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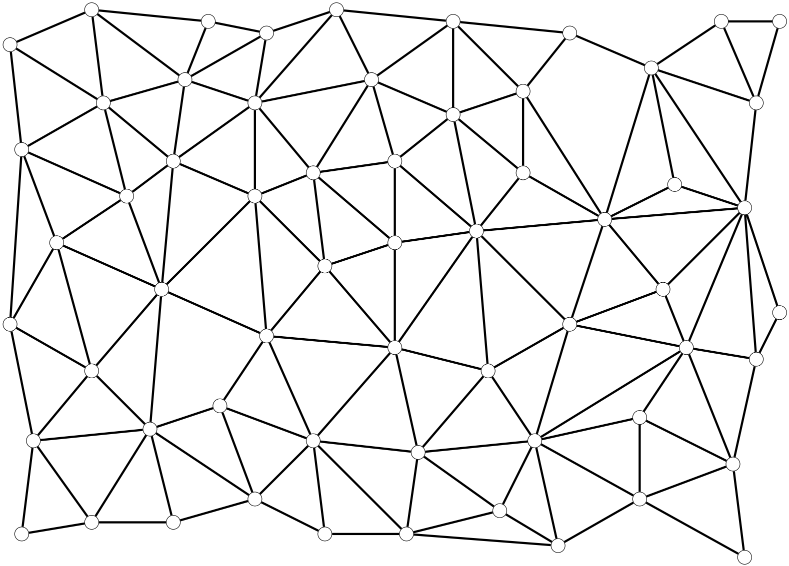

