Clique-width of graphs defined by one-vertex extensions [1]

Michaël Rao

LIRMM - CNRS, Université Montpellier 2,
161 rue Ada,
34392 Montpellier Cedex 5 - France

Abstract

Let G be a graph and u be a vertex of G. We consider the following operation: add a new vertex v such that v does not distinguish any two vertices which are not distinguished by u. We call this operation a *one-vertex extension*. Adding a true twin, a false twin or a pendant vertex are cases of one-vertex extensions. We are interested here in graph classes defined by a subset of allowed one-vertex extension. Examples are trees, cographs and distance-hereditary graphs. We give a complete classification of theses classes with respect to their clique-width. We also introduce a new graph parameter called the *modular-width*, and we give a relation with the clique-width.

Key words: Graph classes, vertex extension, cographs, distance-hereditary graphs, clique-width.

1 Introduction

Many classes of graphs can be defined by one or more vertex extensions. One can cite the class of trees (the vertex extension is to add a pendant vertex), the cographs (add a true or a false twin) and the chordal graphs (add a simplicial vertex). We are interested here by *one-vertex extensions*, *i.e.* for a vertex u, add a new vertex v which does not distinguish any two vertices which are not distinguished by u. Various classes of graphs can be defined by a set of allowed such extensions, for example trees, cographs and distance-hereditary graphs.

Email address: `Michael.Rao@lirmm.fr` (Michaël Rao).

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Clique-width is a graph parameter introduced by Courcelle *et al.* [4]. It generalize the well-known treewidth introduced by Robertson and Seymour [15]. A motivation on researches about treewidth and clique-width is that a lot of NP-complete problems are polynomial time solvable on graphs of bounded width. NLC-width is a parameter introduced by Wanke [17] and close to the clique-width. Recently Oum and Seymour have introduced the rank-width [14], and showed that a graph class has bounded rank-width if and only if it has bounded clique-width.

We give here a complete classification of classes of graphs defined by a set of allowed one-vertex extensions, with respect to their clique-width. For every class $\mathcal{G}$, we give an upper bound of the clique-width of graphs in $\mathcal{G}$, or we show that for every $k > 0$, there is a graph in $\mathcal{G}$ of clique-width at least k. In particular we show that graph class defined by the two operations "add a pendant vertex" and "add a dominating vertex" has unbounded clique-width. This class of graphs is completely decomposable by the *c-decomposition* introduced by J.M. Lanlignel in his doctoral thesis [11]. The c-decomposition consists in successively decomposing a graph by the modular decomposition and the split decomposition, until every graph is indecomposable for the modular decomposition and the split decomposition. Thus the class of graphs completely decomposable by the c-decomposition has unbounded clique-width. This is notable, since for a large part of known decompositions, the class on completely decomposable graphs has bounded clique-width.

We also introduce in this paper a new graph parameter called the *modular-width*, which can be viewed as a generalisation of the modular decomposition. We show that a graph class has bounded clique-width if and only if it has bounded modular-width. We use this parameter to compute upper bounds for the clique-width of some graph classes, or to show that a graph class has unbounded clique-width.

In section 2, we define one-vertex extensions, and we recall some known results. In section 3 we recall definitions of the clique-width and the NLC-width. In section 4 we introduce the modular-width and give some relations with clique-width and NLC-width. The complete classification of graph classes is given in section 5.

2 Definitions and known results

Every graph is supposed to be finite, undirected, without loops nor multiple edges. A graph G is a pair (V, E) where V is the set of *vertices* and $E \subseteq \{\{u, v\} : (u, v) \in V^2 \text{ and } u \neq v\}$ the set of *edges*. For a graph G, $V(G)$ denotes the set of its vertices and $E(G)$ the set of its edges. Let v be a vertex.

The *neighborhood* of v, denoted by $N_G(v)$, is the set $\{u \in V : \{u,v\} \in E\}$, and the *closed neighborhood*, denoted by $N_G[v]$, is $N(v) \cup \{v\}$. When the graph G is clear in the context, we note V, E, $N(v)$ and $N[v]$ instead of $V(G)$, $E(G)$, $N_G(v)$ and $N_G[v]$.

A vertex u is *adjacent* to a vertex v if $\{u,v\} \in E$. A vertex u is adjacent to $W \subseteq V$ if u is adjacent to every vertex in W, and u is *non-adjacent* to W if u is non adjacent to every vertex in W. A vertex u *distinguishes* v and w (where $v, w \in V \setminus \{u\}$ and $v \neq w$) if either $\{u,v\} \in E$ and $\{u,w\} \notin E$, or $\{u,v\} \notin E$ and $\{u,w\} \in E$. A vertex u distinguishes two sets A and B, with $A \cap B = \emptyset$, if there is a $v \in A$ and a $w \in B$ such that u distinguishes v and w.

Let $W \subseteq V$. The graph *induced by* W, denoted $G[W]$, is the graph of vertex set W and edge set $\{\{u,v\} \in E : (u,v) \in W^2\}$. For $W \subseteq V$, $G - W$ denotes $G[V \setminus W]$, and for $v \in V$, $G - v$ denotes $G[V \setminus \{v\}]$.

