Decomposition of (gem,co-gem)-free graphs

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Abstract

We present a structure theorem for (gem,co-gem)-free graphs using the bi-join decomposition. In particular we show that the prime graphs in the bi-join decomposition tree has at most 9 vertices. This leads to a new proof that this class of graph has bounded clique width. As applications, we show that recognition can be done in linear time, and that GRAPH ISOMORPHISM is polynomial time solvable on this class.

1 Introduction

Graph decompositions have often shown their utility in graph theory and algorithmic. Generally results come from structure theorems, saying that a given class of graph $\mathcal{G}$ can be decomposed by a given decomposition into graphs of a class $\mathcal{G}' \subsetneq \mathcal{G}$.

The *modular decomposition* is a well known graph decomposition. The *cographs* are exactly the class of graphs completely decomposable by the modular decomposition. A lot of problems, NP-hard in the general case, can be solved in polynomial time of this class. There is structure theorems using the modular decomposition for several classes: one can cite the comparability graphs [14], (P_5,$\overline{P_5}$)-free graphs [12], (P_5,gem)-free graphs [2] and graphs with few P_4's [13]. The *split decomposition* is a generalisation of the modular decomposition, meaning that it decomposes larger class of graphs. Structure theorems exist for the distance hereditary graphs, and parity graphs [5]. These decomposition theorems usually lead to efficient algorithms on concerned classes.

The *clique-width* generalize the well known *treewidth*, in the sense that graphs of bounded treewidth have bounded clique-width. The *NLC-width* is graph parameter close to the clique-width: they differ by at most a factor of two. A motivation in the study of theses graph parameters is that many NP-hard problems become polynomial time solvable when restricted to graphs of bounded width.

We present in this paper a decomposition theorem for (gem,co-gem)-free graphs using the *bi-join decomposition*, which is an other generalisation of the modular decomposition. The structure theorem is presented in sections 3 and 4. This theorem leads to a linear time recognition algorithm for (gem,co-gem)-free graphs. We show in section 5 that the NLC-width of (gem,co-gem)-free graphs is at most 4, and their clique-width is at most 8, improving the bound given in [3]. Moreover a corresponding k-expression can be computed in linear time. Finally we show in section 6 that GRAPH ISOMORPHISM is $O(nm)$ time solvable on this class.

2 Preliminaries

A *graph* G is a pair (V, E) where V is the set of *vertices* and $E \subseteq \{\{u, v\} : (u, v) \in V^2 \text{ and } u \neq v\}$ is the set of *edges*. Every graph is supposed finite, undirected, without loops nor multiple edges.

For a graph G, $V(G)$ denote the set of its vertices and $E(G)$ the set of its edges. We denote by $n(G) = |V(G)|$ and by $m(G) = |E(G)|$. When the graph G is clear in the context, we note V, E, n and m instead of $V(G)$, $E(G)$, $n(G)$ and $m(G)$. Let v be a vertex; the *neighborhood* of v, denoted by $N_G(v)$, is the set $\{u \in V : \{u, v\} \in E\}$, and the *closed neighborhood*, denoted by $N_G[v]$, is $N(v) \cup \{v\}$. Let the *degree* of v in G be $d_G(v) = |N_G(v)|$. If the graph G is clear in the context, we note $N(v)$, $N[v]$ and $d(v)$ instead of $N_G(v)$, $N_G[v]$ and $d_G(v)$. The *complement* of $G = (V, E)$ is the graph $\overline{G} = (V, E')$ where $\{u, v\} \in E'$ if and only if $\{u, v\} \notin E$.

A vertex u is *adjacent* to a vertex v if $\{u, v\} \in E$, otherwise u is *non-adjacent* to w. Two sets of vertices A and B, with $A \cap B = \emptyset$, are adjacent (respectively, non-adjacent) if every vertex in A is adjacent (resp. non-adjacent) to every vertex in B. A vertex $v \in V$ is adjacent (resp. non-adjacent) to $W \subseteq V \setminus \{v\}$ if $\{v\}$ is adjacent (resp. non-adjacent) to W. A graph G is *complete* (resp. *edgeless*) if for every $u, v \in V(G)$ with $u \neq v$, u and v are adjacent (resp. non-adjacent).

Let G and H be two graphs, and let π be a bijection from $V(E)$ to $V(H)$. We note $G \simeq_\pi H$ if for every $u, v \in V(G)$, $\{u, v\} \in E(G)$ if and only if $\{\pi(u), \pi(v)\} \in E(H)$. G and H are *isomorphic* (denoted by $G \simeq H$) if there is a bijection $\pi : V(G) \to V(H)$ such that $G \simeq_\pi H$.

Let $W \subseteq V$. The *subgraph induced by* W, denoted $G[W]$, is the graph of vertex set W and edge set $\{\{u, v\} : \{u, v\} \in E \text{ and } (u, v) \in W^2\}$. For $W \subseteq V$, $G - W$ denote $G[V \setminus W]$, and for $v \in V$, $G - v$ denote $G[V \setminus \{v\}]$. We say that G is *H-free* if g has no induced subgraph isomorphic to H, and G is *$(H_1, H_2, \ldots, H_k)$-free* if G has no subgraph isomorphic to H_1, H_2, $\ldots$ or H_k.

Figure 1 presents some graphs used in this paper. Note that the bull (resp. the gem, the co-gem) has only one subgraph isomorphic to the P_4. This unique P_4 is called *the P_4 of the bull* (resp. gem, co-gem). We say that (a, b, c, d) is (or induce) a P_4 if $\{\{a, b\}, \{b, c\}, \{c, d\}\} \subseteq E$, and there is no other edges between a, b, c and d. Similarly we say that (a, b, c, d, e) is a bull (resp. a gem, a co-gem, a C_5) if (a, b, c, d) is a P_4 and $G[\{a, b, c, d, e\}]$ induce a bull (resp. gem, co-gem, C_5).

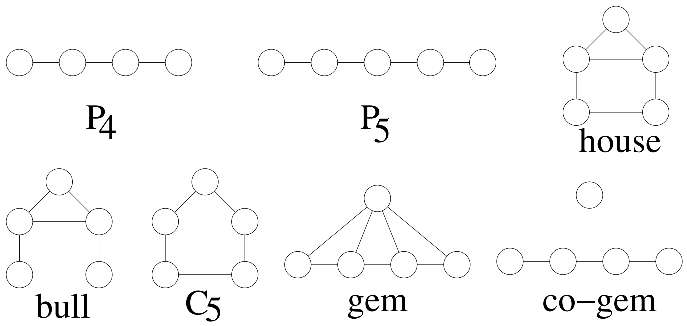

Figure 1: Some graphs.

2.1 Modular decomposition

A *module* in a graph is a non-empty subset $X \subseteq V$ such that for all $u \in V \setminus X$, then either $N(u) \cap X = \emptyset$ or $X \subseteq N(u)$. A module is *trivial* if $|X| = 1$ or $X = V$. A graph is *prime* w.r.t. the modular decomposition if all its modules are trivial. We denote by $\mathcal{M}(G)$ be the set of modules of G.

A module X *overlaps* a module Y if $X \cap Y$, $X \setminus Y$ and $Y \setminus X$ are non-empty. A module is *strong* if there is no module Y such that X overlaps Y. (Note that a trivial module is strong.) Let $\mathcal{S}(G)$ be the set of strong modules of G. $\mathcal{S}(G)$ defines an inclusion family of subsets of V. Let $\mathcal{P}(G)$ be the set of maximal elements (w.r.t. inclusion) in $\mathcal{S}(G) \setminus \{V\}$. It is easy to see that $\mathcal{P}(G)$ is a partition of V. Let $U \subseteq V$ such that for every $P \in \mathcal{P}$, $|U \cap P| = 1$. The *characteristic graph* of G, denoted G^*, is the graph $G[U]$. Note that G^* is independent of the choice of U, since every $P \in \mathcal{P}$ is a module in G.

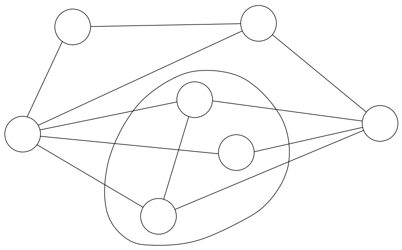

Figure 2: A module in a graph.

The *modular decomposition theorem*, generally attributed to Gallai [11], says that given a graph G we have:

- if G is not connected, then $\mathcal{P}(G)$ is the set of connected components of G, and G^* is an edgeless graph;
- if $\overline{G}$ is not connected, then $\mathcal{P}(G)$ is the set of connected components of $\overline{G}$, and G^* is a complete graph;
- otherwise, $\mathcal{P}$ is the maximal elements of $\mathcal{M}(G) \setminus \{V\}$ (w.r.t. inclusion), and G^* is a prime graph (w.r.t. modular decomposition).

The *modular decomposition* is the recursive decomposition of G into $G[M_1], \ldots G[M_k]$, where $\mathcal{P} = \{M_1, \ldots, M_k\}$, until every graph has one vertex. It can be represented by a rooted tree, called the *modular decomposition tree*. If the graph is not connected or not co-connected, the corresponding node is said *degenerated*, otherwise it is said *prime*. To summarize, we have:

Theorem 1 ([11]). *Let G be a graph. There is an unique rooted tree T, called the* modular decomposition tree*, such that the set of leaves of T is the set of vertices in G, each internal node in T has at least two children, and each internal node is labeled* `prime` *or* `degenerated`*, such that:*

- *for every node α, V_α is a strong module of G, and there is no other strong modules in G,*
- *for every* `degenerated` *node α, and for every subset C of children of α, $\bigcup_{\beta \in C} V_\beta$ is a module of G,*
- *there is no other module in G,*

where for a node α, V_α is the set of leaves in the subtree rooted at α.

Theorem 2 ([16]). *The modular decomposition can be computed in linear time.*

The class of *cographs* is tightly related to the modular decomposition: it is the class of graphs completely decomposable.

