Dejean words with frequency constraint

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Abstract

We prove all cases of Ochem's conjecture on Dejean words involving letter frequency. In all cases, our construction for the Pansiot code is a morphic word. Moreover, for alphabets with at least 9 letters, this morphic word is of the form $h(w_{TM})$, where w_{TM} is the Thue-Morse word, and h is a uniform morphism.

1 Introduction

A *repetition* in a word w is a pair of words (p, e) where p non empty, e is a prefix of pe, and pe is a factor of w. The *exponent* of the repetition is $\frac{|pe|}{|p|}$. A word is said *x-free* (resp. x^+-free) if it does not contain a repetition of exponent y with $y \geq x$ (resp. $y > x$). For an integer $k \geq 2$, the *repetition threshold* for k letters, denoted by $\mathrm{RT}(k)$, is the infimum over the set of x such that there exists an infinite x-free word over a k-letter alphabet, or equivalently the smallest x such that there exists an infinite x^+-free word over a k-letter alphabet. Dejean [9] conjectured the following:

$$\mathrm{RT}(k) = \begin{cases} \frac{7}{4} & \text{if } k = 3 \\ \frac{7}{5} & \text{if } k = 4 \\ \frac{k}{k-1} & \text{otherwise.} \end{cases}$$

Dejean's conjecture is now completely proved, after the work of several authors. The case $k = 2$ follows directly from the fact that the Thue-Morse word is overlap-free. Dejean proved the case $k = 3$, and Pansiot introduced a binary coding to prove the case $k = 4$ [14]. This code was used by the other authors to prove all remaining cases [11, 10, 4, 6, 7, 15, 8].

Ochem introduced a stronger version of the conjecture involving letter frequencies. The *frequency* of a letter x in an infinite word w is $\lim_{n\to\infty} \frac{|w[1;n]|_x}{n}$, if the limit exists.

Conjecture 1 ([12]).

1. *For every $k \geq 5$, there exists an infinite $\left(\frac{k}{k-1}^+\right)$-free word over a k-letter alphabet with letter frequency $\frac{1}{k+1}$.*
2. *For every $k \geq 6$, there exists an infinite $\left(\frac{k}{k-1}^+\right)$-free word over a k-letter alphabet with letter frequency $\frac{1}{k-1}$.*

Based on computer checks, Rao gave a stronger version of Ochem's conjecture for $k \geq 9$, using the Thue-Morse word w_{TM}.

Conjecture 2 ([15]). *For every $k \geq 9$, there is a morphism h such that $h(w_{TM})$ is the Pansiot code of a Dejean word on k letters with a letter of frequency $\frac{1}{k-1}$ (resp. $\frac{1}{k+1}$).*

This conjecture is already proved for $9 \leq k \leq 38$ in [15]. We present in Section 3 a construction which prove Rao's conjecture for every $k \geq 24$. The section 4 is devoted to the proof of result of Section 3 when $k \geq 32$. This give an alternative proof of Dejean's conjecture for large alphabets. The cases when $k < 9$ are treated by specific constructions in Section 5. This prove all cases of Ochem's conjecture.

2 Preliminaries

We denote by Σ_k the set $\{0, 1, \dots, k-1\}$ for $k \geq 2$, and by $B = \Sigma_2 = \{0, 1\}$. Let w be a word of Σ_k. If w is finite, $|w|$ denote its size, otherwise $|w| = \infty$. For $i \geq 0$, $w[i]$ is the $(i+1)$-th letter of w and $w[a; b] = w[a] \dots w[b]$. Note that unusually, indices start at 0 instead of 1. Two words $u, v \in \Sigma^*$ are *isomorphic* if $|u| = |v|$ and there is a permutation $\sigma : \Sigma \to \Sigma$ such that $u[i] = \sigma(v[i])$ for every $i \in \{0, |u| - 1\}$.

An *occurrence* of a factor u in w is an integer $i \in \mathbb{N}$ such that $w[i; i+|u|-1] = u$. The distance between two occurrences i and j is $|i - j|$. For example, the distance between the two occurrences of a in aba is 2.

The Thue-Morse w_{TM} is the fixed point with first letter 0 of the morphism $\nu : B^* \to B^*$ such that $\nu(0) = 01$ and $\nu(1) = 10$.

A *repetition* in a word w is a pair of words (p, e) where p non empty, e is a prefix of pe, and pe is a factor of w. The *period* of the repetition is $|p|$, its *excess* is $|e|$ and its *exponent* is $\frac{|pe|}{|p|}$. By abuse of notation, we identify sometimes the repetition (p, e) with the factor pe of w.

From now on, we fix an size alphabet $k \geq 2$. A repetition (p, e) in w over the alphabet Σ_k is a *short repetition* if $|e| < k - 1$, otherwise it is a *kernel repetition*. A repetition is *forbidden* if its exponent is greater than $\mathrm{RT}(k)$.

Let $\mathbb{S}_k$ be the group of permutations of Σ_k, and $\mathrm{Id}_k \in \mathbb{S}_k$ be the identity permutation. The *invariants* of a permutation σ are the $i \in \{0, \dots, k\}$ such that $\sigma(i) = i$. The set of invariants of σ is denoted $I(\sigma)$. A Ψ-*kernel repetition* in w (where $\Psi : \Sigma^* \to \mathbb{S}_k$ is a morphism) is a pair (p, q) such that (p, q) is a repetition in w and $\Psi(p) = \mathrm{Id}_k$.

2.1 Pansiot code

Pansiot [14] noticed that if a word on the alphabet Σ_k is $\frac{k-1}{k-2}$-free, then it can be encoded by a binary word. Let $k \geq 3$ and let w be a (possibly infinite) $\frac{k-1}{k-2}$-free word over the alphabet Σ_k, of length at least $k-1$. Then every factor of length $k-1$ consists of $k-1$ different letters. The *Pansiot code* of w is the binary word $P_k(w)$ such that for all $i \in \{1, \dots, |w| - k + 1\}$ (for all $i \geq 1$ if w is infinite):

$$P_k(w)[i] = \begin{cases} 0 & \text{if } w[i+k-1] = w[i] \\ 1 & \text{if } w[i+k-1] \notin \{w[i], \dots, w[i+k-2]\}. \end{cases}$$

Note that w is uniquely defined by $P_k(w)$ and $w[1;k-1]$. One can define an inverse operation: for a (possibly infinite) binary word w, $M_k(w)$ is the word on the alphabet Σ_k such that for all $i \in \{1, \ldots, |w|+k-1\}$ (for all $i \geq 1$ if w is infinite):

$$M_k(w)[i] = \begin{cases} i & \text{if } i < k, \\ M_k(w)[i-k+1] & \text{if } i \geq k \text{ and } w[i-k+1] = 0, \\ \alpha & \text{otherwise} \end{cases}$$

where $\{\alpha\} = \Sigma_k \setminus \{M_k(w)[i-k+1], \ldots, M_k(w)[i-1]\}$. Note that if $w[i] = i$ for all $i < k$, then $M_k(P_k(w)) = w$.

Moulin-Ollagnier showed that Pansiot's coding can also be viewed by the way of an action on the symmetric group $\mathbb{S}_k$ [11]. Let φ be the morphism between the free monoid B^* and $\mathbb{S}_k$ such that $\varphi(0) = \sigma_0$ and $\varphi(1) = \sigma_1$, where:

$$\sigma_0 = \begin{bmatrix} 1 & 2 & \cdots & k-2 & k-1 & k \\ 2 & 3 & \cdots & k-1 & 1 & k \end{bmatrix} \quad \text{and} \quad \sigma_1 = \begin{bmatrix} 1 & 2 & \cdots & k-2 & k-1 & k \\ 2 & 3 & \cdots & k-1 & k & 1 \end{bmatrix}.$$

One can easily show that for all $i \geq 0$ and $1 \leq j \leq k-1$, $M_k(w)[i+j] = \varphi(w[1;i])(j)$.

Moulin Ollagnier [11] showed that:

Proposition 1 ([11])**.** *Let $k \geq 3$ and w be a binary word. Then $M_k(w)$ has a kernel repetition (p,q) if and only if w has a φ-kernel repetition (p',q') with $|p'| = |p|$ and $p'q' = P(pq)$. (Note that $|q'| = |q| - k + 1$.)*

Remark 1. *If the Pansiot code of a Dejean word w has a 0 at every position $i \pmod{k-1}$ for an $i \in \{0, \ldots k-1\}$, then w has the same letter at position $i \pmod{k-1}$, thus w has a letter with frequency $\frac{1}{k-1}$. Similarly if the Pansiot code of w has a 0 at every position $i \pmod{k+1}$ for an $i \in \{0, \ldots k+1\}$, then w has a letter with frequency $\frac{1}{k+1}$.*

We now introduce some notations in order to simplify forthcoming proofs.