A graph class $\mathcal{G}$ is *hereditary* if for $G \in \mathcal{G}$ and $W \subseteq V$, $G[W] \in \mathcal{G}$. The *hereditary closure* of a graph class $\mathcal{G}$ is $\{G[V'] : G \in \mathcal{G} \text{ and } V' \subseteq V(G)\}$ (*i.e.* the class of all induced sub-graphs of graphs in $\mathcal{G}$). A graph G is *H-free* (where H is a graph) if for every $W \subseteq V$, $G[W]$ is not isomorphic to H. A graph G is *$\mathcal{H}$-free* (where $\mathcal{H}$ is a set of graphs) if for every $H \in \mathcal{H}$, G is H-free.

Figure 1 presents some graphs used in this paper. A *hole* is a chordless cycle with at least 5 vertices.

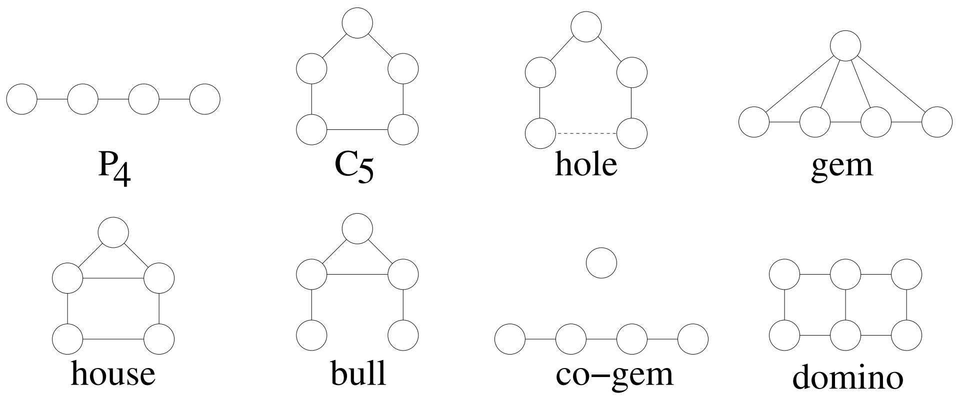

Fig. 1. Some graphs.

One-vertex extensions. Let $G = (V,E)$ be a graph and $u \in V$. We want to add to G a new vertex v such that v does not distinguish any two vertices which are not distinguished by u. That is, v is either adjacent or non-adjacent to $N(v)$, and v is either adjacent or non-adjacent to $V \setminus N[v]$. We have eight possibilities of extensions: for $o \in \{\mathrm{TT}, \mathrm{FT}, \mathrm{PV}, \mathrm{TA}, \mathrm{FA}, \mathrm{CP}, \mathrm{IV}, \mathrm{DV}\}$, we define by $G' = \mathrm{ext}_o(G,u)$ the graph with vertex set $V \cup \{v\}$ such that $G'[V] = G$ and

- $N_{G'}(v) = \emptyset$ if $o = \mathrm{IV}$ (Independent Vertex)
- $N_{G'}(v) = V$ if $o = \mathrm{DV}$ (Dominating Vertex)
- $N_{G'}(v) = N_G[u]$ if $o = \mathrm{TT}$ (True Twin)
- $N_{G'}(v) = N_G(u)$ if $o = \mathrm{FT}$ (False Twin)
- $N_{G'}(v) = V \setminus N_G(u)$ if $o = \mathrm{TA}$ (True Anti-twin)
- $N_{G'}(v) = V \setminus N_G[u]$ if $o = \mathrm{FA}$ (False Anti-twin)
- $N_{G'}(v) = u$ if $o = \mathrm{PV}$ (Pendant Vertex)
- $N_{G'}(v) = V \setminus \{u\}$ if $o = \mathrm{CP}$ (Complement of pendant vertex)

The TT and FT extensions correspond respectively to add a true twin or a false twin to u. The PV extension corresponds to add a pendant vertex. The extension CP corresponds to add a pendant vertex in the complement. TA and FA add a true or false anti-twin to u. Finally, IV and DV are two degenerate cases: add an independent vertex and add a dominating vertex.

For $\mathcal{O} \subseteq \{\mathrm{TT}, \mathrm{FT}, \mathrm{PV}, \mathrm{TA}, \mathrm{FA}, \mathrm{CP}, \mathrm{IV}, \mathrm{DV}\}$, let $\mathcal{G}_{\mathcal{O}}$ be the class of graph such that:

- K_1 (the graph with one vertex) is in $\mathcal{G}_{\mathcal{O}}$ and
- for all $G \in \mathcal{G}_{\mathcal{O}}$, $u \in V(G)$ and $o \in \mathcal{O}$, then $\mathrm{ext}_o(G, u)$ is in $\mathcal{G}_{\mathcal{O}}$.

Note that for an arbitrary $\mathcal{O} \subseteq \{\mathrm{TT}, \mathrm{FT}, \mathrm{PV}, \mathrm{TA}, \mathrm{FA}, \mathrm{CP}, \mathrm{IV}, \mathrm{DV}\}$, $\mathcal{G}_{\mathcal{O}}$ is not necessary a hereditary class (for example, the class of trees $\mathcal{G}_{\{\mathrm{PV}\}}$). We denote by $\mathcal{G}'_{\mathcal{O}}$ the hereditary closure of $\mathcal{G}_{\mathcal{O}}$.