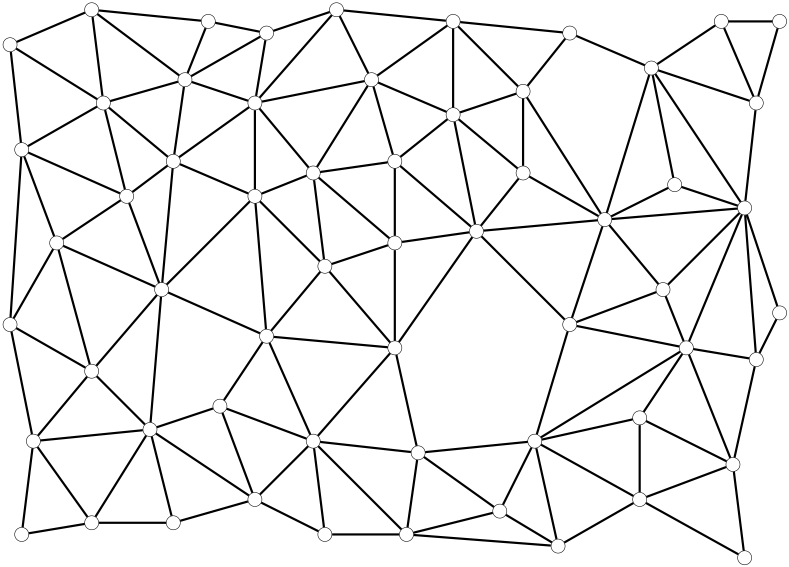

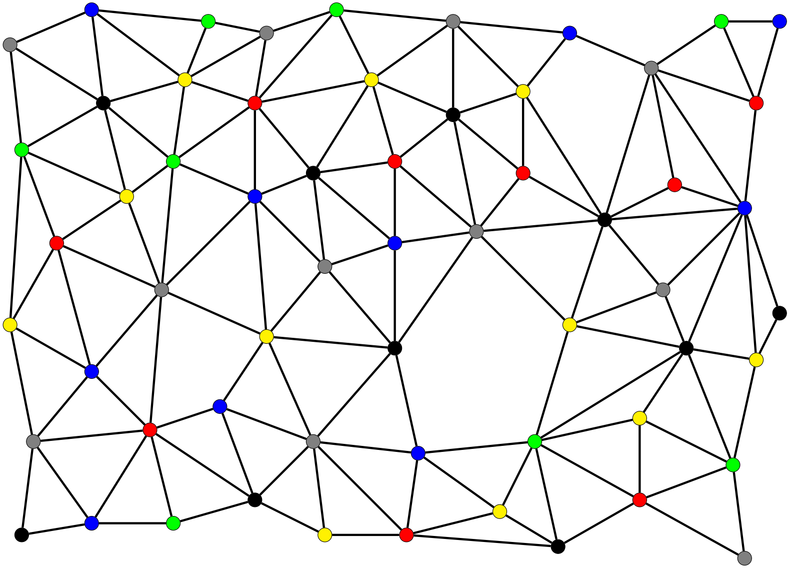

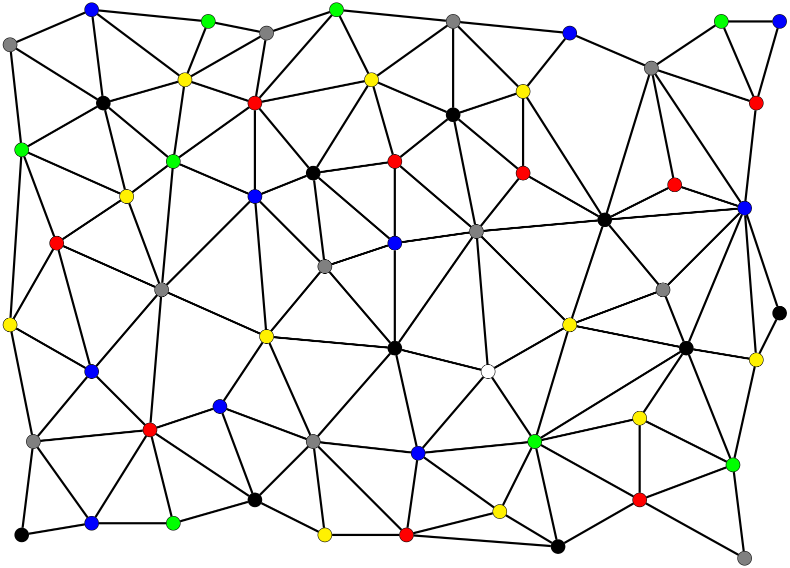