2.2 Odd double crossed cycles

In this section, we fix a $k \in \mathbb{N}$.

Definition 1 (Odd double crossed cycle)**.** *A pair $(\sigma_a, \sigma_b) \in \mathbb{S}_k$ is a* double crossed cycle *if there is a $3 \leq K \leq \frac{k}{2}$ and two sequences $(a_0, \ldots a_{K-1})$ and $(b_0, \ldots, b_{K-1})$ of pairwise disjoint elements of Σ_k such that $\sigma_a = (a_0, a_1, a_2, \ldots, a_{K-1})(b_0, b_1, b_2, \ldots, b_{K-1})$ and $\sigma_b = (b_0, a_1, a_2, \ldots, a_{K-1})(a_0, b_1, b_2, \ldots, b_{K-1})$. Moreover, (σ_a, σ_b) is an* odd double crossed cycle (ODCC) *if K is odd. The* length *of the double crossed cycle is K.*

Lemma 1. *If (σ_a, σ_b) is an ODCC, then there is a $\sigma \in \mathbb{S}_k$ such that*

$$\sigma_a = \sigma \sigma_a \sigma_b \sigma^{-1}$$

and

$$\sigma_b = \sigma \sigma_b \sigma_a \sigma^{-1}.$$

Proof. Clearly, if $(\sigma_1, \sigma_2) \in \mathbb{S}_k^2$ and $(\sigma_3, \sigma_4) \in \mathbb{S}_k^2$ are two ODCC with the same size, then it exists a $\sigma \in \mathbb{S}_k$ such that $\sigma_1 = \sigma\sigma_3\sigma^{-1}$ and $\sigma_2 = \sigma\sigma_4\sigma^{-1}$. So it suffice to show that $(\sigma_a\sigma_b, \sigma_b\sigma_a)$ is an ODCC of size K. Let $\sigma_a = (a_0, a_1, \ldots, a_{K-1})(b_0, b_1, \ldots b_{K-1})$ and $\sigma_b = (b_0, a_1, \ldots, a_{K-1})(a_0, b_1, \ldots b_{K-1})$. Then

$$\sigma_b\sigma_a = (a_0, a_2, a_4 \ldots a_{K-1}, b_1, b_3 \ldots b_{K-2})(b_0, b_2, \ldots b_{K-1}, a_1, a_3, \ldots a_{K-2}),$$

and

$$\sigma_a\sigma_b = (b_0, a_2, a_4 \ldots a_{K-1}, b_1, b_3 \ldots b_{K-2})(a_0, b_2, \ldots b_{K-1}, a_1, a_3, \ldots a_{K-2}).$$

Thus $(\sigma_a\sigma_b, \sigma_b\sigma_a)$ is an ODCC of size K. □

Let (σ_a, σ_b) be an ODCC. The invariants of the ODCC are the invariants of σ_a and σ_b, and are denoted I when (σ_a, σ_b) is clear in the context. For every $i \in \{0, \ldots, K-1\}$, the *couple* i (or the couple number i) is (a_i, b_i).

Let $\varphi' : B^* \to \mathbb{S}_k$ be the morphism such that $\varphi'(0) = \sigma_a$, and $\varphi'(1) = \sigma_b$. The couple i is *good* in a binary word w if $\sigma[a_i] \in \{a_0, \ldots, a_{K-1}\}$, where $\sigma = \varphi'(w)$. Every couple is good in the empty word, and in 0^l for every l. Every couple is good, except 0 and $K-1$ in 1. Note that $\sigma = \mathrm{Id}_k$ if and only if $|w|$ is a multiple of K, and every couple is good in w.

On can easily compute good couples in w. Every 0 in w don't change the set of good couples, when every 1 change the state of exactly two couples. Formally, we have:

Proposition 2. *The couple i is good in w, where w is a binary word of size multiple of K, if and only if the set $\{x \in \{0 \ldots |w|-1\} : x \equiv i$ or $i-1 \mod K$ and $w[i] = 1\}$ has even cardinality.*

Let w' be the *derivative word* of w, that is w' is the binary word of size $|w|-1$ such that for every $i \in \{0, \ldots, |w'|-1\}$, $w'[i] = w[i] + w[i{+}1]$ (in GF(2)).

Proposition 3. *The couple i, $i \neq 0$, is good in w, where w is a binary word of size multiple of K, if the set $\{x \in \{0 \ldots |w|-1\} : x \equiv i-1 \mod K$ and $w[i] = 1\}$ has even cardinality.*

3 Main theorem and plan of the proof

Let $k \geq 24$ be an integer. Let m be such that $k \equiv m \mod 8$. Let us define the uniform morphism $h_k^+ : \{a,b\}^* \to \{0,1\}^*$ and $h_k^- : \{a,b\}^* \to \{0,1\}^*$ as follows

$$h_k^+ : \begin{cases} 0 \mapsto & \iota_m (10)^c 101 \kappa_m (10)^c 110 \\ 1 \mapsto & \iota_m (10)^c 110 \kappa_m (10)^c 101 \end{cases},$$

$$h_k^- : \begin{cases} 0 \mapsto & \iota_{m-2} (10)^{c'} 101 \kappa_{m-2} (10)^{c'} 110 \\ 1 \mapsto & \iota_{m-2} (10)^{c'} 110 \kappa_{m-2} (10)^{c'} 101 \end{cases},$$

where ι_m and κ_m are finite words over alphabet $\{0,1\}$ defined in the Table 1, $c = \frac{1}{2}(2k-4-|\iota_m|-|\kappa_m|)$, and $c' = \frac{1}{2}(2k-8-|\iota_m|-|\kappa_m|)$.

m	ι_m	κ_m
0	1011010101101011101	1010110111010110101
1	101101010110101	101011011010110
2	1011010101101101	1010110110101101
3	1011010101101	1010110110101
4	1011010101	1010110110
5	10110101101	10101101101
6	10110101	10101101
7	10110	10101

Table 1: Construction for the infinite case.

Theorem 1. *For any integer $k \geq 24$, the morphisms h_k^+ and h_k^- defined previously are such that the infinite words $w_k^+ = M_k(h_k^+(w_{TM}))$ and $w_k^- = M_k(h_k^-(w_{TM}))$ over Σ_k are $RT(k)^+$-free and have a letter of frequency respectively $\frac{1}{k+1}$ and $\frac{1}{k+1}$.*

The previous theorem is proved by computer for every $24 \leq k \leq 31$, using the method presented in [15]. We present in here a proof for every $k \geq 32$.

To prove that w_k^+ and w_k^- are $RT(k)^+$-free, we will use a method similar to Pansiot's one, by considering different type of repetitions and dealing with each type separately. We give here the structure of the proof, which will be completed in next section.

- We show that w has no forbidden short repetition with $|e| < 6$, using characterizations of Carpi, Shur and Gorbunova.
- If w has a forbidden repetition of excess at least 5, then the period is a multiple of $s{\cdot}K$ (Section 4.1).
- w has no forbidden repetition of period $s{\cdot}K$ (Section 4.1).
- A forbidden repetition of period at least $2{\cdot}s{\cdot}K$ is a kernel repetition. Using properties of the derivative of the Thue-Morse word, we show that a forbidden kernel repetition has period at least $18{\cdot}s{\cdot}K$ (Section 4.2).
- If w has a forbidden kernel repetition of period $|p| \geq 18{\cdot}s{\cdot}K$, then w has a forbidden repetition of period $|p|/2$ (Section 4.3). That part follows the proof in [15].

4 The Proof

We fix $k \geq 32$ and $\Delta \in \{-1, 1\}$. We prove Ochem's conjecture for k letters and frequency $\frac{1}{k+\Delta}$.

Let $\iota_a = \iota_m(10)^c101$, $\iota_a = \iota_m(10)^c110$, $\kappa_a = \kappa_m(10)^c101$ and $\kappa_a = \kappa_m(10)^c110$, where $c = \frac{1}{2}(2(k+\Delta) - 6 - |\iota_m| - |\kappa_m|)$ and $k + 2(\Delta - 1) \equiv m \mod 8$. Let $\lambda_a = \iota_a\kappa_b$

and $\lambda_b = \iota_b \kappa_a$. Let $h : B^* \to \Sigma_k^*$ be the morphism such that $h(0) = \lambda_a$ and $h(1) = \lambda_b$. Let $s = |h(0)| = |h(1)| = 2(k + \Delta)$. The largest common prefix L of $h(0)$ and $h(1)$ has size $\ell = \frac{s}{2} - 2$, and their last letter differ.