Theorem 3 ([6, 21]). *The following conditions are equivalent:*

- *G is a* cograph, i.e. *every node in its modular decomposition tree is* `degenerated`*.*
- *Every induced subgraph of G with at least* 3 *vertices has a non-trivial module.*
- *G is P_4-free.*

2.2 Bi-join decomposition

To present the bi-join decomposition theorem, we need to introduce some notations. A *bi-partition* of a set V is a $\{X, Y\}$ such that $X \cup Y = V$ and $X \cap Y = \emptyset$. Let $T = (V, E)$ be an unrooted tree. An edge $e = \{\alpha, \beta\}$ in T is an *internal edge* if α and β are internal nodes. We denote by $\mathrm{Leaves}(T)$ the set of leaves in T. For two nodes α and β such that α is adjacent to β, we define by $A_{\alpha\to\beta}$ is the set of nodes in the connected component of $T - \alpha$ that contains β, and we define by $L_{\alpha\to\beta} = A_{\alpha\to\beta} \cap \mathrm{Leaves}(T)$. For an edge $e = \{\alpha, \beta\}$ in E, let $\mathrm{Part}(e)$ be $\{L_{\alpha\to\beta}, L_{\alpha\to\beta}\}$, and for an internal node α, let $\mathrm{Part}(\alpha) = \{L_{\alpha\to\beta} : \beta \in N_T(\alpha)\}$. Note that $\mathrm{Part}(e)$ and $\mathrm{Part}(\alpha)$ are proper partitions of $\mathrm{Leaves}(T)$.

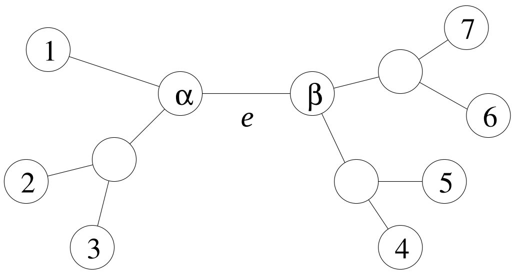

Figure 3: $L_{\alpha\to\beta} = \{4, 5, 6, 7\}$, $\mathrm{Part}(e) = \{\{1, 2, 3\}, \{4, 5, 6, 7\}\}$ and $\mathrm{Part}(\alpha) = \{\{1\}, \{2, 3\}, \{4, 5, 6, 7\}\}$.

A *bi-join* in a graph is a bi-partition $\{X, Y\}$ such that for all $u, v \in X$, $\{N(u) \cap Y, Y \setminus N(u)\} = \{N(v) \cap Y, Y \setminus N(v)\}$. A bi-join is *trivial* if $|X| = 1$ or $|Y| = 1$. A graph is *prime* w.r.t. the bi-join decomposition if all its bi-join are trivial. Note that the gem, co-gem, bull and C_5 are prime graphs w.r.t. bi-join decomposition.

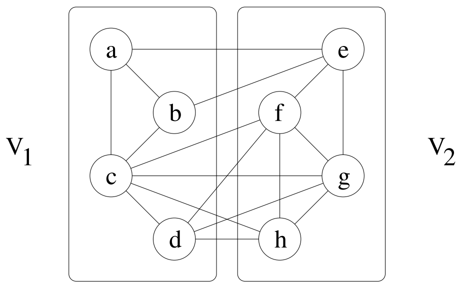

Figure 4: A bi-join in a graph. The neighborhood of a vertex $v \in V_1$ in V_2 is either $\{e\}$ or $\{f, g, h\}$, and $\{\{e\}, \{f, g, h\}\}$ is a partition of V_2.

The family of bi-join in a graph has the *bi-partitive property* [18, 19], close to the properties of the decomposition frame of Cunningham and Edmonds [10]. A family with this property can be represented by an unique unrooted tree. An other well known example is the family of splits in a connected graph, and the unique unrooted representative tree of splits is exactly the split decomposition of the graph [9].

Theorem 4 ([18, 19]). *Let G be a graph. Then there is an unique unrooted tree $T = (V_T, E_T)$, called the* bi-join decomposition tree, *such that* $\mathrm{Leaves}(T) = V$, *each internal node has at least* 3 *neighbors and each internal node is labeled* `prime` *or* `degenerated`, *such that:*

- *For all $e \in E_T$,* $\mathrm{Part}(e)$ *is a bi-join.*
- *Let $\alpha \in V_T$ be an internal node labelled* `degenerated`. *Then for all $\emptyset \subsetneq I \subsetneq N_T(\alpha)$, $\{\cup_{\beta\in I} L_{\alpha\to\beta}, \cup_{\beta\in N_T(\alpha)\setminus I} L_{\alpha\to\beta}\}$ is a bi-join of G.*

- *G has no other bi-join.*

We can easily see that $\{X, Y\}$ is a bi-join of G if and only if it is a bi-join of $\overline{G}$. Thus the bi-join decomposition tree of $\overline{G}$ is exactly the bi-join decomposition tree of G. This observation will be useful in order to reduce the number of cases to analyse in some forthcoming proofs.

Characteristic graphs Let G be a graph, and T be its bi-join decomposition tree. For every internal node α, and for every $\beta \in N_T(\alpha)$, we choose a vertex $v_{\alpha\to\beta} \in L_{\alpha\to\beta}$. We say that $v_{\alpha\to\beta}$ is a *marker* if $|L_{\alpha\to\beta}| > 1$; in other words, $v_{\alpha\to\beta}$ is a marker if it is a representative for more than one vertex.

Let $C(\alpha)$ be the graph $(\mathcal{N}, \mathcal{E})$ where $\mathcal{N} = \{\{\alpha, \beta\} : \beta \in N_T(\alpha)\}$ is the set of edges incident to α, and for every $\beta, \beta' \in N_T(\alpha)$, $\{\{\alpha, \beta\}, \{\alpha, \beta'\}\} \in \mathcal{E} \Leftrightarrow \{v_{\alpha\to\beta}, v_{\alpha\to\beta'}\} \in E(G)$. We say that $C(\alpha)$ is a *characteristic graph* of α. Note that $C(\alpha)$ is isomorphic to $G[\{v_{\alpha\to\beta} : \beta \in N_T(\alpha)\}]$, which is an induced subgraph of G.

Lemma 5 ([18, 19]). *If α is `prime` then its characteristic graph $C(\alpha)$ is prime. If α is `degenerated` then either $C(\alpha)$ or $\overline{C(\alpha)}$ is the disjoin union of two complete graphs.*

A *bi-join decomposition* of G is a triplet (V_T, E_T, C) such that (V_T, E_T) is the bi-join decomposition tree of G, and for every internal node α, $C(\alpha)$ is a characteristic graph of α. Note that a graph has an unique bi-join decomposition tree, but can has several bi-join decompositions since characteristic graphs depend from the choice of $v_{\alpha\to\beta}$, $\beta \in N_T(\alpha)$.

Completely decomposable graphs

Theorem 6 ([18, 19]). *The following conditions are equivalent:*

- *Every node in the bi-join decomposition tree of G is `degenerated`,*
- *Every induced subgraph of G with at least 4 vertices has a non-trivial bi-join,*
- *G is (C_5, bull, gem, co-gem)-free.*

Decomposition algorithm

Theorem 7 ([18, 19]). *The bi-join decomposition can be computed in linear time.*

2.3 Seidel switch

We recall in this section a strong relation between the modular decomposition and the bi-join decomposition by the way of a graph transformation close to the Seidel switch.

The *Seidel switch* [20] of a graph G with switch subset $W \subseteq V$ is the graph $\overline{G}^W = (V, E')$ such that $\{u, v\} \in E'$ if either $\{x, y\} \in E(G)$ and $\{x, y\}$ does not overlaps W, or if $\{x, y\} \notin E(G)$ and $\{x, y\}$ overlaps W (that is, $E' = E\Delta\{\{u, v\} : u \in W \text{ and } v \in V \setminus W\}$, where $A\Delta B = (A \setminus B) \cup (B \setminus A)$). Let the *Seidel reduction* of a graph G at vertex v, denoted $\overline{G}^v$, be $\overline{G}^{N(v)} - v$ (*i.e.* remove v after a Seidel switch with switch subset $N(v)$).

Lemma 8 ([18, 19]). *Let $\{V_1, V_2\}$ be a bi-partition of V and $v \in V_1$. Then $\{V_1, V_2\}$ is a bi-join of G if and only if V_2 is a module of $\overline{G}^v$.*

Theorem 9 ([18, 19]). *Let T be the bi-join decomposition tree of G, and let T' the modular decomposition of $\overline{G}^v$. Then T is isomorphic to T' with an additional leaf for the vertex v adjacent to the root of T'. Moreover, an internal node of T is `degenerated` (resp. `prime`) if and only if the corresponding node is `degenerated` (resp. `prime`) in T'.*

Since the gem, the co-gem, the bull and the C_5 are the only prime graphs (w.r.t the bi-join decomposition) with 5 vertices, we have:

Corollary 10. *The Seidel switch of a gem, a co-gem, a bull or a C_5 is a gem, a co-gem, a bull or a C_5.*

The following theorem says that a characteristic graph of a node α in the bi-join decomposition tree of a graph is unique *modulo* the Seidel switch.

Theorem 11. *Let α be a node in the bi-join decomposition tree of $G = (V, E)$. For all $\beta \in N_T(\alpha)$, let $v_\beta, v'_\beta \in L_{\alpha\to\beta}$. Let $H = G[\{v_{\alpha\to\beta} : \beta \in N_T(\alpha)\}]$ and $H' = G[\{v'_{\alpha\to\beta} : \beta \in N_T(\alpha)\}]$. Then there is a $W \subseteq V(H')$ such that $H \simeq \overline{H'}^W$.*

Proof. Note that $\overline{\overline{G}^A}^B = \overline{G}^{A\Delta B}$. Thus is it sufficient to show theorem when there is a β such that for every $\beta' \neq \beta$, $v_\beta = v'_\beta$. By definition $L_{\alpha\to\beta}$ is a bi-join, thus either $N(v_\beta) \setminus L_{\alpha\to\beta} = N(v'_\beta) \setminus L_{\alpha\to\beta}$ or $N(v_\beta) \setminus L_{\alpha\to\beta} = (V \setminus N(v'_\beta)) \setminus L_{\alpha\to\beta}$. In the first case $H \simeq H'$, and in the second case $H \simeq \overline{H'}^{\{v'_\beta\}}$. Theorem 11 follows. □

3 Prime (gem,co-gem)-free graphs

In this section we prove the following theorem. It leads directly to corollary 13.