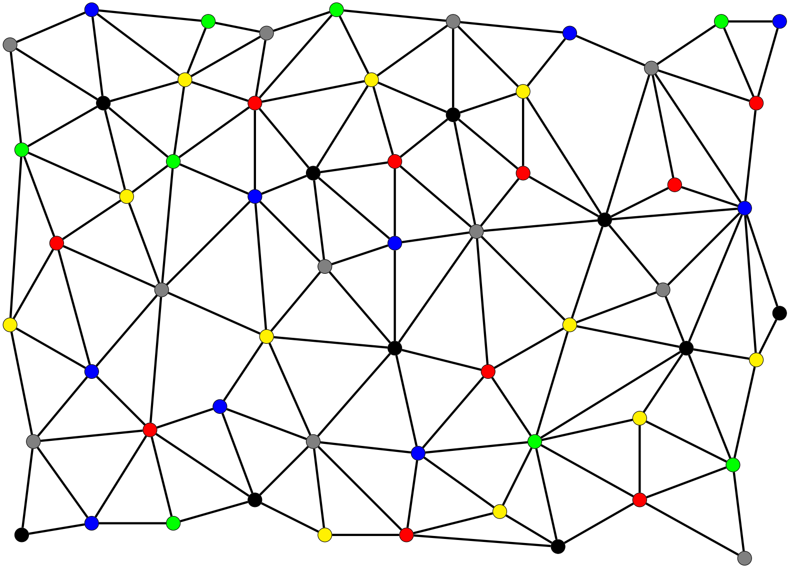

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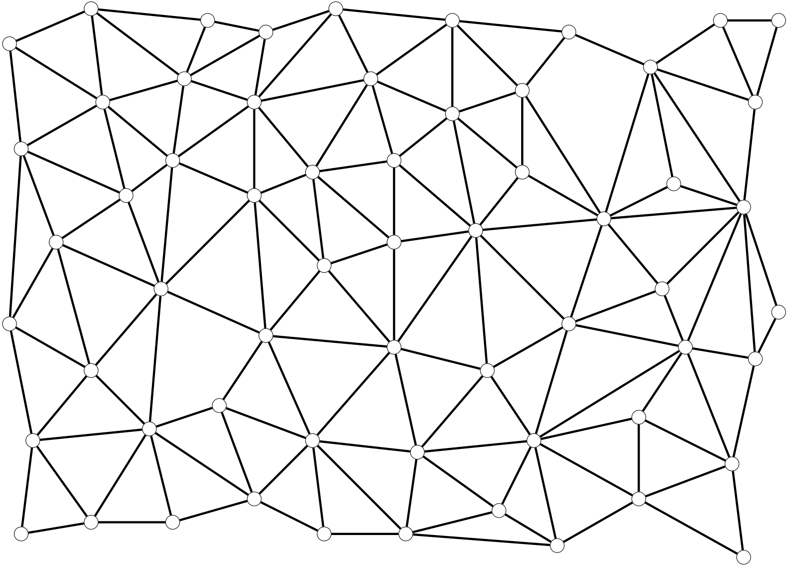