By Remark 1, we know that $M_k(h(w_{TM}))$ has a letter of frequency $\frac{1}{k+\Delta}$, since $\lambda_a[1] = \lambda_b[1] = \lambda_a[1 + \frac{s}{2}] = \lambda_b[1 + \frac{s}{2}] = 0$. We will prove now that it is a Dejean word.

Let $\sigma_a = \varphi(h(0))$, $\sigma_b = \varphi(h(1))$ and $\varphi' : B^* \to \mathbb{S}_k$ be the morphism such that $\varphi'(0) = \sigma_a$, and $\varphi'(1) = \sigma_b$. The permutations σ_a and σ_b are given in Table 2 for every k modulo 8 and Δ. Each permutations can be easily computed by hand using the definition of φ, or by a computer program. We see directly:

Fact 1. *(σ_a, σ_b) is an ODCC of size at least $\left\lceil \frac{k}{4} \right\rceil$.*

Table 2 also present the size of the ODCC and its invariants.

Case $k \equiv 0 \mod 8$, $\Delta = 1$:

$$
\begin{array}{rcl}
\sigma_a & = & (k{-}2\ \ 1\ \ 3\ \ 6\ \ 7\ \ 11\ \ 12\ \ 16\ \ 19\ \ ..(4)..\ \ k{-}9\ \ k{-}5) \\
 & & (k{-}1\ \ 2\ \ 4\ \ 8\ \ 9\ \ 13\ \ 14\ \ 17\ \ 21\ \ ..(4)..\ \ k{-}7\ \ k{-}3) \\
\sigma_b & = & (k{-}1\ \ 1\ \ 3\ \ 6\ \ 7\ \ 11\ \ 12\ \ 16\ \ 19\ \ ..(4)..\ \ k{-}9\ \ k{-}5) \\
 & & (k{-}2\ \ 2\ \ 4\ \ 8\ \ 9\ \ 13\ \ 14\ \ 17\ \ 21\ \ ..(4)..\ \ k{-}7\ \ k{-}3) \\
I & = & \{0,5,10,15,18,..(2)..,k{-}6,k{-}4\} \quad K = \frac{k+12}{4}
\end{array}
$$

1 2 3 4 5 6 7 8 -2 -1 0 1

$u_a = \mathbf{a}\,\mathrm{b\,c\,d\,e}\,\mathbf{f}\,\mathrm{g\,h\,i\,j}\,\mathbf{k}\,\mathrm{l\,m\,n\,o}\,\mathbf{p}\,\mathrm{q\,r}\,\mathbf{s}\,\mathrm{t}\,\mathbf{u}\,\mathrm{v}\ \ldots\ \underline{\mathbf{q}}\,\underline{\mathrm{r}}\,\underline{\mathbf{s}}\,\underline{\mathrm{t}}\,\underline{\mathbf{u}}\,\underline{\mathrm{v}}\,\underline{\mathbf{w}}\,\underline{\mathrm{x}}\,\underline{\mathrm{y}}\,\underline{\mathrm{z}}\,\mathrm{b}$

1 2 3 4 5 6 7 8 9 -2 -1 0 1

$\mathbf{a}\,\mathrm{c\,e\,d\,g}\,\mathbf{f}\,\mathrm{i\,h\,j\,l}\,\mathbf{k}\,\mathrm{n\,m\,o\,q}\,\mathbf{p}\,\mathrm{r\,t}\,\mathbf{s}\,\mathrm{v}\,\mathbf{u}\,\mathrm{x}\ \ldots\ \underline{\mathbf{q}}\,\underline{\mathrm{t}}\,\underline{\mathbf{s}}\,\underline{\mathrm{v}}\,\underline{\mathbf{u}}\,\underline{\mathrm{x}}\,\underline{\mathbf{w}}\,\underline{\mathrm{y}}\,\mathrm{b}\,\underline{\mathrm{z}}\,\mathrm{c}$

1 2 3 4 5 6 7 8 -2 -1 0 1

$u_b = \mathbf{a}\,\mathrm{b\,c\,d\,e}\,\mathbf{f}\,\mathrm{g\,h\,i\,j}\,\mathbf{k}\,\mathrm{l\,m\,n\,o}\,\mathbf{p}\,\mathrm{q\,r}\,\mathbf{s}\,\mathrm{t}\,\mathbf{u}\,\mathrm{v}\ \ldots\ \underline{\mathbf{q}}\,\underline{\mathrm{r}}\,\underline{\mathbf{s}}\,\underline{\mathrm{t}}\,\underline{\mathbf{u}}\,\underline{\mathrm{v}}\,\underline{\mathbf{w}}\,\underline{\mathrm{x}}\,\underline{\mathrm{y}}\,\underline{\mathrm{z}}\,\mathrm{b}$

1 2 3 4 5 6 7 8 9 -2 -1 0 1

$\mathbf{a}\,\mathrm{c\,e\,d\,g}\,\mathbf{f}\,\mathrm{i\,h\,j\,l}\,\mathbf{k}\,\mathrm{n\,m\,o\,q}\,\mathbf{p}\,\mathrm{r\,t}\,\mathbf{s}\,\mathrm{v}\,\mathbf{u}\,\mathrm{x}\ \ldots\ \underline{\mathbf{q}}\,\underline{\mathrm{t}}\,\underline{\mathbf{s}}\,\underline{\mathrm{v}}\,\underline{\mathbf{u}}\,\underline{\mathrm{x}}\,\underline{\mathbf{w}}\,\underline{\mathrm{z}}\,\underline{\mathrm{y}}\,\mathrm{b\,c}$

Case $k \equiv 1 \mod 8$, $\Delta = 1$:

$$
\begin{array}{rcl}
\sigma_a & = & (k{-}2\ \ 1\ \ 3\ \ 6\ \ 7\ \ 11\ \ 12\ \ 16\ \ ..(4)..\ \ k{-}9\ \ k{-}5) \\
 & & (k{-}1\ \ 2\ \ 4\ \ 8\ \ 9\ \ 13\ \ 14\ \ 18\ \ ..(4)..\ \ k{-}7\ \ k{-}3) \\
\sigma_b & = & (k{-}1\ \ 1\ \ 3\ \ 6\ \ 7\ \ 11\ \ 12\ \ 16\ \ ..(4)..\ \ k{-}9\ \ k{-}5) \\
 & & (k{-}2\ \ 2\ \ 4\ \ 8\ \ 9\ \ 13\ \ 14\ \ 18\ \ ..(4)..\ \ k{-}7\ \ k{-}3) \\
I & = & \{0,5,10,15,..(2)..,k{-}6,k{-}4\} \quad K = \frac{k+11}{4}
\end{array}
$$

1 2 3 4 5 6 7 -2 -1 0 1

$u_a = \mathbf{a}\,\mathrm{b\,c\,d\,e}\,\mathbf{f}\,\mathrm{g\,h\,i\,j}\,\mathbf{k}\,\mathrm{l\,m\,n\,o}\,\mathbf{p}\,\mathrm{q}\,\mathbf{r}\,\mathrm{s}\ \ldots\ \underline{\mathbf{q}}\,\underline{\mathrm{r}}\,\underline{\mathbf{s}}\,\underline{\mathrm{t}}\,\underline{\mathbf{u}}\,\underline{\mathrm{v}}\,\underline{\mathbf{w}}\,\underline{\mathrm{x}}\,\underline{\mathrm{y}}\,\underline{\mathrm{z}}\,\mathrm{b}$

1 2 3 4 5 6 7 8 -2 -1 0 1

$\mathbf{a}\,\mathrm{c\,e\,d\,g}\,\mathbf{f}\,\mathrm{i\,h\,j\,l}\,\mathbf{k}\,\mathrm{n\,m\,o\,q}\,\mathbf{p}\,\mathrm{s}\,\mathbf{r}\,\mathrm{u}\ \ldots\ \underline{\mathbf{q}}\,\underline{\mathrm{t}}\,\underline{\mathbf{s}}\,\underline{\mathrm{v}}\,\underline{\mathbf{u}}\,\underline{\mathrm{x}}\,\underline{\mathbf{w}}\,\underline{\mathrm{y}}\,\mathrm{b}\,\underline{\mathrm{z}}\,\mathrm{c}$