Theorem 12. *Let G be a (gem,co-gem)-free graph which is prime w.r.t. the bi-join decomposition. Then G is an induced subgraph of one of the graphs in figure 5, and thus it has at most 9 vertices.*

Corollary 13. *Every prime node in a bi-join decomposition tree of a (gem,co-gem)-free graph has degree at most* 9.

We characterize in a first time prime (C_5,gem,co-gem)-free graphs.

3.1 Prime (C_5,gem,co-gem)-free graphs

Lemma 14. *Let G be a (C_5,gem,co-gem)-free graph which is prime w.r.t. the bi-join decomposition. Then G is an induced subgraph of the graph G_0 in figure 5.*

Proof. Suppose that G is bull-free. By theorem 6, G is completely decomposable by the bi-join decomposition, and thus it has at most 3 vertices since G is prime w.r.t. the bi-join decomposition. All graphs with at most 3 vertices are induced subgraphs of the graph G_0.

Otherwise, let (a, b, c, d, e) be a bull of G. For $W \subseteq \{a, b, c, d, e\}$, let M_W be the set $\{v \in V \setminus \{a, b, c, d, e\} : N_G(v) \cap \{a, b, c, d, e\} = V'\}$.

Note that $M_\emptyset$ and $M_{\{e\}}$ are empty, otherwise G has a co-gem, and $M_{\{a,b,c,d\}}$ and $M_{\{a,b,c,d,e\}}$ are empty, otherwise G has a gem. If $M_{\{a\}}$ is non-empty then (u, a, b, e, d) is a co-gem (with $u \in M_{\{a\}}$); thus $M_{\{a\}}$ is empty. By symmetry (exchange a with d, and b with c) $M_{\{d\}}$ is empty. By symmetry when we take the complement (exchange (a, b, c, d, e) with (b, d, a, c, e)) $M_{\{b,e\}}$ and $M_{\{c,e\}}$ are empty. Similarly, if $M_{\{a,c\}}$ is non-empty, then (u, a, b, e, d) is a co-gem (with $u \in M_{\{a,c\}}$). Thus $M_{\{a,c\}}$ is empty, and with the symmetries, $M_{\{b,d\}}$, $M_{\{a,b,e\}}$ and $M_{\{c,d,e\}}$ are empty. Moreover $M_{\{a,b,c\}}$ is empty, otherwise (a, u, c, e, b) is a gem (with $u \in M_{\{a,b,c\}}$), and with the symmetries, $M_{\{b,c,d\}}$, $M_{\{a,b,d,e\}}$ and $M_{\{a,c,d,e\}}$ are empty. Finally if $M_{\{a,d\}}$ or $M_{\{a,d,e\}}$ is non-empty then G has a C_5. To summarize, $\{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, M_{\{b\}}, M_{\{a,b\}}, M_{\{c\}}, M_{\{b,c\}}, M_{\{a,b,d\}}, M_{\{c,d\}}, M_{\{a,c,d\}}, M_{\{a,e\}}, M_{\{a,c,e\}}, M_{\{b,c,e\}}, M_{\{a,b,c,e\}}, M_{\{d,e\}}, M_{\{b,d,e\}}, M_{\{b,c,d,e\}}\}$ is a partition of V.

Now suppose that a vertex $u \in M_{\{b\}}$ is non-adjacent to a vertex $v \in M_{\{a,c,e\}}$. Then (u, b, a, v, d) is a co-gem, contradiction. Thus every vertex in $M_{\{b\}}$ is adjacent to every vertex in $M_{\{a,c,e\}}$. Similarly, a vertex $u \in M_{\{b\}}$ is adjacent to every vertex in $M_{\{a,b,c,e\}} \cup M_{\{a,e\}} \cup M_{\{a,b,d\}} \cup M_{\{a,c,d\}}$, and non-adjacent to every vertex in $M_{\{c\}} \cup M_{\{c,d\}} \cup M_{\{b,c\}} \cup M_{\{b,c,e\}} \cup M_{\{b,d,e\}} \cup M_{\{b,c,d,e\}} \cup M_{\{d,e\}}$. The figure 6 show forbidden subgraphs in the opposite cases. A vertex in $M_{\{a,b\}}$ is adjacent to every vertex in $M_{\{a,c,e\}} \cup M_{\{a,b,c,e\}} \cup M_{\{a,e\}} \cup M_{\{a,b,d\}} \cup M_{\{a,c,d\}}$, and non-adjacent to every vertex in $M_{\{c\}} \cup M_{\{c,d\}} \cup M_{\{b,c\}} \cup M_{\{b,c,e\}} \cup M_{\{b,d,e\}} \cup M_{\{b,c,d,e\}} \cup M_{\{d,e\}}$ (figure 7), and a vertex in $M_{\{a,e\}}$

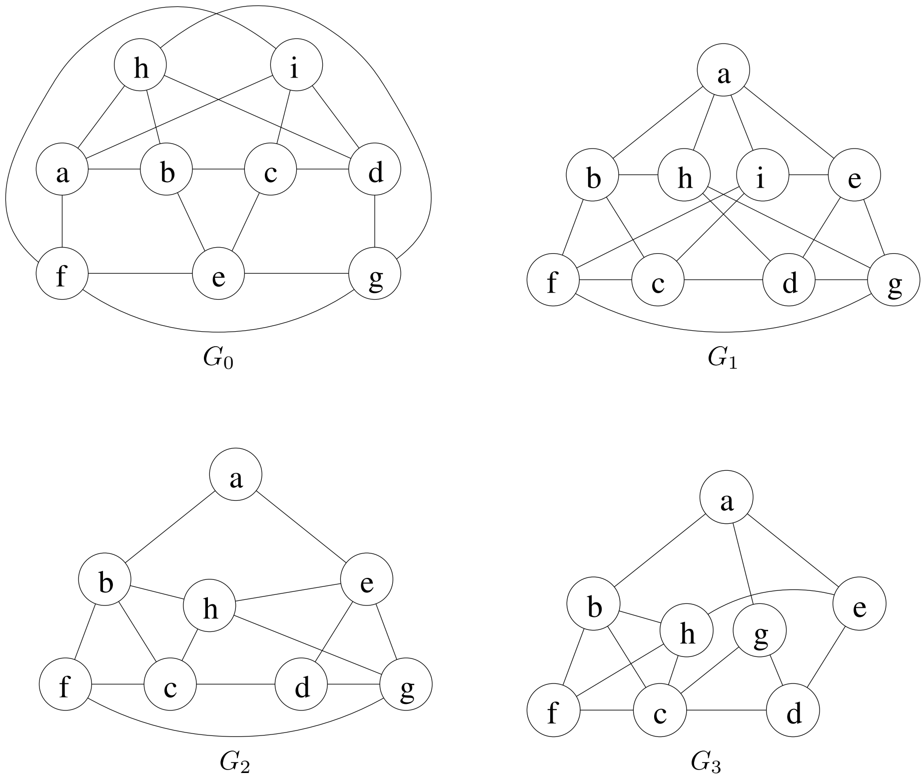

Figure 5: Prime (gem,co-gem)-free graphs (w.r.t. bi-join decomposition).

is adjacent to every vertex in $M_{\{b\}} \cup M_{\{a,b\}} \cup M_{\{b,c\}} \cup M_{\{b,c,e\}} \cup M_{\{d,e\}} \cup M_{\{a,c,d\}}$ and non-adjacent to every vertex in $M_{\{c\}} \cup M_{\{c,d\}} \cup M_{\{a,c,e\}} \cup M_{\{a,b,c,e\}} \cup M_{\{b,d,e\}} \cup M_{\{b,c,d,e\}} \cup M_{\{a,b,d\}}$ (figure 8).

All adjacencies between the classes of the partition of V are summarised in the table in figure 9. The others adjacencies in this table are obtained by symmetries.

We can see that $M_{\{b\}} \cup M_{\{a,b\}} \cup \{a\}$ is a module of G. Since G is prime w.r.t. the bi-join decomposition, every module of size at most $n-2$ is trivial, and thus $M_{\{b\}} = \emptyset$ and $M_{\{a,b\}} = \emptyset$. $M_{\{c\}} \cup M_{\{c,d\}} \cup \{d\}$, $M_{\{b,c\}} \cup M_{\{b,c,e\}} \cup \{e\}$, $M_{\{a,c,e\}} \cup M_{\{a,b,c,e\}} \cup \{b\}$ and $M_{\{b,d,e\}} \cup M_{\{b,c,d,e\}} \cup \{c\}$ are also modules of G, thus with the same argument, $M_{\{c\}} = \emptyset$, $M_{\{c,d\}} = \emptyset$, $M_{\{b,c\}} = \emptyset$, $M_{\{b,c,e\}} = \emptyset$, $M_{\{a,c,e\}} = \emptyset$, $M_{\{a,b,c,e\}} = \emptyset$, $M_{\{b,d,e\}} = \emptyset$ and $M_{\{b,c,d,e\}} = \emptyset$.