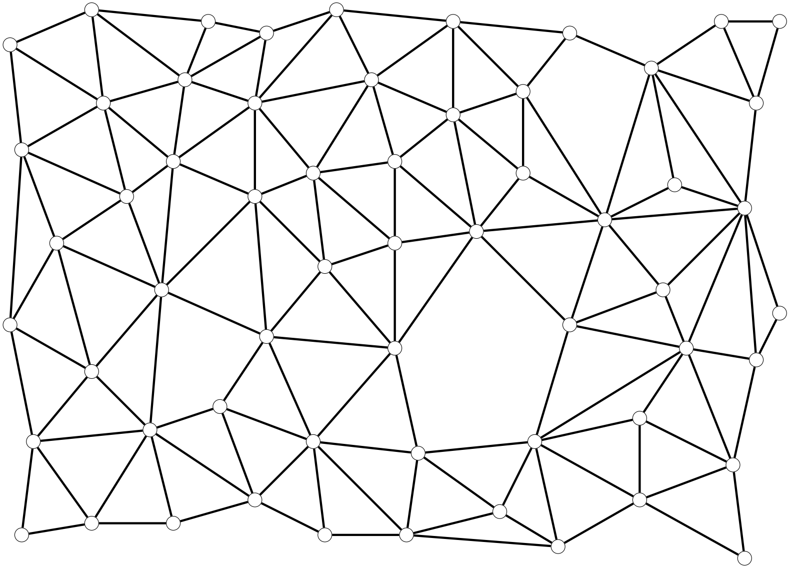

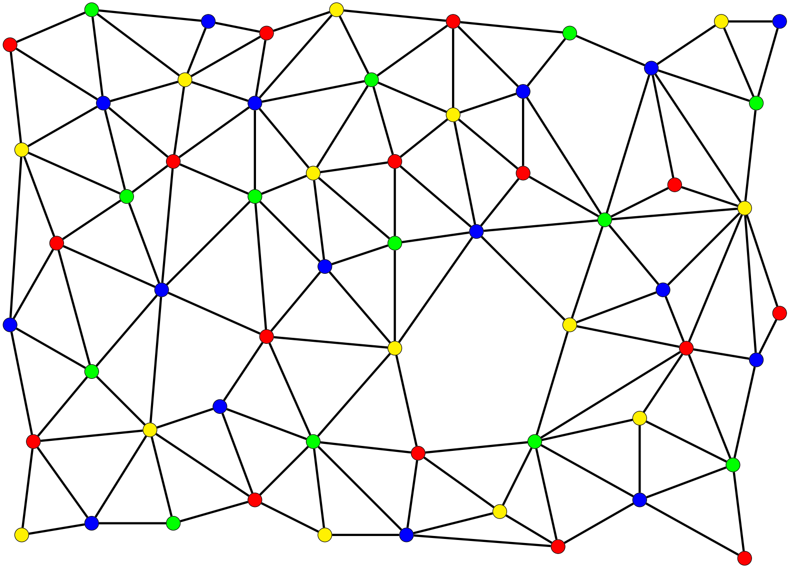

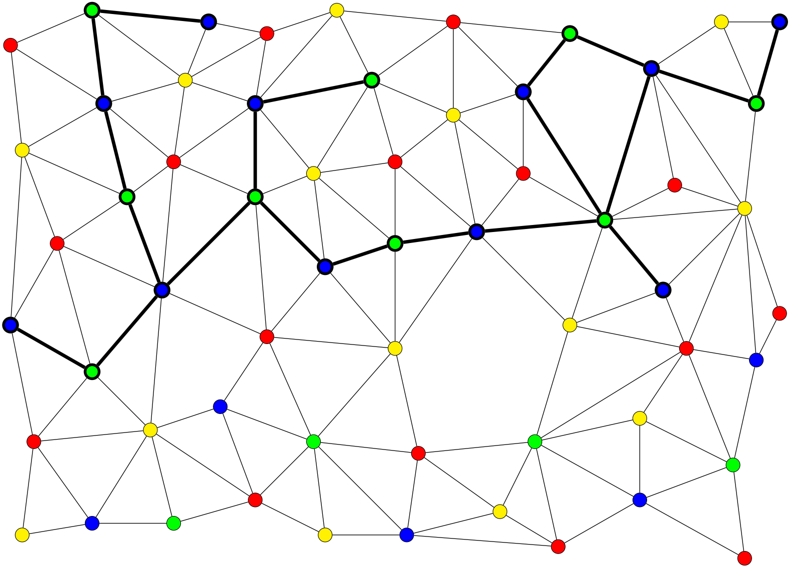

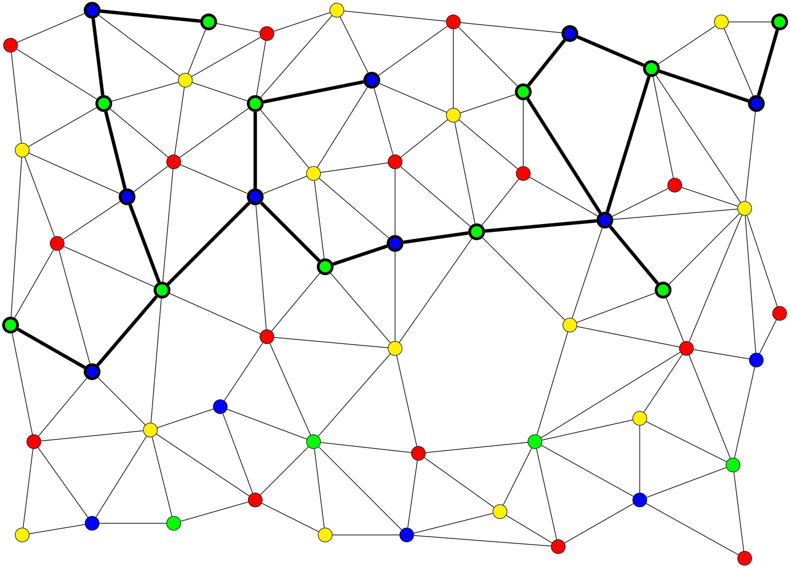

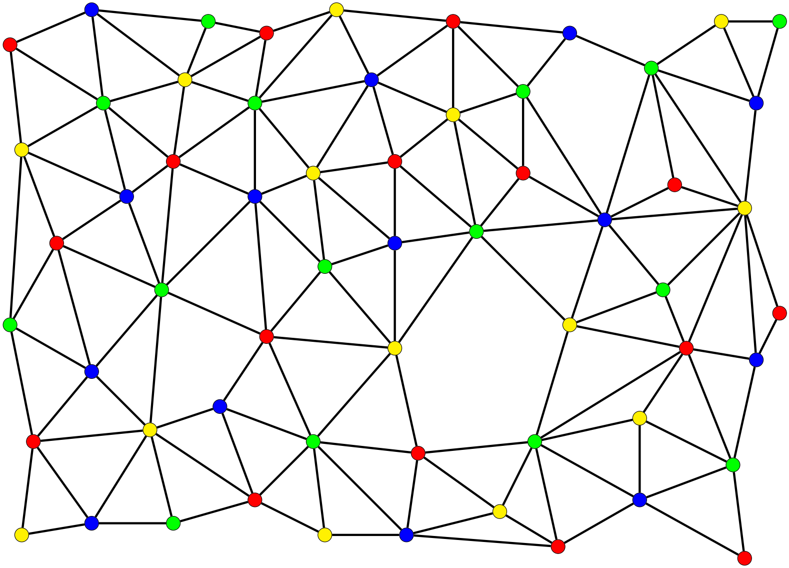

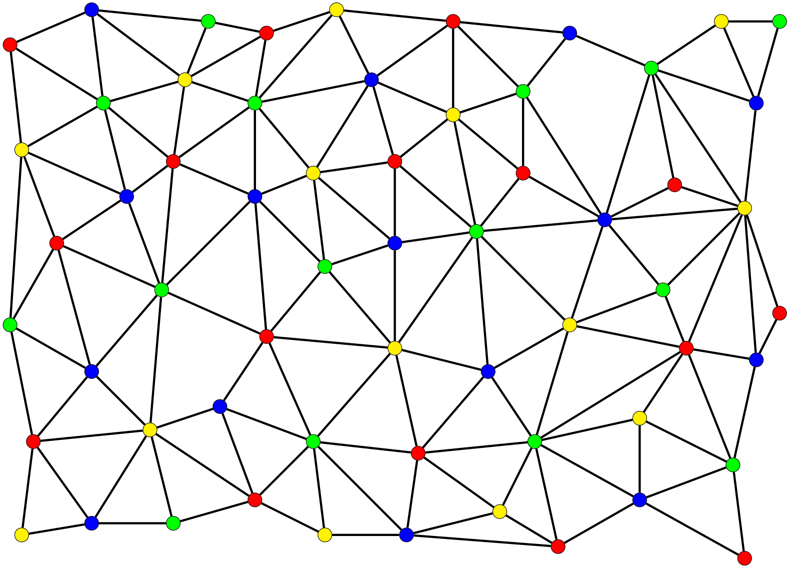

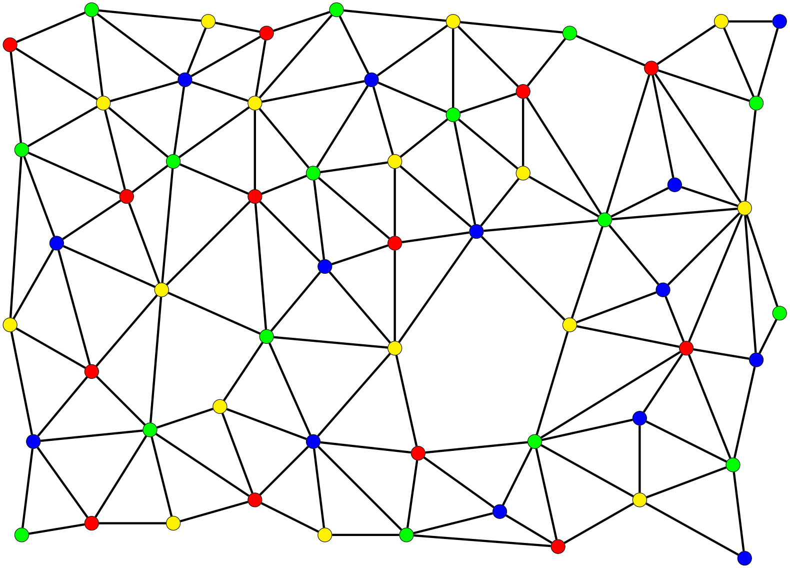

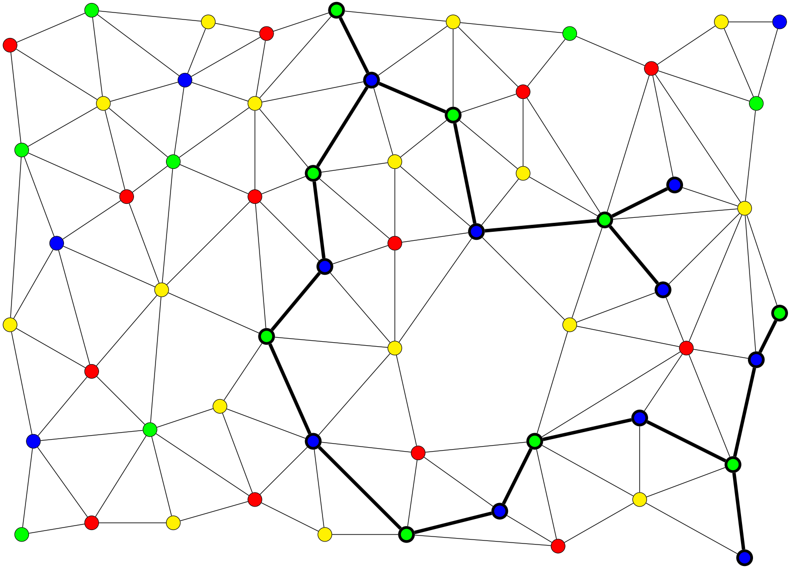

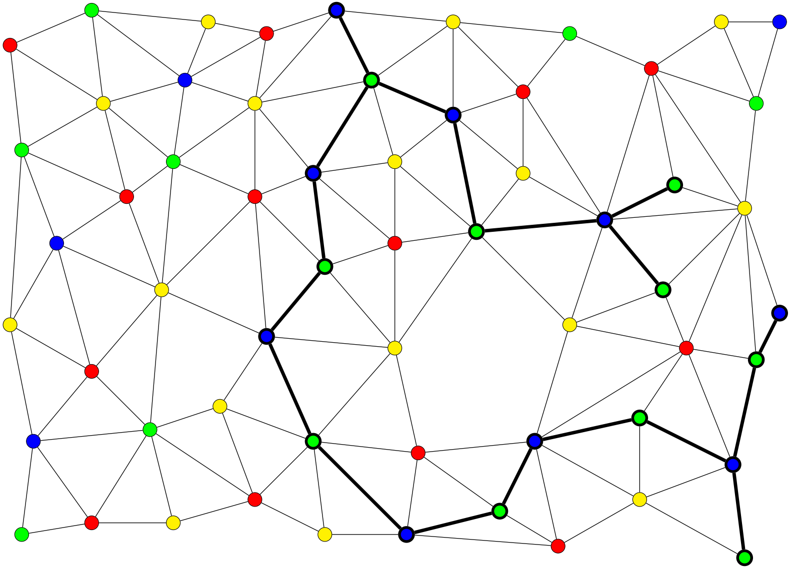

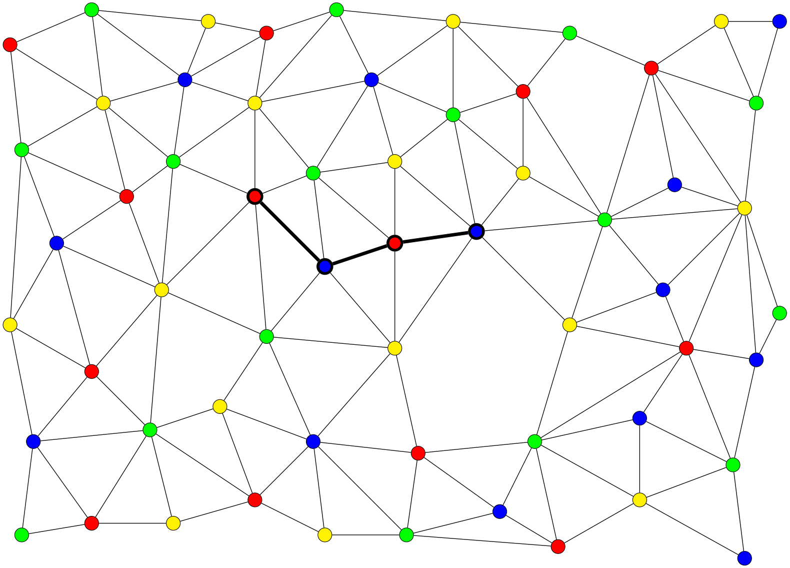

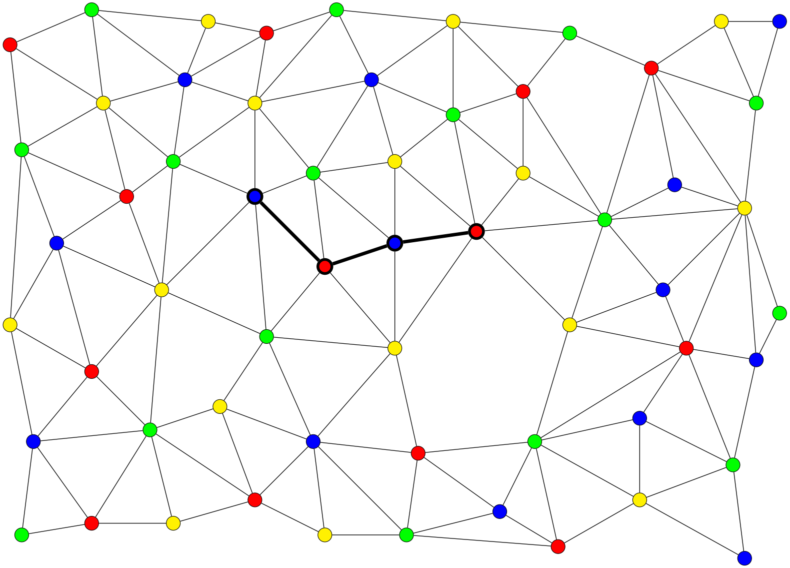