1 2 3 4 5 6 7 -2 -1 0 1

$u_b = \mathbf{a}\,\mathrm{b\,c\,d\,e}\,\mathbf{f}\,\mathrm{g\,h\,i\,j}\,\mathbf{k}\,\mathrm{l\,m\,n\,o}\,\mathbf{p}\,\mathrm{q}\,\mathbf{r}\,\mathrm{s}\ \ldots\ \underline{\mathbf{q}}\,\underline{\mathrm{r}}\,\underline{\mathbf{s}}\,\underline{\mathrm{t}}\,\underline{\mathbf{u}}\,\underline{\mathrm{v}}\,\underline{\mathbf{w}}\,\underline{\mathrm{x}}\,\underline{\mathrm{y}}\,\underline{\mathrm{z}}\,\mathrm{b}$

1 2 3 4 5 6 7 8 -2 -1 0 1

$\mathbf{a}\,\mathrm{c\,e\,d\,g}\,\mathbf{f}\,\mathrm{i\,h\,j\,l}\,\mathbf{k}\,\mathrm{n\,m\,o\,q}\,\mathbf{p}\,\mathrm{s}\,\mathbf{r}\,\mathrm{u}\ \ldots\ \underline{\mathbf{q}}\,\underline{\mathrm{t}}\,\underline{\mathbf{s}}\,\underline{\mathrm{v}}\,\underline{\mathbf{u}}\,\underline{\mathrm{x}}\,\underline{\mathbf{w}}\,\underline{\mathrm{z}}\,\underline{\mathrm{y}}\,\mathrm{b\,c}$

Case $k \equiv 2 \mod 8$, $\Delta = 1$:

$$
\begin{array}{rcl}
\sigma_a & = & (k{-}2\ \ 1\ \ 3\ \ 6\ \ 7\ \ 11\ \ 14\ \ 17\ \ ..(4)..\ \ k{-}9\ \ k{-}5) \\
 & & (k{-}1\ \ 2\ \ 4\ \ 8\ \ 9\ \ 12\ \ 15\ \ 19\ \ ..(4)..\ \ k{-}7\ \ k{-}3) \\
\sigma_b & = & (k{-}1\ \ 1\ \ 3\ \ 6\ \ 7\ \ 11\ \ 14\ \ 17\ \ ..(4)..\ \ k{-}9\ \ k{-}5) \\
 & & (k{-}2\ \ 2\ \ 4\ \ 8\ \ 9\ \ 12\ \ 15\ \ 19\ \ ..(4)..\ \ k{-}7\ \ k{-}3) \\
I & = & \{0,5,10,13,16,..(2)..,k{-}6,k{-}4\} \quad K = \frac{k+10}{4}
\end{array}
$$

1 2 3 4 5 6 7 -2 -1 0 1

$u_a = \mathbf{a}\,\mathrm{b\,c\,d\,e}\,\mathbf{f}\,\mathrm{g\,h\,i\,j}\,\mathbf{k}\,\mathrm{l\,m}\,\mathbf{n}\,\mathrm{o\,p}\,\mathbf{q}\,\mathrm{r}\,\mathbf{s}\,\mathrm{t}\ \ldots\ \underline{\mathbf{q}}\,\underline{\mathrm{r}}\,\underline{\mathbf{s}}\,\underline{\mathrm{t}}\,\underline{\mathbf{u}}\,\underline{\mathrm{v}}\,\underline{\mathbf{w}}\,\underline{\mathrm{x}}\,\underline{\mathrm{y}}\,\underline{\mathrm{z}}\,\mathrm{b}$

1 2 3 4 5 6 7 8 -2 -1 0 1

$\mathbf{a}\,\mathrm{c\,e\,d\,g}\,\mathbf{f}\,\mathrm{i\,h\,j\,l}\,\mathbf{k}\,\mathrm{m\,o}\,\mathbf{n}\,\mathrm{p\,r}\,\mathbf{q}\,\mathrm{t}\,\mathbf{s}\,\mathrm{v}\ \ldots\ \underline{\mathbf{q}}\,\underline{\mathrm{t}}\,\underline{\mathbf{s}}\,\underline{\mathrm{v}}\,\underline{\mathbf{u}}\,\underline{\mathrm{x}}\,\underline{\mathbf{w}}\,\underline{\mathrm{y}}\,\mathrm{b}\,\underline{\mathrm{z}}\,\mathrm{c}$

1 2 3 4 5 6 7 -2 -1 0 1

$u_b = \mathbf{a}\,\mathrm{b\,c\,d\,e}\,\mathbf{f}\,\mathrm{g\,h\,i\,j}\,\mathbf{k}\,\mathrm{l\,m}\,\mathbf{n}\,\mathrm{o\,p}\,\mathbf{q}\,\mathrm{r}\,\mathbf{s}\,\mathrm{t}\ \ldots\ \underline{\mathbf{q}}\,\underline{\mathrm{r}}\,\underline{\mathbf{s}}\,\underline{\mathrm{t}}\,\underline{\mathbf{u}}\,\underline{\mathrm{v}}\,\underline{\mathbf{w}}\,\underline{\mathrm{x}}\,\underline{\mathrm{y}}\,\underline{\mathrm{z}}\,\mathrm{b}$

1 2 3 4 5 6 7 8 -2 -1 0 1

$\mathbf{a}\,\mathrm{c\,e\,d\,g}\,\mathbf{f}\,\mathrm{i\,h\,j\,l}\,\mathbf{k}\,\mathrm{m\,o}\,\mathbf{n}\,\mathrm{p\,r}\,\mathbf{q}\,\mathrm{t}\,\mathbf{s}\,\mathrm{v}\ \ldots\ \underline{\mathbf{q}}\,\underline{\mathrm{t}}\,\underline{\mathbf{s}}\,\underline{\mathrm{v}}\,\underline{\mathbf{u}}\,\underline{\mathrm{x}}\,\underline{\mathbf{w}}\,\underline{\mathrm{z}}\,\underline{\mathrm{y}}\,\mathrm{b\,c}$

Case $k \equiv 3 \mod 8$, $\Delta = 1$:

$$
\begin{array}{rcl}
\sigma_a & = & (k{-}2\ \ 1\ \ 3\ \ 6\ \ 7\ \ 11\ \ 14\ \ ..(4)..\ \ k{-}9\ \ k{-}5) \\
 & & (k{-}1\ \ 2\ \ 4\ \ 8\ \ 9\ \ 12\ \ 16\ \ ..(4)..\ \ k{-}7\ \ k{-}3) \\
\sigma_b & = & (k{-}1\ \ 1\ \ 3\ \ 6\ \ 7\ \ 11\ \ 14\ \ ..(4)..\ \ k{-}9\ \ k{-}5) \\
 & & (k{-}2\ \ 2\ \ 4\ \ 8\ \ 9\ \ 12\ \ 16\ \ ..(4)..\ \ k{-}7\ \ k{-}3) \\
I & = & \{0,5,10,13,..(2)..,k{-}6,k{-}4\} \quad K = \frac{k+9}{4}
\end{array}
$$

1 2 3 4 5 6 -2 -1 0 1

$u_a = \mathbf{a}\,\mathrm{b\,c\,d\,e}\,\mathbf{f}\,\mathrm{g\,h\,i\,j}\,\mathbf{k}\,\mathrm{l\,m}\,\mathbf{n}\,\mathrm{o}\,\mathbf{p}\,\mathrm{q}\ \ldots\ \underline{\mathbf{q}}\,\underline{\mathrm{r}}\,\underline{\mathbf{s}}\,\underline{\mathrm{t}}\,\underline{\mathbf{u}}\,\underline{\mathrm{v}}\,\underline{\mathbf{w}}\,\underline{\mathrm{x}}\,\underline{\mathrm{y}}\,\underline{\mathrm{z}}\,\mathrm{b}$

1 2 3 4 5 6 7 -2 -1 0 1

$\mathbf{a}\,\mathrm{c\,e\,d\,g}\,\mathbf{f}\,\mathrm{i\,h\,j\,l}\,\mathbf{k}\,\mathrm{m\,o}\,\mathbf{n}\,\mathrm{q}\,\mathbf{p}\,\mathrm{s}\ \ldots\ \underline{\mathbf{q}}\,\underline{\mathrm{t}}\,\underline{\mathbf{s}}\,\underline{\mathrm{v}}\,\underline{\mathbf{u}}\,\underline{\mathrm{x}}\,\underline{\mathbf{w}}\,\underline{\mathrm{y}}\,\mathrm{b}\,\underline{\mathrm{z}}\,\mathrm{c}$