$M_{\{a,e\}}$, $M_{\{d,e\}}$, $M_{\{a,b,d\}}$ and $M_{\{a,c,d\}}$ are modules of G, then $|M_{\{a,e\}}| \leq 1$, $|M_{\{d,e\}}| \leq 1$, $|M_{\{a,b,d\}}| \leq 1$ and $|M_{\{a,c,d\}}| \leq 1$. Thus G is an induced subgraph of the graph G_0 given in figure 5. $\square$

3.2 Prime (gem,co-gem)-free graphs

Lemma 15. *Let G be a (gem,co-gem)-free graph which is prime w.r.t. the bi-join decomposition. Then either G is (C_5,gem,co-gem)-free, or G is an induced subgraph of G_1, G_2 or G_3.*

Proof. Suppose that G is not C_5-free. Let $C = \{a_0, a_1, a_2, a_3, a_4\}$ be a C_5 of G such that $\{a_i, a_j\} \in E$ if and only if $i - j \in \{-1, 1\}$ (all indices are taken modulo 5).

For all $i \in \{0,1,2,3,4\}$, let :

- $X_i = \{v \in V \setminus C : N(v) \cap C = \{a_{i-2}, a_{i+2}\}\}$,
- $Y_i = \{v \in V \setminus C : N(v) \cap C = \{a_{i-2}, a_i, a_{i+2}\}\}$,
- $Z_i = \{v \in V \setminus C : N(v) \cap C = \{a_{i-1}, a_{i+1}\}\}$,

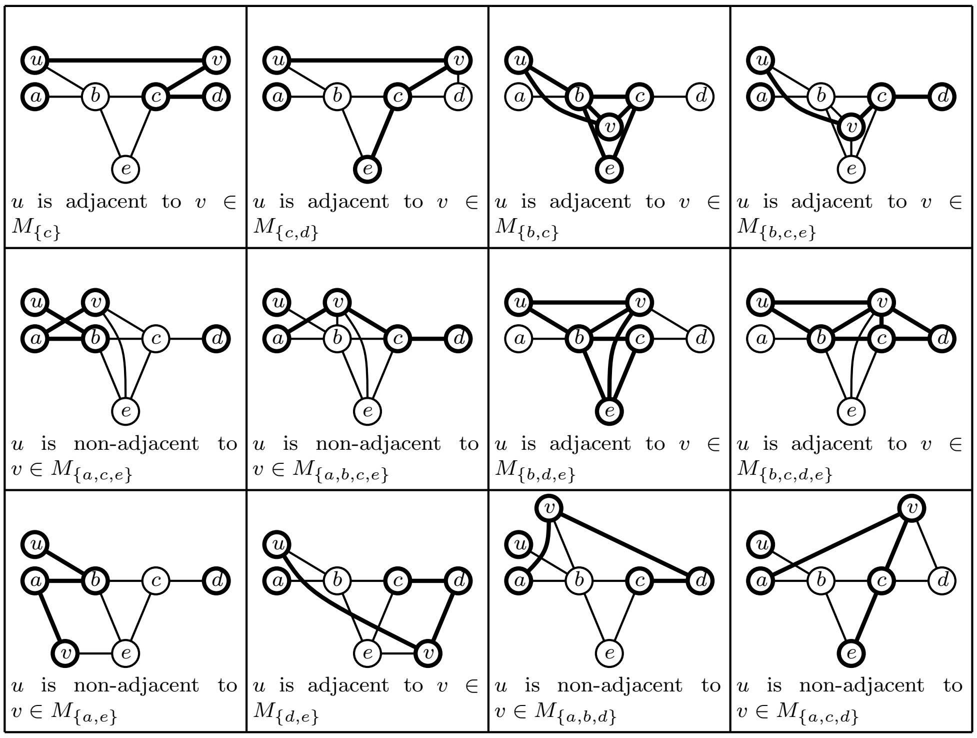

Figure 6: Forbidden adjacencies between $u \in M_{\{b\}}$ and $v \notin \{a\} \cup M_{\{b\}} \cup M_{\{a,b\}}$. Forbidden subgraphs are drawn in bold.

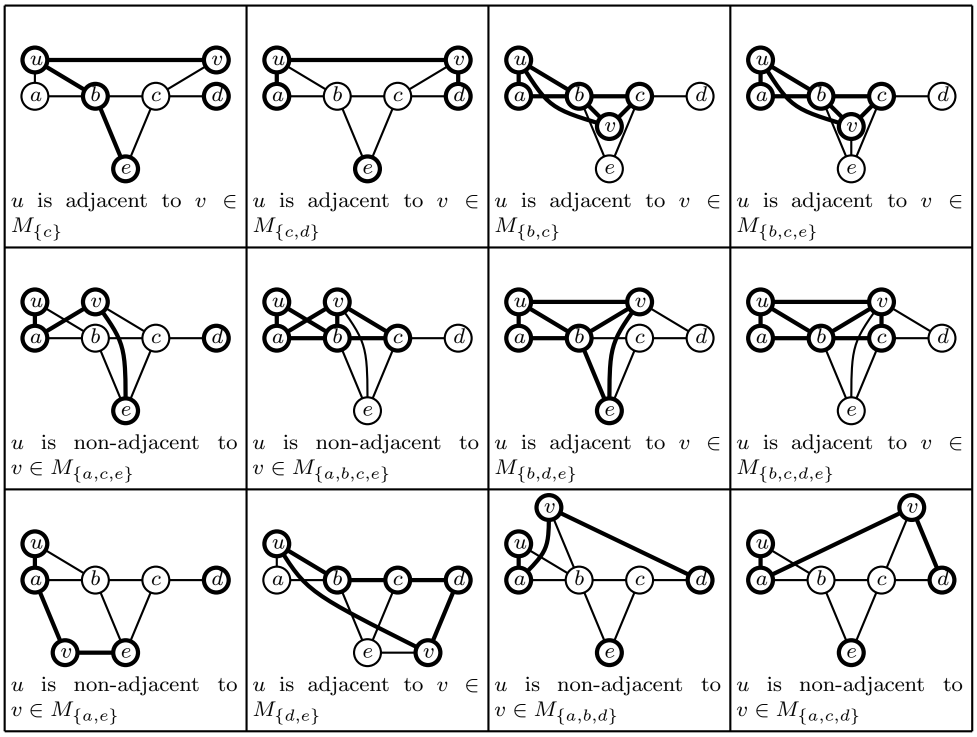

Figure 7: Forbidden adjacencies between $u \in M_{\{a,b\}}$ and $v \notin \{a\} \cup M_{\{b\}} \cup M_{\{a,b\}}$.

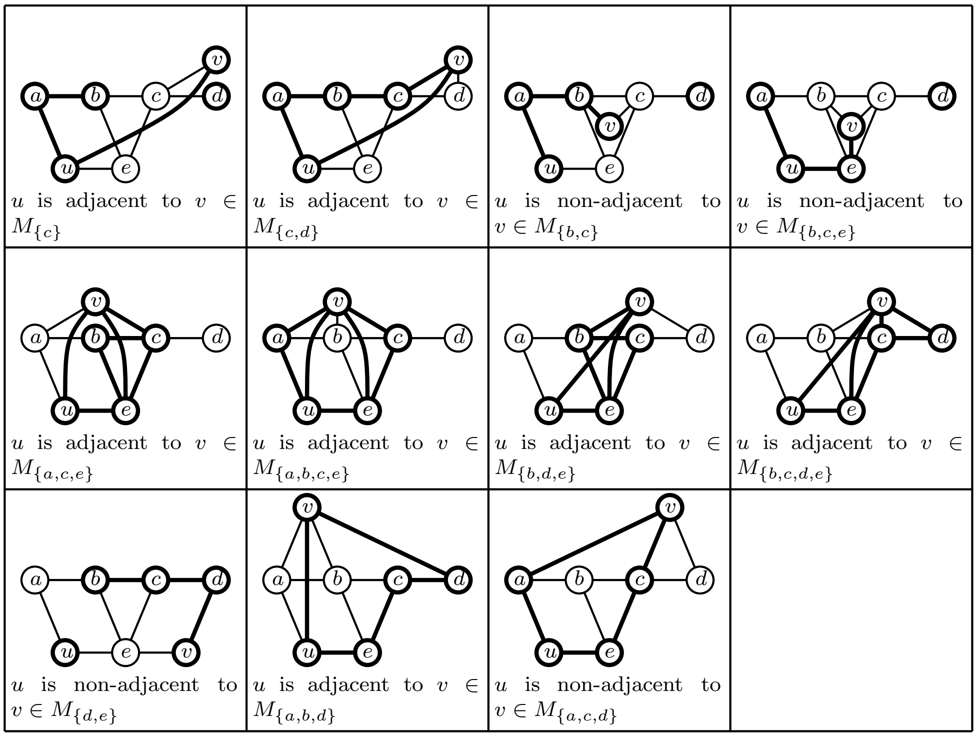

Figure 8: Forbidden adjacencies between $u \in M_{\{a,e\}}$ and $v \notin M_{\{b\}} \cup M_{\{a,b\}} \cup M_{\{a,e\}}$.