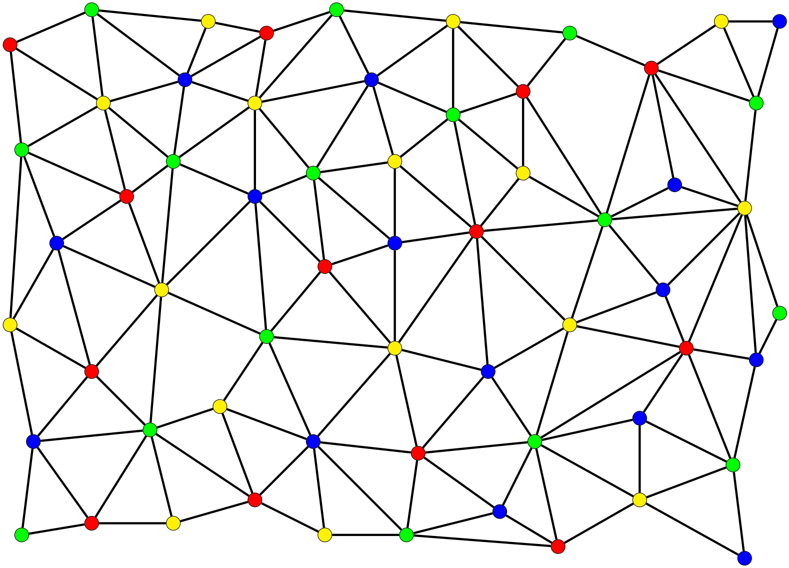

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- Verification of the proof by Gonthier in 2005, using a "Formal proof assistant" (Coq)

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But there are (much) more cases to manage...

Idea for a proof (which is also an algorithm)

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Thus : a reducible configuration cannot appear in a minimal counter example !

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(this tool is often used in planar graph theory)

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4. We get a contradiction !
   Such a graph G cannot exists !

Theorem

A planar graph with minimum degree 5 has either an edge 5-5 or an edge 5-6

Give the weight $d(v) - 6$ to each vertex v and and the weight $2d(f) - 6$ to each face f.

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Give the weight $d(v) - 6$ to each vertex v and and the weight $2d(f) - 6$ to each face f.

The overall sum is -12.

Now move $1/5$ from every neighboor of a 5 degree vertex v to v.

If we suppose that there is no $5 - 5$ nor $5 - 6$, every weight becomes positive.
Contratiction !

Proving the 4CT

Two "big" computational parts in this approach :

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The two parts are slightly different.