1 2 3 4 5 6 -2 -1 0 1

$u_b = \mathbf{a}\,\mathrm{b\,c\,d\,e}\,\mathbf{f}\,\mathrm{g\,h\,i\,j}\,\mathbf{k}\,\mathrm{l\,m}\,\mathbf{n}\,\mathrm{o}\,\mathbf{p}\,\mathrm{q}\ \ldots\ \underline{\mathbf{q}}\,\underline{\mathrm{r}}\,\underline{\mathbf{s}}\,\underline{\mathrm{t}}\,\underline{\mathbf{u}}\,\underline{\mathrm{v}}\,\underline{\mathbf{w}}\,\underline{\mathrm{x}}\,\underline{\mathrm{y}}\,\underline{\mathrm{z}}\,\mathrm{b}$

1 2 3 4 5 6 7 -2 -1 0 1

$\mathbf{a}\,\mathrm{c\,e\,d\,g}\,\mathbf{f}\,\mathrm{i\,h\,j\,l}\,\mathbf{k}\,\mathrm{m\,o}\,\mathbf{n}\,\mathrm{q}\,\mathbf{p}\,\mathrm{s}\ \ldots\ \underline{\mathbf{q}}\,\underline{\mathrm{t}}\,\underline{\mathbf{s}}\,\underline{\mathrm{v}}\,\underline{\mathbf{u}}\,\underline{\mathrm{x}}\,\underline{\mathbf{w}}\,\underline{\mathrm{z}}\,\underline{\mathrm{y}}\,\mathrm{b\,c}$

Table 2: σ_a, σ_b, I, u_a and u_b. "x ..(y).. z" means every integer between x and z, with a step of y. In u_a and u_b, letters $a, b, c, \ldots$ are letters $0, 1, 2 \ldots$, $\underline{z}, \underline{y}, \underline{x}, \ldots$ are letters $k{-}1, k{-}2, k{-}3, \ldots$ and letters in bold are letters in I. The over-brackets give the ODCC couples, with their number (modulo K).

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Case $k \equiv 4 \mod 8$, $\Delta = 1$:

$$
\begin{array}{rcl}
\sigma_a & = & (k{-}2 \;\; 1 \;\; 3 \;\; 6 \;\; 7 \;\; 11 \;\; ..(4).. \;\; k{-}9 \;\; k{-}5) \\
 & & (k{-}1 \;\; 2 \;\; 4 \;\; 8 \;\; 9 \;\; 13 \;\; ..(4).. \;\; k{-}7 \;\; k{-}3) \\
\sigma_b & = & (k{-}1 \;\; 1 \;\; 3 \;\; 6 \;\; 7 \;\; 11 \;\; ..(4).. \;\; k{-}9 \;\; k{-}5) \\
 & & (k{-}2 \;\; 2 \;\; 4 \;\; 8 \;\; 9 \;\; 13 \;\; ..(4).. \;\; k{-}7 \;\; k{-}3) \\
I & = & \{0, 5, 10, ..(2).., k{-}6, k{-}4\} \quad K = \frac{k+8}{4}
\end{array}
$$

$u_a =$ 1 2 3 4 5 … -2 -1 0 1
$\mathbf{a}$ b c d e $\mathbf{f}$ g h i j $\mathbf{k}$ l $\mathbf{m}$ n … $\underline{\mathbf{q}}$ $\underline{r}$ $\underline{\mathbf{s}}$ $\underline{t}$ $\underline{\mathbf{u}}$ $\underline{v}$ $\underline{\mathbf{w}}$ $\underline{x}$ $\underline{y}$ $\underline{z}$ b
1 2 3 4 5 6 … -2 -1 0 1
$\mathbf{a}$ c e d g $\mathbf{f}$ i h j l $\mathbf{k}$ n $\mathbf{m}$ p … $\underline{\mathbf{q}}$ $\underline{t}$ $\underline{\mathbf{s}}$ $\underline{v}$ $\underline{\mathbf{u}}$ $\underline{x}$ $\underline{\mathbf{w}}$ $\underline{y}$ b $\underline{z}$ c

$u_b =$ 1 2 3 4 5 … -2 -1 0 1
$\mathbf{a}$ b c d e $\mathbf{f}$ g h i j $\mathbf{k}$ l $\mathbf{m}$ n … $\underline{\mathbf{q}}$ $\underline{r}$ $\underline{\mathbf{s}}$ $\underline{t}$ $\underline{\mathbf{u}}$ $\underline{v}$ $\underline{\mathbf{w}}$ $\underline{x}$ $\underline{y}$ $\underline{z}$ b
1 2 3 4 5 6 … -2 -1 0 1
$\mathbf{a}$ c e d g $\mathbf{f}$ i h j l $\mathbf{k}$ n $\mathbf{m}$ p … $\underline{\mathbf{q}}$ $\underline{t}$ $\underline{\mathbf{s}}$ $\underline{v}$ $\underline{\mathbf{u}}$ $\underline{x}$ $\underline{\mathbf{w}}$ $\underline{z}$ $\underline{y}$ b c

Case $k \equiv 5 \mod 8$, $\Delta = 1$:

$$
\begin{array}{rcl}
\sigma_a & = & (k{-}2 \;\; 1 \;\; 3 \;\; 6 \;\; 9 \;\; 12 \;\; ..(4).. \;\; k{-}9 \;\; k{-}5) \\
 & & (k{-}1 \;\; 2 \;\; 4 \;\; 7 \;\; 10 \;\; 14 \;\; ..(4).. \;\; k{-}7 \;\; k{-}3) \\
\sigma_b & = & (k{-}1 \;\; 1 \;\; 3 \;\; 6 \;\; 9 \;\; 12 \;\; ..(4).. \;\; k{-}9 \;\; k{-}5) \\
 & & (k{-}2 \;\; 2 \;\; 4 \;\; 7 \;\; 10 \;\; 14 \;\; ..(4).. \;\; k{-}7 \;\; k{-}3) \\
I & = & \{0, 5, 8, 11, ..(2).., k{-}6, k{-}4\} \quad K = \frac{k+7}{4}
\end{array}
$$

$u_a =$ 1 2 3 4 5 … -2 -1 0 1
$\mathbf{a}$ b c d e $\mathbf{f}$ g h $\mathbf{i}$ j k $\mathbf{l}$ m $\mathbf{n}$ o … $\underline{\mathbf{q}}$ $\underline{r}$ $\underline{\mathbf{s}}$ $\underline{t}$ $\underline{\mathbf{u}}$ $\underline{v}$ $\underline{\mathbf{w}}$ $\underline{x}$ $\underline{y}$ $\underline{z}$ b
1 2 3 4 5 6 … -2 -1 0 1
$\mathbf{a}$ c e d g $\mathbf{f}$ h j $\mathbf{i}$ k m $\mathbf{l}$ o $\mathbf{n}$ q … $\underline{\mathbf{q}}$ $\underline{t}$ $\underline{\mathbf{s}}$ $\underline{v}$ $\underline{\mathbf{u}}$ $\underline{x}$ $\underline{\mathbf{w}}$ $\underline{y}$ b $\underline{z}$ c

$u_b =$ 1 2 3 4 5 … -2 -1 0 1
$\mathbf{a}$ b c d e $\mathbf{f}$ g h $\mathbf{i}$ j k $\mathbf{l}$ m $\mathbf{n}$ o … $\underline{\mathbf{q}}$ $\underline{r}$ $\underline{\mathbf{s}}$ $\underline{t}$ $\underline{\mathbf{u}}$ $\underline{v}$ $\underline{\mathbf{w}}$ $\underline{x}$ $\underline{y}$ $\underline{z}$ b
1 2 3 4 5 6 … -2 -1 0 1
$\mathbf{a}$ c e d g $\mathbf{f}$ h j $\mathbf{i}$ k m $\mathbf{l}$ o $\mathbf{n}$ q … $\underline{\mathbf{q}}$ $\underline{t}$ $\underline{\mathbf{s}}$ $\underline{v}$ $\underline{\mathbf{u}}$ $\underline{x}$ $\underline{\mathbf{w}}$ $\underline{z}$ $\underline{y}$ b c