	$\{a\}$	$M_{\{b\}}$	$M_{\{a,b\}}$	$\{d\}$	$M_{\{c\}}$	$M_{\{c,d\}}$	$\{e\}$	$M_{\{b,c\}}$	$M_{\{b,c,e\}}$	$\{b\}$	$M_{\{a,c,e\}}$	$M_{\{a,b,c,e\}}$	$\{c\}$	$M_{\{b,d,e\}}$	$M_{\{b,c,d,e\}}$	$M_{\{a,e\}}$	$M_{\{d,e\}}$	$M_{\{a,b,d\}}$	$M_{\{a,c,d\}}$
$\{a\}$		0	1	0	0	0	0	0	0	1	1	1	0	0	0	1	0	1	1
$M_{\{b\}}$	0			0	0	0	0	0	0	1	1	1	0	0	0	1	0	1	1
$M_{\{a,b\}}$	1			0	0	0	0	0	0	1	1	1	0	0	0	1	0	1	1
$\{d\}$	0	0	0		0	1	0	0	0	0	0	0	1	1	1	0	1	1	1
$M_{\{c\}}$	0	0	0	0			0	0	0	0	0	0	1	1	1	0	1	1	1
$M_{\{c,d\}}$	0	0	0	1			0	0	0	0	0	0	1	1	1	0	1	1	1
$\{e\}$	0	0	0	0	0	0		0	1	1	1	1	1	1	1	1	1	0	0
$M_{\{b,c\}}$	0	0	0	0	0	0	0			1	1	1	1	1	1	1	1	0	0
$M_{\{b,c,e\}}$	0	0	0	0	0	0	1			1	1	1	1	1	1	1	1	0	0
$\{b\}$	1	1	1	0	0	0	1	1	1		0	1	1	1	1	0	0	1	0
$M_{\{a,c,e\}}$	1	1	1	0	0	0	1	1	1	0			1	1	1	0	0	1	0
$M_{\{a,b,c,e\}}$	1	1	1	0	0	0	1	1	1	1			1	1	1	0	0	1	0
$\{c\}$	0	0	0	1	1	1	1	1	1	1	1	1		0	1	0	0	0	1
$M_{\{b,d,e\}}$	0	0	0	1	1	1	1	1	1	1	1	1	0			0	0	0	1
$M_{\{b,c,d,e\}}$	0	0	0	1	1	1	1	1	1	1	1	1	1			0	0	0	1
$M_{\{a,e\}}$	1	1	1	0	0	0	1	1	1	0	0	0	0	0	0		1	0	1
$M_{\{d,e\}}$	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	1		1	0
$M_{\{a,b,d\}}$	1	1	1	1	1	1	0	0	0	1	1	1	0	0	0	0	1		0
$M_{\{a,c,d\}}$	1	1	1	1	1	1	0	0	0	0	0	0	1	1	1	1	0	0	

Figure 9: Adjacencies between the classes of the partition in a (C_5,gem,co-gem)-free graph having a bull (0 signify that the two sets are non-adjacent and 1 signify that they are adjacent).

- $T_i = \{v \in V \setminus C : N(v) \cap C = \{a_{i-1}, a_i, a_{i+1}\}\}$.

Since G is (gem,co-gem)-free, a vertex not in C cannot be adjacent to 0, 1, 4 or 5 vertices of C. Thus we have $V = \bigcup_{i=0}^{4}(X_i \cup Y_i \cup Z_i \cup T_i \cup \{a_i\})$.

Figure 10 gives some forbidden adjacencies between X_0, Y_0, Z_0, T_0 and X_1, Y_1, Z_1, T_1. Note that a vertex in X_0 cannot be adjacent nor non-adjacent to a vertex in X_1. Thus if X_0 is non-empty, then X_1 is empty. The table in figure 11 show the adjacencies between the previously defined sets, for $i \in \{0, 1, 2, 3, 4\}$. All this adjacencies are obtained by symmetries (add a constant to the indice, and/or take the complement).

Now we do some easy observations. Observations (t1)-(t4) come directly from adjacencies show in figure 11.

(t1) If $X_i \neq \emptyset$, then $X_{i+1} = X_{i-1} = \emptyset$.

(t2) If $Y_i \neq \emptyset$, then $Y_{i+2} = Y_{i-2} = \emptyset$.

(t3) If $X_i = Y_i = \emptyset$, then $Z_i \cup T_i \cup \{a_i\}$ is a module of G, and thus $Z_i = T_i = \emptyset$ since G is prime w.r.t. the bi-join decomposition.

(t4) If there is a i such that for all $j \neq i$, $X_j = Y_j = \emptyset$, then $X_i \cup Y_i \cup Z_i \cup T_i \cup \{a_i\}$ is a bi-join of G, and thus $X_i = Y_i = Z_i = T_i = \emptyset$, and G is a C_5.

(t5) If $X_i \neq \emptyset$ and $X_{i+2} \neq \emptyset$, then $Y_{i+1} = \emptyset$, otherwise G has a co-gem (a_{i+2}, u, v, a_i, w), where $u \in X_i, v \in X_{i+2}$ and $w \in Y_{i+1}$.

(t6) Si $Y_i \neq \emptyset$ and $Y_{i+1} \neq \emptyset$, then $X_{i-2} = \emptyset$, otherwise G has a gem (u, a_i, a_{i+1}, v, w), where $u \in Y_i, v \in Y_{i+1}$ and $w \in X_{i-2}$.

Suppose now that $X_i \neq \emptyset$, and let $u \in X_i$.

(x1) X_{i+2} is non-adjacent to Y_{i+2}, otherwise $(a_{i+3}, a_{i+2}, w, v, u)$ induce a gem, where $v \in X_{i+2}$ is adjacent to $w \in Y_{i+2}$.

(x2) X_{i+2} is non-adjacent to Z_{i+2}, otherwise $(a_{i+2}, a_{i+3}, w, v, u)$ induce a gem, where $v \in X_{i+2}$ is adjacent to $w \in Z_{i+2}$.

(x3) X_{i+2} is non-adjacent to T_{i+2}, otherwise $(v, u, a_{i+2}, a_{i+1}, w)$ induce a gem, where $v \in X_{i+2}$ is adjacent to $w \in T_{i+2}$.

(x4) Y_{i+2} is adjacent to Z_{i+2}, otherwise $(w, a_{i+3}, a_{i+2}, v, u)$ is a gem, where $v \in Y_{i+2}$ is non adjacent to $w \in Z_{i+2}$.

(x5) Y_{i+2} is adjacent to T_{i+2}, otherwise $(v, u, w, a_{i+1}, a_{i+2})$ is a gem, where $v \in Y_{i+2}$ is non adjacent to $w \in T_{i+2}$.

(x6) Y_{i+1} is adjacent to Z_{i+1} and T_{i+1}, otherwise (v, a_{i+4}, a_i, w, u) is a co-gem, where $v \in Y_{i+1}$ is non-adjacent to $w \in Z_{i+1} \cup T_{i+1}$.

Suppose that X_i is non-empty. By observations x2, x3, x4 and x5, $Z_{i+2} \cup T_{i+2} \cup \{a_{i+2}\}$ is a module, and thus $Z_{i+2} = T_{i+2} = \emptyset$. By observations x6 and t1 then $Z_{i+1} \cup T_{i+1} \cup \{a_{i+1}\}$ is a module, and thus $Z_{i+1} = T_{i+1} = \emptyset$. By observations x1, x2, x3, x4 and x5, X_{i+2} and Y_{i+2} are modules, and thus $|X_{i+2}| \leq 1$ and $|Y_{i+2}| \leq 1$. By observations x6 and t1 then X_{i+1} and Y_{i+1} are modules, and thus $|X_{i+1}| \leq 1$ and $|Y_{i+1}| \leq 1$. With the symmetries, for all $j \neq i$, $Z_j = T_j = \emptyset$, $|X_j| \leq 1$ and $|Y_j| \leq 1$.

With sames arguments on the complement, if $Y_i \neq \emptyset$, then for all $j \neq i$, $Z_j = T_j = \emptyset$, $|X_j| \leq 1$ and $|Y_j| \leq 1$.

If G is not a C_5, then by observation t4 there is i and j such that $i \neq j$, $X_i \cup Y_i \neq \emptyset$ and $X_j \cup Y_j \neq \emptyset$. Then by the previous observations, for all k, $Z_k = T_k = \emptyset$, $|X_k| \leq 1$ and $|Y_k| \leq 1$.

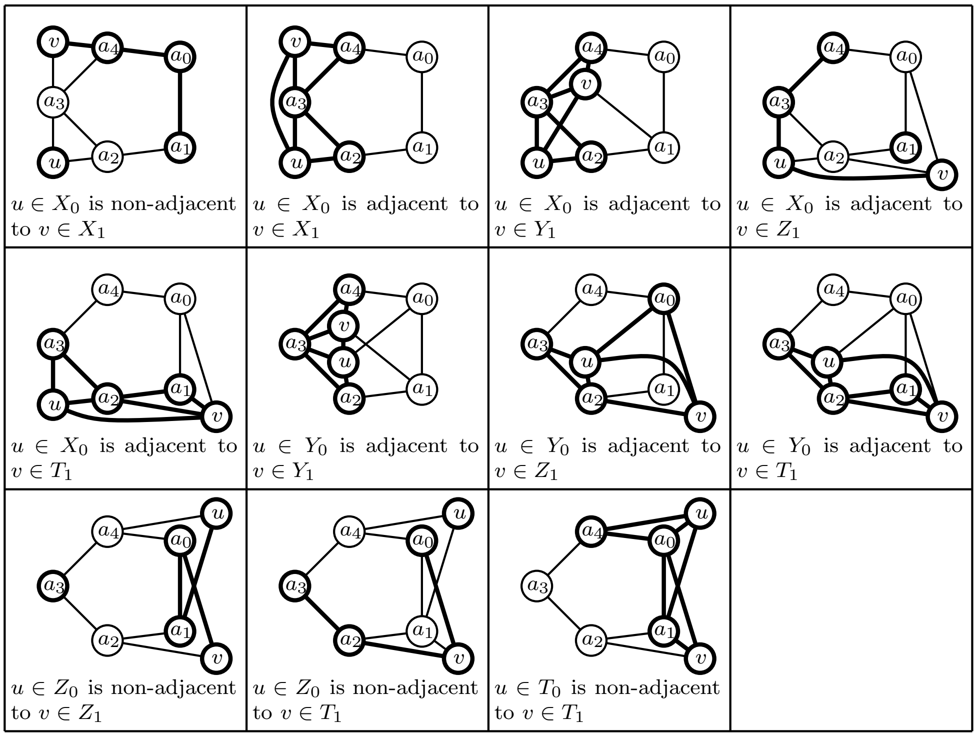

Figure 10: Forbidden adjacencies between X_0, Y_0, Z_0, T_0 and X_1, Y_1, Z_1, T_1.