Modular decomposition and cographs. A *module* in a graph $G = (V, E)$ is a non empty subset $M \subseteq V$ such that for every $v \in V \setminus M$, either $N(v) \cap M = \emptyset$ or $M \subseteq N(v)$. A *modular decomposition tree* for a graph $G = (V, E)$ is a rooted tree T such that the set of leaves of T is V, and for every node α in T, the set V_α of leaves of the sub-tree rooted at α is a module of G. Note that this is not the usual way to define the modular decomposition tree, and one can define a unique canonical modular decomposition tree [6] which verifies our definition. Nevertheless, the uniqueness is not needed here. A graph G is a *cograph* (or equivalently G is completely decomposable by modular decomposition) if and only of it has a binary modular decomposition tree.

Theorem 1 ([3,16]) *The following conditions are equivalent:*

- *G is a cograph (*i.e. *is completely decomposable by the modular decomposition),*
- *G is P_4-free,*
- *G can be obtained by a sequence of extensions: add a true twin and add a false twin (*i.e. *$G \in \mathcal{G}_{\{\mathrm{TT}, \mathrm{FT}\}}$).*

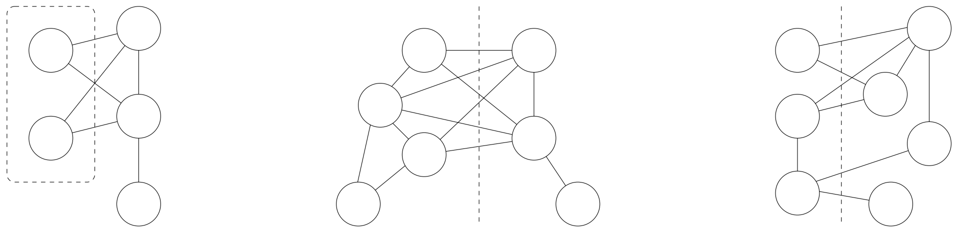

Fig. 2. A module, a split and a bi-join.

Split decomposition and distance-hereditary graphs. A *split* in a graph G is a bi-partition $\{X, Y\}$ of V such that the vertices in X with a neighbor in Y have the same neighborhood in Y (that is for every $u, v \in X$ such that $N(u) \cap Y \neq \emptyset$ and $N(v) \cap Y \neq \emptyset$, then $N(u) \cap Y = N(v) \cap Y$). Let T be an unrooted tree. Leaves(T) denotes the set of leaves of T, and for an edge e in T, Part(T, e) denotes the partition $\{\text{Leaves}(T) \cap V_1, \text{Leaves}(T) \cap V_2\}$, where V_1 and V_2 are the connected components of T without the edge e. A *split decomposition tree* of a graph is an unrooted tree T such that the set of leaves in T is V, and for each edge e in T, Part(T, e) is a split in the graph. Splits and split decomposition are generalizations of modules and modular decomposition. There is also a way to define a unique split decomposition tree [5]. A graph is completely decomposable by the split decomposition if there is a split decomposition tree in which every internal node has degree 3.

A graph is *distance-hereditary* if every chordless path between two vertices has the same length.

Theorem 2 ([1,9]) *The following conditions are equivalent:*

- *G is distance-hereditary*
- *G is completely decomposable by the split decomposition*
- *G is (house, hole, domino, gem)-free*
- *G can be obtained by a sequence of extension: add a true twin, add a false twin and add a pendant vertex (*i.e. $G \in \mathcal{G}_{\{\text{TT},\text{FT},\text{PV}\}}$*).*

Bi-join decomposition and (C_5,bull,gem,co-gem)-free graphs. The bi-join decomposition is an other generalization of the modular decomposition [12,13]. A *bi-join* in a graph is a bi-partition $\{X, Y\}$ such that for every $v, v' \in X$, $\{N(v) \cap Y, Y \setminus N(v)\} = \{N(v') \cap Y, Y \setminus N(v')\}$. Equivalently there is a partition $\{X_1, X_2\}$ of X and $\{Y_1, Y_2\}$ of Y such that the edges between X and Y are exactly the edges between X_1 and Y_1 and the edges between X_2 and Y_2, and there is no others edges between X and Y. A *bi-join decomposition tree* of a graph is an unrooted tree T such that the leaves of T is V, and for each edge e in T, Part(T, e) is a bi-join in the graph. There is a way to define a unique bi-join decomposition tree [12,13]. A graph is completely decomposable by the bi-join decomposition if there is a bi-join decomposition tree in which every internal node has degree 3. Theorem 3 gives several characterisations of the graphs completely decomposable by the bi-join decomposition.

Theorem 3 ([12,13]) *The following conditions are equivalent:*

- *G is completely decomposable by the bi-join decomposition*
- *G is (C_5, bull, gem, co-gem)-free*
- *G can be obtained by a sequence of extensions: add a true twin, add a false twin, add a true anti-twin and add a false anti-twin (*i.e. $G \in \mathcal{G}_{\{\mathrm{TT},\mathrm{FT},\mathrm{TA},\mathrm{FA}\}}$*).*

3 Clique-width and NLC-width

A *labeled graph* is graph in which every vertex has a label into $\{1, 2, \ldots\}$. It is denoted by (V, E, l), where V is the set of vertex, E the set of edges, and $l : V \to \{1, 2, \ldots\}$ the labeling function. If G is a labeled graph, $l(G)$ denotes the labeling function of G. A labeled graph is a *k-labeled graph* (where $k \in \{1, 2, \ldots\}$) if every label is of $\{1, 2, \ldots, k\}$.

Let us define the following operations:

(1) For $i \in \{1, 2, \ldots\}$, $(v)_i$ is the graph with one vertex v labeled i.