Case $k \equiv 6 \mod 8$, $\Delta = 1$:

$$
\begin{array}{rcl}
\sigma_a & = & (k{-}2 \;\; 1 \;\; 3 \;\; 6 \;\; 9 \;\; ..(4).. \;\; k{-}9 \;\; k{-}5) \\
 & & (k{-}1 \;\; 2 \;\; 4 \;\; 7 \;\; 11 \;\; ..(4).. \;\; k{-}7 \;\; k{-}3) \\
\sigma_b & = & (k{-}1 \;\; 1 \;\; 3 \;\; 6 \;\; 9 \;\; ..(4).. \;\; k{-}9 \;\; k{-}5) \\
 & & (k{-}2 \;\; 2 \;\; 4 \;\; 7 \;\; 11 \;\; ..(4).. \;\; k{-}7 \;\; k{-}3) \\
I & = & \{0, 5, 8, ..(2).., k{-}6, k{-}4\} \quad K = \frac{k+6}{4}
\end{array}
$$

$u_a =$ 1 2 3 4 … -2 -1 0 1
$\mathbf{a}$ b c d e $\mathbf{f}$ g h $\mathbf{i}$ j $\mathbf{k}$ l … $\underline{\mathbf{q}}$ $\underline{r}$ $\underline{\mathbf{s}}$ $\underline{t}$ $\underline{\mathbf{u}}$ $\underline{v}$ $\underline{\mathbf{w}}$ $\underline{x}$ $\underline{y}$ $\underline{z}$ b
1 2 3 4 5 … -2 -1 0 1
$\mathbf{a}$ c e d g $\mathbf{f}$ h j $\mathbf{i}$ l $\mathbf{k}$ n … $\underline{\mathbf{q}}$ $\underline{t}$ $\underline{\mathbf{s}}$ $\underline{v}$ $\underline{\mathbf{u}}$ $\underline{x}$ $\underline{\mathbf{w}}$ $\underline{y}$ b $\underline{z}$ c

$u_b =$ 1 2 3 4 … -2 -1 0 1
$\mathbf{a}$ b c d e $\mathbf{f}$ g h $\mathbf{i}$ j $\mathbf{k}$ l … $\underline{\mathbf{q}}$ $\underline{r}$ $\underline{\mathbf{s}}$ $\underline{t}$ $\underline{\mathbf{u}}$ $\underline{v}$ $\underline{\mathbf{w}}$ $\underline{x}$ $\underline{y}$ $\underline{z}$ b
1 2 3 4 5 … -2 -1 0 1
$\mathbf{a}$ c e d g $\mathbf{f}$ h j $\mathbf{i}$ l $\mathbf{k}$ n … $\underline{\mathbf{q}}$ $\underline{t}$ $\underline{\mathbf{s}}$ $\underline{v}$ $\underline{\mathbf{u}}$ $\underline{x}$ $\underline{\mathbf{w}}$ $\underline{z}$ $\underline{y}$ b c

Case $k \equiv 7 \mod 8$, $\Delta = 1$:

$$
\begin{array}{rcl}
\sigma_a & = & (k{-}2 \;\; 1 \;\; 3 \;\; 6 \;\; ..(4).. \;\; k{-}9 \;\; k{-}5) \\
 & & (k{-}1 \;\; 2 \;\; 4 \;\; 8 \;\; ..(4).. \;\; k{-}7 \;\; k{-}3) \\
\sigma_b & = & (k{-}1 \;\; 1 \;\; 3 \;\; 6 \;\; ..(4).. \;\; k{-}9 \;\; k{-}5) \\
 & & (k{-}2 \;\; 2 \;\; 4 \;\; 8 \;\; ..(4).. \;\; k{-}7 \;\; k{-}3) \\
I & = & \{0, 5, ..(2).., k{-}6, k{-}4\} \quad K = \frac{k+5}{4}
\end{array}
$$

$u_a =$ 1 2 3 … -2 -1 0 1
$\mathbf{a}$ b c d e $\mathbf{f}$ g $\mathbf{h}$ i … $\underline{\mathbf{q}}$ $\underline{r}$ $\underline{\mathbf{s}}$ $\underline{t}$ $\underline{\mathbf{u}}$ $\underline{v}$ $\underline{\mathbf{w}}$ $\underline{x}$ $\underline{y}$ $\underline{z}$ b
1 2 3 4 … -2 -1 0 1
$\mathbf{a}$ c e d g $\mathbf{f}$ i $\mathbf{h}$ k … $\underline{\mathbf{q}}$ $\underline{t}$ $\underline{\mathbf{s}}$ $\underline{v}$ $\underline{\mathbf{u}}$ $\underline{x}$ $\underline{\mathbf{w}}$ $\underline{y}$ b $\underline{z}$ c

$u_b =$ 1 2 3 … -2 -1 0 1
$\mathbf{a}$ b c d e $\mathbf{f}$ g $\mathbf{h}$ i … $\underline{\mathbf{q}}$ $\underline{r}$ $\underline{\mathbf{s}}$ $\underline{t}$ $\underline{\mathbf{u}}$ $\underline{v}$ $\underline{\mathbf{w}}$ $\underline{x}$ $\underline{y}$ $\underline{z}$ b
1 2 3 4 … -2 -1 0 1
$\mathbf{a}$ c e d g $\mathbf{f}$ i $\mathbf{h}$ k … $\underline{\mathbf{q}}$ $\underline{t}$ $\underline{\mathbf{s}}$ $\underline{v}$ $\underline{\mathbf{u}}$ $\underline{x}$ $\underline{\mathbf{w}}$ $\underline{z}$ $\underline{y}$ b c

Case $k \equiv 0 \mod 8$, $\Delta = -1$:

$$\begin{array}{rcl}
\sigma_a & = & (k{-}3 \quad k{-}4 \quad ..(-4).. \quad 24 \quad 20 \quad 16 \quad 13 \quad 10 \quad 8 \quad 5 \quad 4 \quad 0) \\
 & & (k{-}2 \quad k{-}6 \quad ..(-4).. \quad 22 \quad 18 \quad 15 \quad 12 \quad 9 \quad 7 \quad 3 \quad 2 \quad k{-}1) \\
\sigma_b & = & (k{-}2 \quad k{-}4 \quad ..(-4).. \quad 24 \quad 20 \quad 16 \quad 13 \quad 10 \quad 8 \quad 5 \quad 4 \quad 0) \\
 & & (k{-}3 \quad k{-}6 \quad ..(-4).. \quad 22 \quad 18 \quad 15 \quad 12 \quad 9 \quad 7 \quad 3 \quad 2 \quad k{-}1) \\
I & = & \{1, 6, 11, 14, 17, 19, ..(2).., k{-}5\} \quad K = \frac{k+12}{4}
\end{array}$$

-1 -2 -3 -4 -5 -6 -7 -8 -9 2 1 0

$u_a = \mathrm{a}\,\mathbf{b}\,\mathrm{c\,d\,e\,f}\,\mathbf{g}\,\mathrm{h\,i\,j\,k}\,\mathbf{l}\,\mathrm{m\,n}\,\mathbf{o}\,\mathrm{p\,q}\,\mathbf{r}\,\mathrm{s}\,\mathbf{t}\,\mathrm{u}\,\mathbf{v}\,\mathrm{w}\,\mathbf{x}\ \ldots\ \underline{\mathrm{s}}\,\underline{\mathbf{t}}\,\underline{\mathrm{u}}\,\underline{\mathbf{v}}\,\underline{\mathrm{w}}\,\underline{\mathrm{x}}\,\underline{\mathrm{y}}$

-1 -2 -3 -4 -5 -6 -7 -8 2 1 0

$\underline{\mathrm{z}}\,\mathbf{b}\,\mathrm{a\,c\,e\,d}\,\mathbf{g}\,\mathrm{f\,i\,h\,j}\,\mathbf{l}\,\mathrm{k\,m}\,\mathbf{o}\,\mathrm{n\,p}\,\mathbf{r}\,\mathrm{q}\,\mathbf{t}\,\mathrm{s}\,\mathbf{v}\,\mathrm{u}\,\mathbf{x}\ \ldots\ \underline{\mathrm{q}}\,\underline{\mathbf{t}}\,\underline{\mathrm{s}}\,\underline{\mathbf{v}}\,\underline{\mathrm{u}}\,\underline{\mathrm{w}}\,\underline{\mathrm{y}}$