	Z_{i+1}	T_{i+1}	X_{i+1}	Y_{i+1}	Z_{i+2}	T_{i+2}	X_{i+2}	Y_{i+2}	Z_{i+3}	T_{i+3}	X_{i+3}	Y_{i+3}	Z_{i+4}	T_{i+4}	X_{i+4}	Y_{i+4}
Z_i	1	1	0	0	0	0	1	1	0	0	1	1	1	1	0	0
T_i	1	1	0	0	0	0	1	1	0	0	1	1	1	1	0	0
X_i	0	0	0/1	0	1	1	1	1	1	1	1	1	0	0	0/1	0
Y_i	0	0	0	0	1	1	1	0/1	1	1	1	0/1	0	0	0	0

Figure 11: Adjacencies between the classes of the partition of V in a (gem,co-gem)-free graph having a C_5.

By observation t1, there is at most two indices i and j such that X_i and X_j are non-empty, and in this case we have $j = i + 2$ or $j = i - 2$. Similarly, by t2, there is at most two indices i and j such that Y_i and Y_j are non-empty, and in this case we have $j = i + 1$ or $j = i - 1$. Now we study all the different cases.

Case 1: for all i, $X_i = \emptyset$.

Case 1.1: for all j, $Y_j = \emptyset$. Then G is a C_5.

Case 1.2: $Y_j \neq \emptyset$ for an unique j. Then $Y_j \cup \{a_i\}$ is a non-trivial bi-join. Contradiction.

Case 1.3: $Y_j \neq \emptyset$ and $Y_{j+1} \neq \emptyset$. G is the subgraph of G_1 induced by $\{a, b, c, d, e, h, i\}$.

Case 2: There is an unique i such that $X_i \neq \emptyset$.

Case 2.1: $Y_j = \emptyset$ for all j. Then $X_i \cup \{a_i\}$ is a non-trivial bi-join. Contradiction.

Case 2.2: $Y_j \neq \emptyset$ for an unique j.

Case 2.2.1: $i = j$. Then $X_i \cup Y_i \cup \{a_i\}$ is a non-trivial bi-join. Contradiction.

Case 2.2.2: $i \in \{j + 1, j - 1\}$. Then G is the subgraph of G_1 induced by $\{a, b, c, d, e, f, h\}$.

Case 2.2.3: $i \in \{j + 2, j - 2\}$. Then G is the subgraph of G_1 induced by $\{a, b, c, d, e, f, i\}$.

Case 2.3: $Y_j \neq \emptyset$ and $Y_{j+1} \neq \emptyset$. By the observation t6, $i \neq j - 2$. If $i \in \{j, j+1\}$ then G is the graph G_3. Otherwise $i \in \{j-1, j+2\}$, and G is the subgraph of G_1 induced by $\{a, b, c, d, e, f, h, i\}$.

Case 3: There is an unique i such that $X_i \neq \emptyset$ and $X_{i+2} \neq \emptyset$.

Case 3.1: $Y_j = \emptyset$ for all j. Then G is the subgraph of G_1 induced by $\{a, b, c, d, e, f, g\}$.

Case 3.2: $Y_j \neq \emptyset$ for an unique j. By the observation t5, $j \neq i+1$. If $j \in \{i, i+2\}$ then G is the graph G_2. Otherwise $j \in \{i-1, i+3\}$, and G is the subgraph of G_1 induced by $\{a, b, c, d, e, f, g, h\}$.

Case 3.3: $Y_j \neq \emptyset$ and $Y_{j+1} \neq \emptyset$. By observations t5 and t6, $j = i + 3$. Then G is the graph G_1. □

4 Characterization of (gem,co-gem)-free graphs

We give in this section a characterization theorem of (gem,co-gem)-free graphs using the modular and the bi-join decompositions. Note that even if gem and co-gem are prime w.r.t. the bi-join decomposition, the bi-join decomposition can decompose a graph with a gem or a co-gem into (gem,co-gem)-free graphs. An example is given in figure 12.

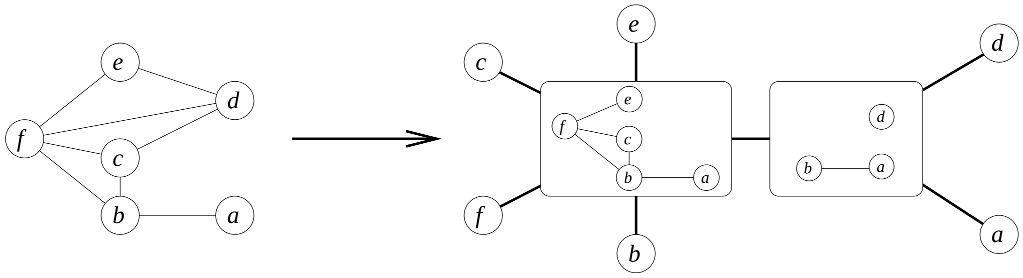

Figure 12: A graph with a gem (b, c, d, e, f) such that all characteristics graphs in the bi-join decomposition are (gem,co-gem)-free.

We need a more precise analysis in order to characterise and recognize (gem,co-gem)-free graphs. This is given by the following theorem.

Theorem 16. *Let G be a graph, let G^* be the characteristic graph of G (w.r.t. the modular decomposition), and let T be the bi-join decomposition tree of G^*. Then G is (gem,co-gem)-free if and only if the following conditions are fulfilled:*

- *Every non trivial module of G induces a cograph.*
- *For every prime node α in T:*
  - *α is adjacent to at most one internal node (*i.e. *a characteristic graph of α has at most one marker vertex), and*

– *every marker vertex in a characteristic graph of α is not a vertex of a P_4 of a bull (*i.e. *there is no bull (a,b,c,d,e) such that a, b, c or d is a marker).*

Moreover, if G^* is not prime w.r.t. the bi-join decomposition, then every characteristic graph of a prime node is a bull or a C_5.

The remainder of this section is devoted to the proof of theorem 16.

Lemma 17. *A graph G is (gem,co-gem)-free if and only if the following conditions are fulfilled:*

- *Every non trivial module induces a cograph.*
- *The characteristic graph (w.r.t. the modular decomposition) of G is (gem,co-gem)-free.*

Proof. Suppose that G is (gem,co-gem)-free. The characteristic graph of G is an induced subgraph of G, and thus is also (gem,co-gem)-free. Moreover if a non trivial module has a P_4 (a,b,c,d), then (a,b,c,d,e) is a gem or a co-gem, where e is a vertex outside of the module. Contradiction.

Suppose now that G has a gem or a co-gem. W.l.o.g. suppose that G has a gem (a,b,c,d,e) (otherwise take the complement). Note that a gem has only one non-trivial module, which is its P_4. If every non trivial module induce a cograph, then for every module $M \neq V$, we have $|M \cap \{a,b,c,d,e\}| \in \{0,1\}$. Thus $G[\{a,b,c,d,e\}]$ is isomorphic to a subgraph of G^*, and G^* has a gem. □

If G is (gem,co-gem)-free and the characteristic graph G^* of G is not prime w.r.t. the modular decomposition, then by lemma 17, G is a cograph. Thus we only have to see when G^* is prime w.r.t. the modular decomposition.

Lemma 18. *A prime graph (w.r.t. modular decomposition) G is (gem,co-gem)-free if and only if for every characteristic graph H of a node in the bi-join decomposition tree of G we have:*

1. *H is (gem,co-gem)-free,*
2. *H has no marker vertex which is a vertex of a P_4 of a bull, and*
3. *H has no induced C_5 having at least two marker vertex.*

Proof. Suppose that there is a characteristic graph H of the node α which does not respect one of the conditions. By definition H is an induced subgraph of G. If H has a gem or a co-gem, then this is also the case for G.

Suppose that H has a bull and v is a marker vertex on the P_4 of the bull. Suppose w.l.o.g. that the bull is (v,a,b,c,d) (otherwise take the complement of G). Let $\beta \in N_T(\alpha)$ such that $v \in L_{\alpha\to\beta}$. By definition $\{L_{\alpha\to\beta}, V \setminus L_{\alpha\to\beta}\}$ is a bi-join. Since v is a marker, $|L_{\alpha\to\beta}| > 1$, and since G is prime w.r.t. the modular decomposition, $L_{\alpha\to\beta}$ is not a module. Thus there is a $v' \in L_{\alpha\to\beta}$ such that $N_G[v'] \setminus L_{\alpha\to\beta} = (V \setminus N_G[v]) \setminus L_{\alpha\to\beta}$, and so $N_G[v'] \cap \{a,b,c,d\} = \{b,c,d\}$. Then (a,d,v',c,b) induce a gem in G.

Finally suppose that H has a C_5 with two markers v and w. Suppose w.l.o.g. than v and w are adjacent (otherwise take the complement). Let (v,w,a,b,c) be this C_5. With the same argument, there is a v' such that $N_G[v'] \cap \{w,a,b,c\} = \{a,b\}$. Also with the same argument on w, there is a w' such that $N_G[w'] \cap \{a,b,c,v'\} = \{b,c,v'\}$. Thus (a,v',w',c,b) induce a gem.

We show now the other direction. Suppose that G has a gem or a co-gem. W.l.o.g. suppose that $W \subseteq V$ induce a gem (otherwise take the complement). Since a gem is prime w.r.t. bi-join decomposition, there is an internal node α in the bi-join decomposition tree T of G such that for all $\beta \in N_T(\alpha)$, $|W \cap L_{\alpha\to\beta}| \leq 1$. Let $W' = \{\{\alpha,\beta\} : \beta \in N_T(\alpha)$ and $L_{\alpha\to\beta} \cap W \neq \emptyset\}$. $C(\alpha)[W']$ is a Seidel switch of a gem (theorem 11), and thus is a gem, a co-gem, a bull or a C_5 by corollary 10.