(2) Let $G = (V, E, l)$ be a labeled graph, and let $i, j \in \{1, 2, \ldots\}$, $i \neq j$. Then $\rho_{i \to j}(G)$ is the labeled graph (V, E, l') where for all $v \in V$:

$$l'(v) = \begin{cases} l(v) & \text{if } l(v) \neq i, \\ j & \text{if } l(v) = i. \end{cases}$$

(3) Let $G = (V, E, l)$ be a labeled graph, and $i, j \in \{1, 2, \ldots\}$, $i \neq j$. Then $\eta_{i,j}(G)$ is the labeled graph (V, E', l) where

$$E' = E \cup \{\{u, v\} : (u, v) \in V^2, l(u) = i \text{ and } l(v) = j\}.$$

(4) Let $G_1 = (V_1, E_1, l_1)$ and $G_2 = (V_2, E_2, l_2)$ be two disjoint labeled graphs (*i.e.* $V_1 \cap V_2 = \emptyset$). Then $G_2 \oplus G_2$ is the disjoint union of G_1 and G_2, *i.e.* the labeled graph $(V_1 \cup V_2, E_1 \cup E_2, l)$ where for all $v \in V$

$$l(v) = \begin{cases} l_1(v) & \text{if } v \in V_1, \\ l_2(v) & \text{if } v \in V_2. \end{cases}$$

A *k-expression* is an expression built with $(v)_i$, $\rho_{i \to j}$, $\eta_{i,j}$ and $\oplus$, with $i, j \in \{1, \ldots, k\}$, and $\mathrm{val}(t)$ is the labeled graph defined by t. The *clique-width* of a labeled graph G is the smallest integer k such that there is a k-expression t with $G = \mathrm{val}(t)$. It is denoted by $\mathrm{cwd}(G)$. The clique-width of an unlabeled graph G is the minimum of the clique-width over all labeling of vertices of G, and a k-expression for G is a k-expression for a labeling of G.

The clique-width of a graph class $\mathcal{G}$ is $\max\{\mathrm{cwd}(G) : G \in \mathcal{G}\}$. We say that the clique-width is unbounded if for every $k > 0$, there is a graph $G \in \mathcal{G}$ such that $\mathrm{cwd}(G) \geq k$. Note that if $\mathcal{G}'$ is the hereditary closure of $\mathcal{G}$, then the clique-width of $\mathcal{G}'$ is unbounded if and only if the clique-width of $\mathcal{G}$ is unbounded, and otherwise the clique-width of $\mathcal{G}'$ is exactly the clique-width of $\mathcal{G}$.

It is not hard to see that graphs of clique-width 2 are exactly the cographs. Moreover we have:

Theorem 4 ([7]) *If G is distance-hereditary then $\mathrm{cwd}(G) \leq 3$, and a 3-expression can be constructed in linear time.*

The *NLC-width*, introduced by Wanke [17], is very close to clique-width. NLC-width of a labeled graph G, denoted $\mathrm{nlc}(G)$, is the smallest k such that there is an expression using the following operations, with labels into $\{1, \ldots, k\}$:

(1) For $i \in \{1, \ldots, k\}$, $(v)_i$ is the graph with one vertex v labeled i.
(2) Let $G = (V, E, l)$ be a k-labeled graph. For $R : \{1, \ldots, k\} \to \{1, \ldots, k\}$, $\rho_R(G)$ is the k labeled graph (V, E, l') where $l'(v) = R(l(v))$ for all $v \in V$.
(3) Let $G_1 = (V_1, E_1, l_1)$ and $G_2 = (V_2, E_2, l_2)$ be two disjoint k-labeled graphs, and $S \subseteq \{1, \ldots, k\}^2$. Then $G_2 \times_S G_2$ is the k-labeled graph $(V_1 \cup V_2, E', l)$ where

$$E' = E_1 \cup E_2 \cup \{\{u, v\} : (u, v) \in V_1 \times V_2 : (l_1(u), l_2(v)) \in S\}$$

and for all $v \in V_1 \cup V_2$,

$$l'(v) = \begin{cases} l_1(v) \text{ if } v \in V_1, \\ l_2(v) \text{ if } v \in V_2. \end{cases}$$

The NLC-width of an unlabeled graph is the minimum of the NLC-width over all labelings of G. The following Theorem gives a strong relation between clique-width and NLC-width. In particular, a graph class has bounded clique-width if and only if it has bounded NLC-width.

Theorem 5 ([10]) *For every graph G, $\mathrm{nlc}(G) \leq \mathrm{cwd}(G) \leq 2 \times \mathrm{nlc}(G)$.*

4 Modular-width

Let k be a positive integer. A *k-module* of a graph $G = (V, E)$ is a subset $M \subseteq V$ such that there is a partition $\{M_1, \ldots, M_k\}$ of M and for every $i \in \{1, \ldots k\}$, M_i is a module in $G[(V \setminus M) \cup M_i]$. A 1-module is exactly a module. If $\{V_1, V_2\}$ is a split or a bi-join, then V_1 and V_2 are 2-modules.

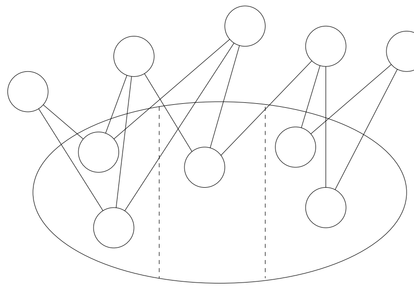

Fig. 3. A 3-module in a graph.