-1 -2 -3 -4 -5 -6 -7 -8 -9 2 1 0

$u_b = \mathrm{a}\,\mathbf{b}\,\mathrm{c\,d\,e\,f}\,\mathbf{g}\,\mathrm{h\,i\,j\,k}\,\mathbf{l}\,\mathrm{m\,n}\,\mathbf{o}\,\mathrm{p\,q}\,\mathbf{r}\,\mathrm{s}\,\mathbf{t}\,\mathrm{u}\,\mathbf{v}\,\mathrm{w}\,\mathbf{x}\ \ldots\ \underline{\mathrm{s}}\,\underline{\mathbf{t}}\,\underline{\mathrm{u}}\,\underline{\mathbf{v}}\,\underline{\mathrm{w}}\,\underline{\mathrm{x}}\,\underline{\mathrm{y}}$

-1 -2 -3 -4 -5 -6 -7 -8 2 1 0

$\underline{\mathrm{z}}\,\mathbf{b}\,\mathrm{a\,c\,e\,d}\,\mathbf{g}\,\mathrm{f\,i\,h\,j}\,\mathbf{l}\,\mathrm{k\,m}\,\mathbf{o}\,\mathrm{n\,p}\,\mathbf{r}\,\mathrm{q}\,\mathbf{t}\,\mathrm{s}\,\mathbf{v}\,\mathrm{u}\,\mathbf{x}\ \ldots\ \underline{\mathrm{q}}\,\underline{\mathbf{t}}\,\underline{\mathrm{s}}\,\underline{\mathbf{v}}\,\underline{\mathrm{u}}\,\underline{\mathrm{x}}\,\underline{\mathrm{w}}$

Case $k \equiv 1 \mod 8$, $\Delta = -1$:

$$\begin{array}{rcl}
\sigma_a & = & (k{-}3 \quad k{-}4 \quad ..(-4).. \quad 21 \quad 17 \quad 13 \quad 10 \quad 8 \quad 5 \quad 4 \quad 0) \\
 & & (k{-}2 \quad k{-}6 \quad ..(-4).. \quad 19 \quad 15 \quad 12 \quad 9 \quad 7 \quad 3 \quad 2 \quad k{-}1) \\
\sigma_b & = & (k{-}2 \quad k{-}4 \quad ..(-4).. \quad 21 \quad 17 \quad 13 \quad 10 \quad 8 \quad 5 \quad 4 \quad 0) \\
 & & (k{-}3 \quad k{-}6 \quad ..(-4).. \quad 19 \quad 15 \quad 12 \quad 9 \quad 7 \quad 3 \quad 2 \quad k{-}1) \\
I & = & \{1, 6, 11, 14, 16, ..(2).., k{-}5\} \quad K = \frac{k+11}{4}
\end{array}$$

-1 -2 -3 -4 -5 -6 -7 -8 2 1 0

$u_a = \mathrm{a}\,\mathbf{b}\,\mathrm{c\,d\,e\,f}\,\mathbf{g}\,\mathrm{h\,i\,j\,k}\,\mathbf{l}\,\mathrm{m\,n}\,\mathbf{o}\,\mathrm{p}\,\mathbf{q}\,\mathrm{r}\,\mathbf{s}\,\mathrm{t}\,\mathbf{u}\ \ldots\ \underline{\mathrm{s}}\,\underline{\mathbf{t}}\,\underline{\mathrm{u}}\,\underline{\mathbf{v}}\,\underline{\mathrm{w}}\,\underline{\mathrm{x}}\,\underline{\mathrm{y}}$

-1 -2 -3 -4 -5 -6 -7 2 1 0

$\underline{\mathrm{z}}\,\mathbf{b}\,\mathrm{a\,c\,e\,d}\,\mathbf{g}\,\mathrm{f\,i\,h\,j}\,\mathbf{l}\,\mathrm{k\,m}\,\mathbf{o}\,\mathrm{n}\,\mathbf{q}\,\mathrm{p}\,\mathbf{s}\,\mathrm{r}\,\mathbf{u}\ \ldots\ \underline{\mathrm{q}}\,\underline{\mathbf{t}}\,\underline{\mathrm{s}}\,\underline{\mathbf{v}}\,\underline{\mathrm{u}}\,\underline{\mathrm{w}}\,\underline{\mathrm{y}}$

-1 -2 -3 -4 -5 -6 -7 -8 2 1 0

$u_b = \mathrm{a}\,\mathbf{b}\,\mathrm{c\,d\,e\,f}\,\mathbf{g}\,\mathrm{h\,i\,j\,k}\,\mathbf{l}\,\mathrm{m\,n}\,\mathbf{o}\,\mathrm{p}\,\mathbf{q}\,\mathrm{r}\,\mathbf{s}\,\mathrm{t}\,\mathbf{u}\ \ldots\ \underline{\mathrm{s}}\,\underline{\mathbf{t}}\,\underline{\mathrm{u}}\,\underline{\mathbf{v}}\,\underline{\mathrm{w}}\,\underline{\mathrm{x}}\,\underline{\mathrm{y}}$

-1 -2 -3 -4 -5 -6 -7 2 1 0

$\underline{\mathrm{z}}\,\mathbf{b}\,\mathrm{a\,c\,e\,d}\,\mathbf{g}\,\mathrm{f\,i\,h\,j}\,\mathbf{l}\,\mathrm{k\,m}\,\mathbf{o}\,\mathrm{n}\,\mathbf{q}\,\mathrm{p}\,\mathbf{s}\,\mathrm{r}\,\mathbf{u}\ \ldots\ \underline{\mathrm{q}}\,\underline{\mathbf{t}}\,\underline{\mathrm{s}}\,\underline{\mathbf{v}}\,\underline{\mathrm{u}}\,\underline{\mathrm{x}}\,\underline{\mathrm{w}}$

Case $k \equiv 2 \mod 8$, $\Delta = -1$:

$$\begin{array}{rcl}
\sigma_a & = & (k{-}3 \quad k{-}4 \quad ..(-4).. \quad 18 \quad 14 \quad 10 \quad 8 \quad 5 \quad 4 \quad 0) \\
 & & (k{-}2 \quad k{-}6 \quad ..(-4).. \quad 16 \quad 12 \quad 9 \quad 7 \quad 3 \quad 2 \quad k{-}1) \\
\sigma_b & = & (k{-}2 \quad k{-}4 \quad ..(-4).. \quad 18 \quad 14 \quad 10 \quad 8 \quad 5 \quad 4 \quad 0) \\
 & & (k{-}3 \quad k{-}6 \quad ..(-4).. \quad 16 \quad 12 \quad 9 \quad 7 \quad 3 \quad 2 \quad k{-}1) \\
I & = & \{1, 6, 11, 13, ..(2).., k{-}5\} \quad K = \frac{k+10}{4}
\end{array}$$

-1 -2 -3 -4 -5 -6 -7 2 1 0

$u_a = \mathrm{a}\,\mathbf{b}\,\mathrm{c\,d\,e\,f}\,\mathbf{g}\,\mathrm{h\,i\,j\,k}\,\mathbf{l}\,\mathrm{m}\,\mathbf{n}\,\mathrm{o}\,\mathbf{p}\,\mathrm{q}\,\mathbf{r}\ \ldots\ \underline{\mathrm{s}}\,\underline{\mathbf{t}}\,\underline{\mathrm{u}}\,\underline{\mathbf{v}}\,\underline{\mathrm{w}}\,\underline{\mathrm{x}}\,\underline{\mathrm{y}}$

-1 -2 -3 -4 -5 -6 2 1 0

$\underline{\mathrm{z}}\,\mathbf{b}\,\mathrm{a\,c\,e\,d}\,\mathbf{g}\,\mathrm{f\,i\,h\,j}\,\mathbf{l}\,\mathrm{k}\,\mathbf{n}\,\mathrm{m}\,\mathbf{p}\,\mathrm{o}\,\mathbf{r}\ \ldots\ \underline{\mathrm{q}}\,\underline{\mathbf{t}}\,\underline{\mathrm{s}}\,\underline{\mathbf{v}}\,\underline{\mathrm{u}}\,\underline{\mathrm{w}}\,\underline{\mathrm{y}}$

-1 -2 -3 -4 -5 -6 -7 2 1 0

$u_b = \mathrm{a}\,\mathbf{b}\,\mathrm{c\,d\,e\,f}\,\mathbf{g}\,\mathrm{h\,i\,j\,k}\,\mathbf{l}\,\mathrm{m}\,\mathbf{n}\,\mathrm{o}\,\mathbf{p}\,\mathrm{q}\,\mathbf{r}\ \ldots\ \underline{\mathrm{s}}\,\underline{\mathbf{t}}\,\underline{\mathrm{u}}\,\underline{\mathbf{v}}\,\underline{\mathrm{w}}\,\underline{\mathrm{x}}\,\underline{\mathrm{y}}$