If there is zero marker in W', then $C(\alpha)[W']$ is a gem and the condition 1 is not fulfilled. If there is at least two markers in W', then the conditions 2 and 3 are not fulfilled. Suppose now that there is exactly one marker w' in W'. If $C(\alpha)[W']$ is a C_5, then $G[W]$ is either isomorphic to

$C(\alpha)[W']$ or to $\overline{C(\alpha)[W']}^{\{w'\}}$ and thus W induce a C_5 or a bull, contradiction. If $C(\alpha)[W']$ is a bull and w' is not on the P_4 of the bull, then W induce a bull, contradiction. Thus $C(\alpha)[W']$ is a bull and w' is on the P_4 of the bull. Condition 2 is not fulfilled. □

Let G be a (gem,co-gem)-free graph, and let α be a prime node in a bi-join decomposition. Let $H = C(\alpha)$ be a characteristic graph of α in the decomposition. Then either:

- H is a C_5, and by lemma 18, H has at most one marker vertex,
- H is a bull, and by lemma 18, H has at most one marker vertex which is not on the P_4 of the bull,
- H is the subgraph of G_0 induced by W such that $\{a,b,c,d,e\} \subsetneq W$. Then by lemma 18, a, b, c and d cannot be a marker. If $f \in W$, then f and e cannot be a marker, since there are on the P_4 of the bull (f,e,c,d,b). By symmetries, g, h and i cannot be a marker. So H has no marker vertex.
- H is the subgraph of G_1 induced by $\{a,b,c,d,e,h,i\}$. The vertices a, b, c and e are on the P_4 of the bull (e,a,b,c,h), d is on the P_4 of the bull (b,a,e,d,i), and h and i are on the P_4 of bull (d,h,a,i,b). So H has no marker vertex.
- H is a subgraph of G_1 induced by W such that $\{a,b,c,d,e,g\} \subseteq W$ and either h or i are in W. Vertices a, c, d and e are on the P_4 of the bull (a,e,d,c,g), b, g and h are on the P_4 of the bull (b,a,c,g,i) or (g,h,b,c,a), and i is on the P_4 of the bull (c,i,e,g,a). In all cases, all the vertices are in a P_4 of a bull.
- H is the subgraph of G_0 induced by $\{a,b,c,d,e,f,g\}$. H is isomorph to the complement of G_0 induced by $\{a,b,c,d,e,h,i\}$. Thus it has no marker.
- H is the subgraph of G_0 induced by $\{a,b,c,d,e,f,g,h\}$. It is the complement case of $\{a,b,c,d,e,f,h,i\}$. It has no marker vertex.
- H is the graph G_2. All the vertices are on a P_4 of the bull (a,b,c,d,f), (f,g,e,a,h) or (a,b,h,g,c). So H has no marker vertex.
- H is the graph G_3. All the vertices are on a P_4 of the bull (a,b,c,d,f), (e,h,c,g,f) or (f,c,d,e,g). So H has no marker vertex.

In all cases, H has at most one marker vertex, and thus α is adjacent to at most one internal node. Moreover if H is not a bull or a C_5, then H has no marker. Thus α is adjacent only to leaves in T, that is G is prime w.r.t. the bi-join decomposition. Theorem 16 follows.

We can compute the modular decomposition in linear time [16], the bi-join decomposition tree in linear time [18, 19], and we can check in constant time if every prime node has the desired property since every prime graph has at most 9 vertices. Thus we have:

Corollary 19. *Recognition of (gem,co-gem)-free graphs can be done in linear time.*

5 NLC-width and clique-width of (gem,co-gem)-free graphs

It is already known that the clique-width of (gem,co-gem)-free graphs is bounded by 16 [3]. We show in this section that their NLC-width is bounded by 4, and thus their clique-width is bounded by 8. We also show that a NLC 4-expression and a clique-width 8-expression can be computed in linear time.

Clique-width is a graph parameter introduced by Courcelle *et al.* [7], and the *NLC-width* is a graph parameter introduced by Wanke [22]. A motivation of researches about clique-width and NLC-width is that a lot of NP-hard problems are polynomial time solvable on graphs of bounded width.

A *labeled graph* is graph in which every vertex has a label into $\{1, 2, \ldots\}$. It is denoted by (V, E, l), or by (G, l), where $G = (V, E)$, V is the set of vertex, E the set of edges, and $l : V \to \{1, 2, \ldots\}$ the labeling function. If G is a labeled graph, $l(G)$ denote the labeling function of G. A labeled graph is a *k-labeled graph* (where $k \in \{1, 2, \ldots\}$) if every label are into $\{1, 2, \ldots, k\}$.

NLC-width Let define the following operations:

1. For $i \in \{1, \ldots, k\}$, $(v)_i$ is the graph with one vertex v labeled i.
2. Let $G = (V, E, l)$ be a k-labeled graph. For $R : \{1, \ldots, k\} \to \{1, \ldots, k\}$, $\circ_R(G)$ is the k labeled graph (V, E, l') where $l'(v) = R(l(v))$ for all $v \in V$.
3. Let $G_1 = (V_1, E_1, l_1)$ and $G_2 = (V_2, E_2, l_2)$ be two disjoint k-labeled graphs, and $S \subseteq \{1, \ldots, k\}^2$. Then $G_2 \times_S G_2$ is the k-labeled graph $(V_1 \cup V_2, E, l)$ where
$$E = E_1 \cup E_2 \cup \{\{u, v\} \in V_1 \times V_2 : (l_1(u), l_2(v)) \in S\}$$
and for all $v \in V_1 \cup V_2$,
$$l(v) = \begin{cases} l_1(v) \text{ if } v \in V_1, \\ l_2(v) \text{ if } v \in V_2. \end{cases}$$

A *k-expression* is an expression built with $(v)_i$, $\times$ and $\circ$ with labels into $\{1, \ldots, k\}$, and $G(t)$ is the labeled graph defined by the expression t. The *NLC-width* of a labeled graph G is the smallest integer k such that there is a k-expression with $G = G(t)$. It is denoted by $\mathrm{nlc}(G)$. The NLC-width of an unlabeled graph G is the minimum of the NLC-width over all labellings of vertices of G, and a k-expression for G is a k-expression for a labeling of G. The NLC-width of a graph class $\mathcal{G}$ is $\max\{\mathrm{nlc}(G) : G \in \mathcal{G}\}$.

It is easy to see that graphs of NLC-width 1 are exactly the cographs. Moreover, we have the following relation with the modular decomposition.

Theorem 20 ([15]). *Let G be a graph. Then $\mathrm{nlc}(G) = \max\{\mathrm{nlc}(H) : H$ is a characteristic graph in the modular decomposition of G $\}$.*

The well known clique-width [7] is defined using a similar graph algebra. The following theorem gives a strong relation between clique-width and NLC-width. In particular, a graph class has bounded clique-width if and only if it has bounded NLC-width.

Theorem 21 ([15]). *For every graph G,* $\mathrm{nlc}(G) \leq \mathrm{cwd}(G) \leq 2 \times \mathrm{nlc}(G)$.

NLC-width of (gem,co-gem)-free graphs

Lemma 22. *Graphs in the figure 5 have NLC-width at most* 4.

Proof. Let G be a graph in the figure 5, and v be a vertex of degree 4 in G. Let $\{X, Y\}$ be a partition of $V \setminus \{v\}$ such that $|X| = 4$ and $|X \cap N_G(v)| = 2$. We have $3 \leq |Y| \leq 4$. We assign a different label $l(w)$ from $\{1, 2, 3, 4\}$ to every $w \in X$, and we assign a different label $l(w)$ from $\{1, 2, 3, 4\}$ to every $w \in Y$, such that $l(w) \in \{1, 2\}$ if and only if $w \in N_G(v)$. We construct a 4-expression t_X for $G[X]$, and a 4-expression t_Y for $G[X]$ (immediate, since every vertex has a different label). Then $t' = t_X \times_S t_Y$ where $S = \{(l(w), l(w')) : w \in X \text{ and } w' \in Y\}$ is a 4-expression for $G - v$. Finally $t = (v)_1 \times_{\{(1,1),(1,2)\}} t'$ is a 4-expression for G. ☐

Theorem 23. *(Gem,co-gem)-free graphs have NLC-width at most* 4. *Moreover a* 4*-expression of a (gem,co-gem)-free can be computed in linear time.*

Proof. Suppose that G is a (gem,co-gem)-free graph. By theorem 20, we suppose w.l.o.g. than G is prime w.r.t. the modular decomposition.

If G is prime w.r.t. the bi-join decomposition, then by lemma 22 the NLC-width of G is at most 4. Otherwise, by lemma 16, the prime nodes in the bi-join decomposition tree T of G have degree 5. Take a vertex v of minimum degree in G. By theorem 9, the prime nodes in the modular decomposition tree T' of $\overline{G}^v$ has 4 children. Thus every characteristic graph of a prime node in T' has 4 vertices. Since the P_4 is the only graph with at most 4 vertices which is not a cograph, then every characteristic graph of a prime node is a P_4. By theorem 20, the NLC-width of $\overline{G}^v$ is at most 2, since the P_4 has NLC-width 2.

Now we use a 2-expression t for $\overline{G}^v$ to construct a 4-expression for G. Let t' be the expression t in where:

- we replace every operator $(w)_i$, for $w \in N_G(v)$, by $(w)_{i+2}$,
- we replace every operator $\circ_R$ by $\circ_{R'}$, where $R'(i) = R(i)$ if $i \in \{1,2\}$, and $R'(i) = 2+R(i-2)$ if $i \in \{3,4\}$, and
- we replace every operator $\times_S$ by $\times_{S'}$ where $S' = S \cup \{(a+2, b+2) : (a,b) \in S\} \cup \{(a+2, b) : (a,b) \in \{1,2\}^2 \setminus S\} \cup \{(a, b+2) : (a,b) \in \{1,2\}^2 \setminus S\}$.

Let $t'' = (v)_1 \times_{\{(1,3),(1,4)\}} t'$. It is not hard to see that t' is a 4-expression for $\overline{\overline{G}^v}^{N_G(v)} = G - v$, and that t'' is a 4-expression for G.

We can compute the modular decomposition of G in linear time. Since v is of minimum degree $\overline{G}^v$ is linear in the size of G, and can be computed linear time. The modular decomposition of $\overline{G}^v$ can be computed in linear time. The NLC 2-expression for a characteristic graph of a prime node in the modular decomposition can be founded in constant time, since this graph has at most 4 vertices. Thus t can be constructed in linear time, and we can construct t'' from t in linear time.