A *k-modular decomposition* of G is a rooted binary tree T such that the set of leaves of T is V, and of every node α in T, the set V_α of leaves of the sub-tree rooted at α is a k-module. The *modular-width* of a graph G, denoted $\mathrm{mod}(G)$, is the smallest k such that there is a k-modular decomposition of G. The graphs of modular-width 1 are exactly the cographs. The following proposition comes immediately.

Proposition 6 *M is a k-module of G if and only if M is a k-module of $\overline{G}$. Therefore* $\mathrm{mod}(G) = \mathrm{mod}(\overline{G})$.

Theorem 7 *For every graph G,* $\mathrm{mod}(G) \leq \mathrm{nlc}(G) \leq \mathrm{cwd}(G) \leq 2 \times \mathrm{mod}(G)$.

PROOF. For an expression t, we denote for short l_t the labeling function $l(\mathrm{val}(t))$. We already know that $\mathrm{nlc}(G) \leq \mathrm{cwd}(G)$ (Theorem 5). Let k be a positive integer, let G be a graph with $\mathrm{nlc}(G) \leq k$, and let t be a NLC k-expression for G. For every sub-expression t' of t, the set of vertices appearing in t' is a k-module, with partition $\{M_1, \ldots, M_k\}$ where $M_i = \{v \in V : l_{t'}(v) = i\}$ for all $i \in \{1, \ldots, k\}$. Then $T(t)$ is a k-modular decomposition tree of G, where $T(t)$ is a rooted binary tree defined inductively on t:

- $T((v)_i)$ is the rooted tree with one vertex v,
- $T(\rho_R(t')) = T(t')$.
- $T(t_1 \times_S t_2)$ is tree with a new root r, and r has two children: the root of $T(t_1)$ and the root of $T(t_2)$.

On the other hand, let T be a k-modular decomposition tree. For every node a in T, let V_a be the set of leaves in the sub-tree rooted at a. We recursively build for every node a in T a $2k$-expression t_a for $G[V_a]$ such that for all $u \in V_a$, $l_{t_a}(u) \in \{1, \ldots, k\}$ and for all $u, v \in V_a$ with $(N_G(u) \setminus V_a) \neq (N_G(v) \setminus V_a)$, then $l_{t_a}(u) \neq l_{t_a}(v)$.

If a is a leaf, then $t_a = (v)_1$. Otherwise, let b and c be the children of a. Let t_b and t_c be the expressions for $G[V_b]$ and $G[V_c]$ with the requested condition. Let $t''_a = t_b \oplus \left(\circ_{i \in \{1,\ldots,k\}} \rho_{i \to i+k}\right)(t_c)$, where $\circ$ is the composition of operations. Let $X_i = \{v \in V_a : l_{t''_a}(v) = i\}$. By hypothesis, for all $i \in \{1, \ldots, k\}$ and

$j \in \{k+1,\dots,2k\}$, either there is no edge between X_i and X_j, or there is every possible edges between X_i and X_j. Let $S = \{(i,j) : i \in \{1,\dots,k\}, j \in \{k+1,\dots,2k\}$ and $\exists(u,v) \in X_i \times X_j$ such that $\{u,v\} \in E(G[V_a])\}$. By the previous observations, $t'_a = (\circ_{(a,b)\in S}\ \eta_{a,b})(t''_a)$ is an $2k$-expression for $G[V_a]$.

Now we know that V_a is a k-module. Then there is a partition into non-empty subsets $Y_1,\dots Y_{k'}$ (where $k' \le k$) of V_a such that for all $j \in \{1,\dots,k'\}$, Y_j is a non-empty module in $G[(V \setminus V_a) \cup Y_j]$, and for every $i,j \in \{1,\dots,k'\}$ with $i \neq j$, there is a vertex in $G \setminus V_a$ which distinguishes Y_i and Y_j. For every $i \in \{1,\dots 2k\}$, there is a $f(i) \in \{1,\dots k\}$ such that $X_i \subseteq Y_{f(i)}$, otherwise there will be a vertex in $V \setminus V_a$ which distinguishes two vertices in X_i, and this contradict the definition of X_i. For every $j \in \{1,\dots,k\}$, let $g(j) = \min\{i \in \{1,\dots,2k\} : f(i) = j\}$.

Let $I = \{f(i) : i \in \{1,\dots,2k\}\}$. Note that $|I| = k'$. Let π be a bijection from $I \setminus \{1,\dots,k'\}$ to $\{1,\dots,k'\} \setminus I$. Let $K = \{i \in \{1,\dots,2k\} : i \neq g(f(i))\}$, and let

$$t_a = \left(\circ_{i\in I\setminus\{1,\dots k'\}}\ \rho_{i\to\pi(i)}\right) \circ \left(\circ_{i\in K}\ \rho_{i\to g(f(i))}\right)(t'_a).$$

t_a is a $2k$-expression for $G[V_a]$, and has the desired property. □

Note that the modular-width and the NLC-width are two different parameters. For example P_{10} has NLC-width 3 [8], modular-width 2 (according to Lemma 8), and clique-width 3 (since it is distance-hereditary). On the other hand, a non empty cograph has modular-width 1, NLC-width 1 and clique-width 2.

Lemma 8 *If G is a distance-hereditary graph then* $\mathrm{mod}(G) \le 2$. *Therefore the clique-width of complements of distance-hereditary graphs is at most* 4.