Approach to show 4CT :

1. Find a big enough set $\mathcal{C}$, with all "reducibility tools" we have
2. Try to show, using the discharging method, that $\mathcal{C}$ is unavoidable
   if we fail, go back in (1)

Estimation of Heesch in 1955

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$\Rightarrow$ the usage of a computer seems unavoidable

Computers are not powerful enough. A "race" begins...

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But, now well accepted...

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But some new problems...

- The second part is done by a computer
- The program is programmed in assembly, on a IBM 370/168
- How can be sure that there is no bug in the computer program ?
- Few people know really what is a computer, how it works, have access to it...
- Almost nobody known how to program a computer
- How can be sure that the computer has no hardware bug ? or no "computation" error ? (computers are not very reliable at this time)

```
HELLO    CSECT               The name of this program is 'HELLO'
*                            Register 15 points here on entry from OPSYS or caller.
         STM   14,12,12(13)  Save registers 14,15, and 0 thru 12 in caller's Save area
         LR    12,15         Set up base register with program's entry point address
         USING HELLO,12      Tell assembler which register we are using for pgm. base
         LA    15,SAVE       Now Point at our own save area
         ST    15,8(13)      Set forward chain
         ST    13,4(15)      Set back chain
         LR    13,15         Set R13 to address of new save area
*                            -end of housekeeping (similar for most programs) -
         WTO   'Hello World' Write To Operator  (Operating System macro)
*
         L     13,4(13)      restore address to caller-provided save area
         XC    8(4,13),8(13) Clear forward chain
         LM    14,12,12(13)  Restore registers as on entry
         DROP  12            The opposite of 'USING'
         SR    15,15         Set register 15 to 0 so that the return code (R15) is Zero
         BR    14            Return to caller
*
SAVE     DS    18F           Define 18 fullwords to save calling program registers
         END   HELLO         This is the end of the program
```

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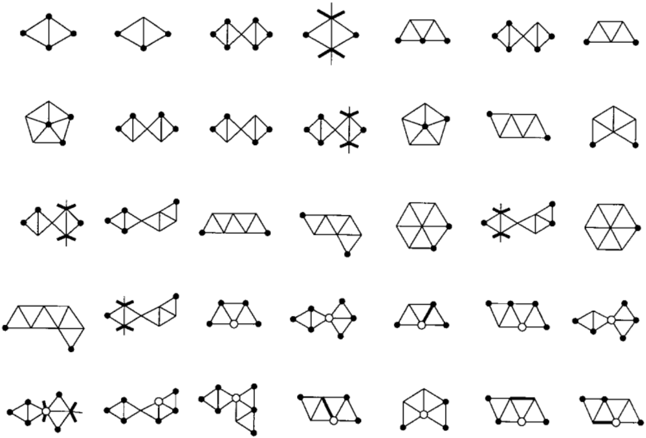
- there is also a companion technical paper to explain the computer program.
- water has flowed under the bridge....

Everything is done to avoid doubts, and indeed, the proof is better accepted

In the proof of Robertson et al. 1997

- 633 reducibles configurations (each one checked by computer)
- 32 discharging rules (found by hand)
- The discharging "check" is done by computer

APPENDIX: THE UNAVOIDABLE SET OF REDUCIBLE CONFIGURATIONS



(and so on...)

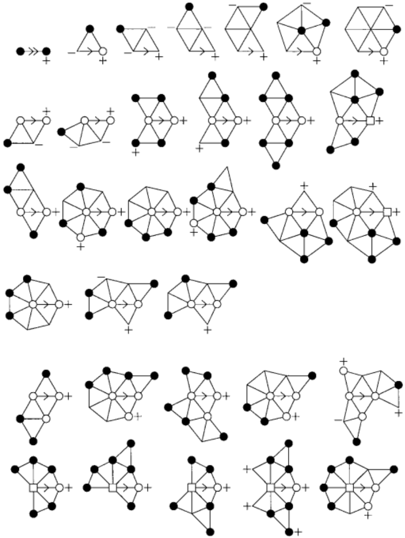


Fig. 4. The rules.

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(But, it's not the subject of this course...)