□

It is easy to see that a clique-width $2k$-expression for G can be constructed in linear time from a NLC k-expression for G (see [15]). Thus we have:

Corollary 24. *(Gem,co-gem)-free graphs have clique-width at most* 8*. Moreover a clique-width* 8*-expression of a (gem,co-geom)-free graph can be computed in linear time.*

6 Graph isomorphism on (gem,co-gem)-free graphs

We present in this section an algorithm to test isomorphism between two (gem,co-gem)-free graphs. This algorithm also works for every graph class for which every prime node in the bi-join decomposition tree has bounded degree. Note that, as far we know, GRAPH ISOMORPHISM is not known to be polynomial on graphs of bounded clique-width.

To test if two (gem,co-gem)-free graphs are isomorphic, it is not sufficient to check if their bi-join decomposition are isomorphic, since a graph can has several different bi-join decomposition. To check isomorphism, we check if there is a $v \in V(G_1)$ and $w \in V(G_2)$ such that $\overline{G_1}^v$ and $\overline{G_2}^w$ are isomorphic, and such that the neighborhoods of v and w match.

Let G_1 be a (gem,co-gem)-free graph and let $v \in V(G_1)$. By theorem 12 every prime node in the bi-join decomposition of G_1 and G_2 has degree at most 9, and by theorem 9, every prime node in the modular decomposition of $\overline{G_1}^{\{v\}}$ has at most 8 children.

We give in a first time a linear time algorithm to test isomorphism when every prime graph in the modular decomposition of the two graphs has at most a bounded number of children.

6.1 Graph isomorphism using the modular decomposition

The algorithm presented in this section (algorithm 1) is a modification of the linear time algorithm to test isomorphism of trees [1]. It checks if two labeled graphs are isomorphic, when the size of the characteristic graphs of prime nodes in the modular decomposition tree is bounded by a $k \in \mathbb{N}$. Thus every characteristic graph is either a complete graph, an edgeless graph, or is member of a finite set $\mathcal{G}$: the set of prime graphs with at most k vertices.

The modular decomposition is unique, thus two graphs G and H are isomorphic if and only if their modular decomposition tree are isomorphic. We deal in this section with *labeled* graphs, *i.e.* graphs with a label function $V \to \mathbb{N}$. To test if two labeled graphs are isomorphic, we check if there is an isomorphism between the tree such that colors of leaves match.

It assign a *long label* to every node in the trees, level by level, from the leaves up to the root. Two nodes α and β in the same level has the same label if and only if $G[V_\alpha] \simeq H[V_\beta]$, where V_α is the set of leaves in the subtree rooted at α. After that, it assigns a *short label* to every node, such that two nodes have the same short label if and only if they have the same long label.

Two labelled graphs (G, l) and (G', l') are *isomorphic* if there is a bijection π from $V(G)$ to $V(G')$ such that for every $u, v \in V(G)$, $\{u, v\} \in E(G) \Leftrightarrow \{\pi(u), \pi(v)\} \in E(G')$, and for every $v \in V(G)$, $l(v) = l'(\pi(v))$.

Let G be the set of prime graphs with at most k vertices. For every $G \in \mathcal{G}$, we assign a different integer $c(G) \in \{1, \dots |\mathcal{G}|\}$, and a graph $G^0 = (V^0, E^0)$ such that $G \simeq G^0$ and such that V^0 is totally ordered.

A *multiset*, denoted $\langle e_1, e_2, \dots \rangle$, is a set in where elements can appear more that once. We explain now how to compute the long labels (LL). Let (G, l) be a labeled graph, where G is a characteristic graph in the modular decomposition of the original graph. If $V(G) = \{v\}$ then $\mathrm{LL}(G, l) = (-1, l(v))$. If G has at least 2 vertices and is edgeless, then $\mathrm{LL}(G, l) = (0, 0, S)$, where S is the multiset $\langle l(v) : v \in V(G)\rangle$. If G has at least 2 vertices and is complete, then $\mathrm{LL}(G, l) = (0, 1, S)$, where S is the multiset $\langle l(v) : v \in V(G)\rangle$. Otherwise $G \in \mathcal{G}$. Let $V(G^0) = \{v_1, \dots, v_n\}$ with $v_1 < v_2 < \dots < v_n$. Let $\mathrm{LL}(G, l) = (1, c(G), \mathrm{cc}(G, l))$, where $\mathrm{cc}(G, l) = \min\{cc_\pi(G, l) :$ π is a bijection from V^0 to V such that $G^0 \simeq_\pi G\}$ and $\mathrm{cc}_\pi(G, l) = (l(\pi(v_1)), \dots, l(\pi(v_n)))$.

Input: Two labeled graphs (G_1, l_1), (G_2, l_2) such that every prime node in modular decompositions of G_1 and G_2 has at most k children (k is fixed)
Output: Yes if $(G_1, l_1) \simeq (G_2, l_2)$, otherwise No

```
Compute the modular decomposition D_1 of G_1
Compute the modular decomposition D_2 of G_2
if there is a prime node in D_1 or D_2 with at least k+1 children then Fail
if the height of D_1 and D_2 differ then return No
Assign label (0, l_1(α)) to all leaves α of level 0 in D_1
Assign label (0, l_2(α)) to all leaves α of level 0 in D_2
foreach level i from 1 to the height of D_1 do
    foreach node in the level i do
        Compute LL(α)
    Let S_1 be the set of long labels of nodes of level i in D_1
    Let S_2 be the set of long labels of nodes of level i in D_2
    Bucket sort S_1 and S_2
    if S_1 ≠ S_2 then return No
    for k from 1 to the size of S_1 do
        Assign label k to nodes of label s_k in D_1 and D_2, where s_k is the k-th element in S_1
if the roots of D_1 and D_2 have the same label then return Yes else return No
```

Algorithm 1: Graph isomorphism using the modular decomposition.

Theorem 25. *Algorithm 1 is correct, and can be implemented to run in linear time.*

Proof. The correctness follow from the observation than $\mathrm{LL}(G,l) = \mathrm{LL}(G',l')$ if and only if $(G,l) \simeq (G',l')$. This observation is immediate if G and G' are complete or edgeless. Otherwise, if $\mathrm{LL}(G,l) = \mathrm{LL}(G',l')$, then $G \simeq G'$, and there is π and π' such that $G' \simeq_{\pi'} G^0 \simeq_{\pi^{-1}} G$, and for all $v \in V(G^0)$, $l(\pi(v)) = l'(\pi'(v))$. Thus $(G,l) \simeq_{\pi' \circ \pi^{-1}} (G',l')$. In the other hand, if $(G,l) \simeq (G',l')$, then sets $\mathcal{S} = \{(l(\pi(v_1)), \ldots, l(\pi(v_n))) : \pi \text{ is a bijection from } V^0 \text{ to } V(G) \text{ such that } G^0 \simeq_\pi G\}$ and $\mathcal{S}' = \{(l(\pi(v_1)), \ldots, l(\pi(v_n))) : \pi \text{ is a bijection from } V^0 \text{ to } V(G') \text{ such that } G^0 \simeq_\pi G'\}$ are equal, thus $\mathrm{cc}(G,l) = \min \mathcal{S}$ and $\mathrm{cc}(G',l') = \min \mathcal{S}'$ are equal.

The bucket sort line 12 can be implemented in time $O(|L_i| + |L_{i-1}|)$ (where L_i is the set of nodes in level i), using a secondary structure to skip empty buckets as explained in [1]. Thus algorithm 1 can be implemented to run in linear time. □

6.2 Graph isomorphism using bi-join decomposition

Lemma 26. *Let G_1 and G_2 be two graphs, and let $v_1 \in V(G_1)$. G_1 and G_2 are isomorphic if and only if there is a $v_2 \in V(G_2)$ such that $(\overline{G_1}^{\{v\}}, l_1)$ and $(\overline{G_2}^{\{v\}}, l_2)$ are isomorphic, where $l_i : V(G_i) \to \{0,1\}$ such that $l_i(u) = 0 \Leftrightarrow u \in N_{G_i}(v_i)$. (In this case, v_1 and v_2 have the same degree.)*

Proof. Suppose that G and H are isomorphic, and let π such that $G \simeq_\pi H$. Then $(\overline{G_1}^{\{v\}}, l_1) \simeq_\pi (\overline{G_2}^{\{v\}}, l_2)$. In the other hand, if $(\overline{G_1}^{\{v\}}, l_1) \simeq_{\pi'} (\overline{G_2}^{\{v\}}, l_2)$, then $G \simeq_\pi H$, where $\pi(v_1) = v_2$ and $\forall u \in V(G_1)$, $\pi(u) = \pi'(u)$. □

Theorem 27. *Algorithm 2 is correct, and can be implemented to run in $O(nm)$ time.*

Proof. The correctness of algorithm 2 follows from previous lemma. Its time complexity is $O(nm)$ since if v is of minimum degree, $\overline{G}^{\{v\}}$ can be computed in linear time. □

Input: Two graphs G_1 and G_2 such that every prime node in the bi-join decompositions of G_1 and G_2 has degree at most k (k is fixed)
Output: Yes if $G_1 \simeq G_2$, otherwise No

Let v_1 be a vertex of minimum degree in G_1
Compute $\overline{G_1}^{\{v\}}$
Compute $l_1 : V \to \{0,1\}$ such that $l_1(u) = 0 \Leftrightarrow u \in N_{G_1}(v_1)$
foreach *vertex v_2 of minimum degree of G_2* **do**
  Compute $\overline{G_2}^{\{v\}}$
  Compute $l_2 : V \to \{0,1\}$ such that $l_2(u) = 0 \Leftrightarrow u \in N_{G_2}(v_2)$
  if $(\overline{G_1}^{\{v\}}, l_1) \simeq (\overline{G_2}^{\{v\}}, l_2)$ *(algorithm 1)* **then return** *Yes*
return *No*

Algorithm 2: Graph isomorphism using the bi-join decomposition.

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