PROOF. Let T be a split decomposition of G in which every internal node has degree 3. We know that such a tree always exists since G is completely decomposable by the split decomposition. We choose an arbitrary root r in T. For every node α in T, let V_α be the set of leaves in the sub-tree rooted by α. If $\alpha \neq r$, $\{V_\alpha, V \setminus V_\alpha\}$ is a split, and thus V_α is a 2-module. Thus T is a 2-modular decomposition of the graph. □

It is known that the class of (gem,co-gem)-free graphs has clique-width bounded by 16 [2]. The following Lemma shows that (C_5,bull,gem,co-gem)-free graphs has clique-width bounded by 4. The proof is very similar to the proof of Lemma 8 (replace "split" by "bi-join").

Lemma 9 *If G is (C_5, bull, gem, co-gem)-free then* $\mathrm{mod}(G) \le 2$. *Therefore the clique-width of (C_5, bull, gem, co-gem)-free graphs is bounded by* 4.

5 Clique-width of $\mathcal{G}_{\mathcal{O}}$ classes

In this section we prove the following Theorem.

Theorem 10 (Main Theorem) *Let* $\mathcal{O} \subseteq \{\mathrm{TT}, \mathrm{FT}, \mathrm{PV}, \mathrm{TA}, \mathrm{FA}, \mathrm{CP}, \mathrm{IV}, \mathrm{DV}\}$ *Then* $\mathcal{G}_{\mathcal{O}}$ *is of bounded clique-width if and only if one of the following conditions hold:*

- $\mathcal{O} \subseteq \{\mathrm{TT}, \mathrm{FT}, \mathrm{IV}, \mathrm{DV}\}$, *and thus* $\mathcal{G}_{\mathcal{O}}$ *is a subclass of cographs, and its clique-width is bounded by* 2,
- $\mathcal{O} \subseteq \{\mathrm{TT}, \mathrm{FT}, \mathrm{PV}, \mathrm{IV}\}$, *and thus* $\mathcal{G}_{\mathcal{O}}$ *is a subclass of distance-hereditary graphs, and its clique-width is bounded by* 3,
- $\mathcal{O} \subseteq \{\mathrm{TT}, \mathrm{FT}, \mathrm{CP}, \mathrm{DV}\}$, *and thus* $\mathcal{G}_{\mathcal{O}}$ *is a subclass of complement of distance-hereditary graphs, and its clique-width is bounded by* 4,
- $\mathcal{O} \subseteq \{\mathrm{TT}, \mathrm{FT}, \mathrm{TA}, \mathrm{FA}\}$, *and thus* $\mathcal{G}_{\mathcal{O}}$ *is a subclass of (*C_5*, bull, gem, co-gem)-free graphs, and its clique-width is bounded by* 4.

The proof comes from the following Lemmas.

Lemma 11 *The graph class* $\mathcal{G}_{\{\mathrm{PV},\mathrm{DV}\}}$ *is of unbounded modular-width (and thus of unbounded clique-width).*

PROOF. Suppose the opposite. Let k be the modular-width of $\mathcal{G}_{\{\mathrm{PV},\mathrm{DV}\}}$.

Let $s > 2k$ and let $G = (V, E)$ be the following graph. V can be partitioned into a clique $U = \{u_1, \ldots, u_s\}$, and s sets $P_i = \{v_{i,1}, \ldots, v_{i,s}\}$ which induce a path $(v_{i,1}, \ldots, v_{i,s})$. Each u_l is adjacent to every $v_{i,j}$ with $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, l\}$. G has no other edge.

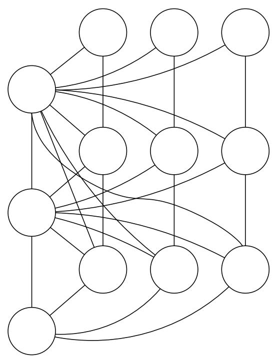

Fig. 4. The graph for $s = 3$.

For every s, G is in $\mathcal{G}_{\{\mathrm{PV},\mathrm{DV}\}}$. It can be constructed by the graph with one vertex u_1, adding a pendant vertex $v_{i,1}$ to u_1 for every $i \in \{1, \ldots, s\}$, and adding successively, for every $i \in \{2, \ldots, s\}$, a dominating vertex u_i and s vertices $v_{i,j}$ pendant to $v_{i,j-1}$ (for $i \in \{1, \ldots, s\}$).

Let T be a k-modular decomposition of G. For a node α in T, we denote by V_α the leaves of T in the sub-tree rooted at α. Let a be an internal node of T, and b and c be its two children, such that there is a $l \in \{1, \ldots, s\}$ with $P_l \subseteq V_a$, and for all $i \in \{1, \ldots, k\}s$, $P_i \not\subseteq V_b$ and $P_i \not\subseteq V_c$. By definition of k-modular decomposition, V_a, V_b and V_c are k-modules of G.

For $M \subseteq V$, the *classes* of M are the sets in the partition $\{M_1, \ldots, M_k\}$, such that for all $i \in \{1, \ldots, k\}$ M_i is a module in $G[(V \setminus M) \cup M_i]$, and for all $i, j \in \{1, \ldots, k\}$ with $i \neq j$, there is a vertex in $V \setminus M$ which distinguishes M_i and M_j. Note that M is a k-module if and only if it has at most k classes.

Claim 1 *$|U \setminus V_a| \leq k$, and $|U \cap V_a| > k$.*

PROOF. By definition of a and l, $P_l \subseteq V_a$. If two vertices $v_{l,i}$ and $v_{l,j}$ in P_l are in the same class of V_a, then every vertex u_m with $m \in \{i, \ldots, j-1\}$ is in V_a, since it distinguishes $v_{l,i}$ and $v_{l,j}$.

Let $i \in \{1, \ldots, s\}$ such that $u_i \notin V_a$. Then there is no class C of V_a such that $C \cap \{u_{l,1}, \ldots, u_{l,i}\} \neq \emptyset$ and $C \cap \{u_{l,i+1}, \ldots, u_{l,s}\} \neq \emptyset$, since $C \cap \{u_{l,1}, \ldots, u_{l,i}\}$ and $C \cap \{u_{l,i+1}, \ldots, u_{l,s}\}$ are distinguished by u_i.

Suppose that $|U \setminus V_a| > k$, and let $\pi_1 < \pi_2 < \ldots < \pi_{k+1}$ such that for all $i \in \{1, \ldots, k+1\}$, $u_{\pi_i} \notin V_a$. Let $\pi_0 = 0$ and $\pi_{k+2} = s$. Then for all $i, j \in \{0, 1, \ldots, k+1\}$, $i < j$, no vertex in $\{v_{l,\alpha}\}_{\pi_i < \alpha \leq \pi_{i+1}}$ are in the same class of V_a than a vertex in $\{v_{l,\alpha}\}_{\pi_j < \alpha \leq \pi_{j+1}}$ since these vertices are distinguished by u_{π_j}. For every $i \in \{0, 1, \ldots, k\}$, $\pi_{i+1} > \pi_i$ (note that π_{k+1} can be equal to π_{k+2}). Then V_a has strictly more than k classes, contradiction. □

Claim 2 *For every $r \in \{1, \ldots, s\}$, $P_r \cap V_a \neq \emptyset$.*

PROOF. Suppose that there is a r such that $P_r \cap V_a = \emptyset$. Then every vertex in $U \cap V_a$ is in a different class of V_a. Contradiction since by Claim 1 $|U \cap V_a| > k$, and there will be at least $k+1$ classes in V_a. □

Claim 3 *The set $A = \{i \in \{1, \ldots, s\} : P_i \subseteq V_a\}$ contains at least $k+1$ elements.*

PROOF. Suppose that $B = \{i \in \{1, \ldots, s\} : P_i \not\subseteq V_a\}$ has at most $k+1$ elements. For every $i \in B$, let x_i be a vertex in $P_i \setminus V_a$ adjacent to a vertex y_i in $P_i \cap V_a$. Then for every $i \in B$, x_i is in a different class of V_a, since it is distinguished by y_i. Contradiction since V_a has at most k classes. Thus $|B|$ is at most k, and $A = \{1, \ldots, s\} \setminus B$ has at least $k+1$ elements. □

Now we have a contradiction. By our choice of a, no P_i is a subset of V_b of V_c. For every $i \in A$, P_i can be partitioned into $\{P_i \cap V_b, P_i \cap V_c\}$. We need at most one class in V_b for each of these P_i, and thus by the last Claim V_b has at least $k+1$ classes. Contradiction. □

Let $o \in \mathrm{TT}, \mathrm{FT}, \mathrm{PV}, \mathrm{TA}, \mathrm{FA}, \mathrm{CP}, \mathrm{IV}, \mathrm{DV}$. We say that a graph class $\mathcal{G}$ is *closed* under o if $\forall G \in \mathcal{G}$ and $\forall u \in V(G)$, $\mathrm{ext}_o(G, u) \in \mathcal{G}$.

Lemma 12 *Let $\mathcal{G}$ be a hereditary class.*

(1) If $\mathcal{G}$ is closed under PV *and* TA*, then $\mathcal{G}$ is closed under* DV*.*
(2) If $\mathcal{G}$ is closed under PV *and* FA*, then $\mathcal{G}$ is closed under* DV*.*
(3) If $\mathcal{G}$ is closed under PV *and* CP*, then $\mathcal{G}$ is closed under* DV*.*
(4) If $\mathcal{G}$ is closed under IV *and* TA*, then $\mathcal{G}$ is closed under* PV*.*
(5) If $\mathcal{G}$ is closed under IV *and* FA*, then $\mathcal{G}$ is closed under* PV*.*

PROOF. Let $G \in \mathcal{G}$, and let u be a vertex of G. Let G' be the graph constructed from G after adding a new vertex v pendant to u, and then a new vertex w pendant to v, and then a (true or false) anti-twin x to v. Then in $G' - \{v, w\}$, x is a dominating vertex. This proves propositions (1) and (2). For proposition (3), add a pendant vertex v to u, and a co-pendant vertex w to v. Then in $G' - v$, w is a dominating vertex.

Now suppose that $\mathcal{G}$ is closed under IV and TA. To construct G', we add a true anti-twin v to u, then an independent vertex w, and then a true anti-twin x to v. It is easy to see that $G' - \{u, v, w\}$ is isomorphic to G, and w is a pendant vertex of x in $G' - \{v, w\}$. Thus $\mathcal{G}$ is closed under PV. For the last proposition, replace "true anti-twin" by "false anti-twin" in the previous construction. □

Lemma 13

(1) $\mathcal{G}_{\{\mathrm{PV},\mathrm{TT},\mathrm{FT},\mathrm{IV}\}} = \mathcal{G}_{\{\mathrm{PV},\mathrm{TT},\mathrm{FT}\}}$.
(2) $\mathcal{G}_{\{\mathrm{TT},\mathrm{FT},\mathrm{IV},\mathrm{DV}\}} = \mathcal{G}_{\{\mathrm{TT},\mathrm{FT}\}}$.

PROOF. If we add a dominating vertex or an independent vertex to a cograph, the new graph is still a cograph. If we add an independent vertex to a distance-hereditary graph, the new graph is still distance-hereditary. The result follows by the characterizations of cographs and distance-hereditary graphs (Theorems 1 and 2). □

Now we can prove the Main Theorem.

PROOF. [Proof of Theorem 10] Let $\mathcal{O} \subseteq \{\mathrm{TT}, \mathrm{FT}, \mathrm{PV}, \mathrm{TA}, \mathrm{FA}, \mathrm{CP}, \mathrm{IV}, \mathrm{DV}\}$ such that $\mathcal{G}_{\mathcal{O}}$ has bounded clique-width. Suppose that PV $\in \mathcal{O}$. Then by Lemma 11, DV $\notin \mathcal{O}$, and by propositions (1), (2) and (3) of Lemma 12, if $\mathcal{O} \cap \{\mathrm{TA}, \mathrm{FA}, \mathrm{CP}\} = \emptyset$, otherwise $\mathcal{G}'_{\mathcal{O}}$ is closed under DV and thus $\mathcal{G}'_{\mathcal{O}}$ has unbounded clique-width. Thus $\mathcal{O} \subseteq \{\mathrm{TT}, \mathrm{FT}, \mathrm{PV}, \mathrm{IV}\}$, and by Lemma 13, $\mathcal{G}_{\mathcal{O}}$ is a subclass of distance-hereditary graphs (Theorem 2), and has clique-width at most 3 (Theorem 4).

If CP $\in \mathcal{O}$, then by same arguments on the complement, $\mathcal{O} \subseteq \{\mathrm{TT}, \mathrm{FT}, \mathrm{CP}, \mathrm{DV}\}$, and $\mathcal{G}_{\mathcal{O}}$ is a subclass of co-distance-hereditary graphs.

Now suppose that TA $\in \mathcal{O}$ and IV $\in \mathcal{O}$. By proposition (4) of Lemma 12, $\mathcal{G}'_{\mathcal{O}}$ is closed under PV, and by proposition (1), $\mathcal{G}'_{\mathcal{O}}$ is closed under DV, and thus $\mathcal{G}'_{\mathcal{O}}$ has unbounded clique-width. Contradiction. Similarly by propositions (5) and (2), if FA $\in \mathcal{O}$ and IV $\in \mathcal{O}$ then $\mathcal{G}'_{\mathcal{O}}$ has unbounded clique-width. Contradiction. Thus if IV $\in \mathcal{O}$, then $\mathcal{O} \subseteq \{\mathrm{TT}, \mathrm{FT}, \mathrm{IV}, \mathrm{DV}\}$, and $\mathcal{G}_{\mathcal{O}}$ is a subclass of the cographs (Lemma 13 and Theorem 1) and has clique-width at most 2.

With the same arguments on the complement, if DV $\in \mathcal{O}$ then $\mathcal{O} \subseteq \{\mathrm{TT}, \mathrm{FT}, \mathrm{IV}, \mathrm{DV}\}$ and $\mathcal{G}_{\mathcal{O}}$ is a subclass of the cographs.

Finally if $\mathcal{O} \subseteq \{\mathrm{TT}, \mathrm{FT}, \mathrm{TA}, \mathrm{FA}\}$ then $\mathcal{G}_{\mathcal{O}}$ is a subclass of (C_5, bull, gem, co-gem)-free graphs (Theorem 3) and its clique-width is bounded by 4 (Lemma 9). □

The only classes of graphs of bounded clique-width which can be defined by one-vertex extensions are subclasses of distance-hereditary graphs, co-distance-hereditary graphs, or (C_5, bull, gem, co-gem)-free graphs.

6 Concluding remarks

It is not hard to see that $\mathcal{G}_{\mathcal{O}}$ is of bounded treewidth if and only if $\mathcal{O} \subseteq \{\mathrm{IV}, \mathrm{PV}\}$ or $\mathcal{O} \subseteq \{\mathrm{IV}, \mathrm{FT}\}$, since we can construct arbitrarily large cliques with DV, CP or FA, and arbitrarily large complete bipartite graphs with TA, or PV and FT.

A graph completely decomposable by the modular decomposition (*i.e.* a cograph) is of clique-width at most 2, and a graph completely decomposable by the split decomposition (*i.e.* a distance-hereditary graph) is of clique-width at most 3. Some graphs are prime for the modular decomposition and decomposable by the split decomposition (*e.g.* P_4), and conversely some graphs are prime for the split decomposition and decomposable by the modular decomposition (*e.g.* gem). One can successively use the modular decomposition and the split decomposition to decompose larger classes of graph. This decomposition has been called *c-decomposition* in the doctoral dissertation of J.M. Lanlignel [11]. This decomposition completely decomposes graphs in $\mathcal{G}_{\{\mathrm{PV},\mathrm{DV}\}}$, and so the class of graphs completely decomposable by the c-decomposition has unbounded clique-width (by Lemma 11). This is notable, since for a large part of known decompositions, the class on completely decomposable graphs has bounded clique-width (*e.g.* tree decomposition for fixed k, modular, split and bi-join decompositions).

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