-1 -2 -3 -4 -5 -6 2 1 0

$\underline{\mathrm{z}}\,\mathbf{b}\,\mathrm{a\,c\,e\,d}\,\mathbf{g}\,\mathrm{f\,i\,h\,j}\,\mathbf{l}\,\mathrm{k}\,\mathbf{n}\,\mathrm{m}\,\mathbf{p}\,\mathrm{o}\,\mathbf{r}\ \ldots\ \underline{\mathrm{q}}\,\underline{\mathbf{t}}\,\underline{\mathrm{s}}\,\underline{\mathbf{v}}\,\underline{\mathrm{u}}\,\underline{\mathrm{x}}\,\underline{\mathrm{w}}$

Case $k \equiv 3 \mod 8$, $\Delta = -1$:

$$\begin{array}{rcl}
\sigma_a & = & (k{-}3 \quad k{-}4 \quad ..(-4).. \quad 19 \quad 15 \quad 11 \quad 8 \quad 5 \quad 4 \quad 0) \\
 & & (k{-}2 \quad k{-}6 \quad ..(-4).. \quad 17 \quad 13 \quad 10 \quad 7 \quad 3 \quad 2 \quad k{-}1) \\
\sigma_b & = & (k{-}2 \quad k{-}4 \quad ..(-4).. \quad 19 \quad 15 \quad 11 \quad 8 \quad 5 \quad 4 \quad 0) \\
 & & (k{-}3 \quad k{-}6 \quad ..(-4).. \quad 17 \quad 13 \quad 10 \quad 7 \quad 3 \quad 2 \quad k{-}1) \\
I & = & \{1, 6, 9, 12, 14, ..(2).., k{-}5\} \quad K = \frac{k+9}{4}
\end{array}$$

-1 -2 -3 -4 -5 -6 -7 2 1 0

$u_a = \mathrm{a}\,\mathbf{b}\,\mathrm{c\,d\,e\,f}\,\mathbf{g}\,\mathrm{h\,i}\,\mathbf{j}\,\mathrm{k\,l}\,\mathbf{m}\,\mathrm{n}\,\mathbf{o}\,\mathrm{p}\,\mathbf{q}\,\mathrm{r}\,\mathbf{s}\ \ldots\ \underline{\mathrm{s}}\,\underline{\mathbf{t}}\,\underline{\mathrm{u}}\,\underline{\mathbf{v}}\,\underline{\mathrm{w}}\,\underline{\mathrm{x}}\,\underline{\mathrm{y}}$

-1 -2 -3 -4 -5 -6 2 1 0

$\underline{\mathrm{z}}\,\mathbf{b}\,\mathrm{a\,c\,e\,d}\,\mathbf{g}\,\mathrm{f\,h}\,\mathbf{j}\,\mathrm{i\,k}\,\mathbf{m}\,\mathrm{l}\,\mathbf{o}\,\mathrm{n}\,\mathbf{q}\,\mathrm{p}\,\mathbf{s}\ \ldots\ \underline{\mathrm{q}}\,\underline{\mathbf{t}}\,\underline{\mathrm{s}}\,\underline{\mathbf{v}}\,\underline{\mathrm{u}}\,\underline{\mathrm{w}}\,\underline{\mathrm{y}}$

-1 -2 -3 -4 -5 -6 -7 2 1 0

$u_b = \mathrm{a}\,\mathbf{b}\,\mathrm{c\,d\,e\,f}\,\mathbf{g}\,\mathrm{h\,i}\,\mathbf{j}\,\mathrm{k\,l}\,\mathbf{m}\,\mathrm{n}\,\mathbf{o}\,\mathrm{p}\,\mathbf{q}\,\mathrm{r}\,\mathbf{s}\ \ldots\ \underline{\mathrm{s}}\,\underline{\mathbf{t}}\,\underline{\mathrm{u}}\,\underline{\mathbf{v}}\,\underline{\mathrm{w}}\,\underline{\mathrm{x}}\,\underline{\mathrm{y}}$

-1 -2 -3 -4 -5 -6 2 1 0

$\underline{\mathrm{z}}\,\mathbf{b}\,\mathrm{a\,c\,e\,d}\,\mathbf{g}\,\mathrm{f\,h}\,\mathbf{j}\,\mathrm{i\,k}\,\mathbf{m}\,\mathrm{l}\,\mathbf{o}\,\mathrm{n}\,\mathbf{q}\,\mathrm{p}\,\mathbf{s}\ \ldots\ \underline{\mathrm{q}}\,\underline{\mathbf{t}}\,\underline{\mathrm{s}}\,\underline{\mathbf{v}}\,\underline{\mathrm{u}}\,\underline{\mathrm{x}}\,\underline{\mathrm{w}}$

Case $k \equiv 4 \mod 8$, $\Delta = -1$:

$$\begin{aligned}
\sigma_a &= (k{-}3 \quad k{-}4 \quad ..(-4).. \quad 16 \quad 12 \quad 8 \quad 5 \quad 4 \quad 0) \\
& (k{-}2 \quad k{-}6 \quad ..(-4).. \quad 14 \quad 10 \quad 7 \quad 3 \quad 2 \quad k{-}1) \\
\sigma_b &= (k{-}2 \quad k{-}4 \quad ..(-4).. \quad 16 \quad 12 \quad 8 \quad 5 \quad 4 \quad 0) \\
& (k{-}3 \quad k{-}6 \quad ..(-4).. \quad 14 \quad 10 \quad 7 \quad 3 \quad 2 \quad k{-}1) \\
I &= \{1, 6, 9, 11, ..(2).., k{-}5\} \quad K = \tfrac{k+8}{4}
\end{aligned}$$

u_a =
-1 -2 -3 -4 -5 -6 2 1 0
a **b** c d e f **g** h i **j** k **l** m **n** o **p** … s **t** u **v** w x y
-1 -2 -3 -4 -5 2 1 0
z **b** a c e d **g** f h **j** i **l** k **n** m **p** … q **t** s **v** u w y

u_b =
-1 -2 -3 -4 -5 -6 2 1 0
a **b** c d e f **g** h i **j** k **l** m **n** o **p** … s **t** u **v** w x y
-1 -2 -3 -4 -5 2 1 0
z **b** a c e d **g** f h **j** i **l** k **n** m **p** … q **t** s **v** u x w

Case $k \equiv 5 \mod 8$, $\Delta = -1$:

$$\begin{aligned}
\sigma_a &= (k{-}3 \quad k{-}4 \quad ..(-4).. \quad 9 \quad 5 \quad 4 \quad 0) \\
& (k{-}2 \quad k{-}6 \quad ..(-4).. \quad 7 \quad 3 \quad 2 \quad k{-}1) \\
\sigma_b &= (k{-}2 \quad k{-}4 \quad ..(-4).. \quad 9 \quad 5 \quad 4 \quad 0) \\
& (k{-}3 \quad k{-}6 \quad ..(-4).. \quad 7 \quad 3 \quad 2 \quad k{-}1) \\
I &= \{1, 6, ..(2).., k{-}5\} \quad K = \tfrac{k+7}{4}
\end{aligned}$$

u_a =
-1 -2 -3 -4 2 1 0
a **b** c d e f **g** h **i** j **k** … s **t** u **v** w x y
-1 -2 -3 -4 2 1 0
z **b** a c e d **g** f **i** h **k** … q **t** s **v** u w y

u_b =
-1 -2 -3 -4 2 1 0
a **b** c d e f **g** h **i** j **k** … s **t** u **v** w x y
-1 -2 -3 -4 2 1 0
z **b** a c e d **g** f **i** h **k** … q **t** s **v** u x w

Case $k \equiv 6 \mod 8$, $\Delta = -1$:

$$\begin{aligned}
\sigma_a &= (k{-}3 \quad k{-}4 \quad ..(-4).. \quad 26 \quad 22 \quad 18 \quad 15 \quad 13 \quad 10 \quad 8 \quad 5 \quad 4 \quad 0) \\
& (k{-}2 \quad k{-}6 \quad ..(-4).. \quad 24 \quad 20 \quad 17 \quad 14 \quad 12 \quad 9 \quad 7 \quad 3 \quad 2 \quad k{-}1) \\
\sigma_b &= (k{-}2 \quad k{-}4 \quad ..(-4).. \quad 26 \quad 22 \quad 18 \quad 15 \quad 13 \quad 10 \quad 8 \quad 5 \quad 4 \quad 0) \\
& (k{-}3 \quad k{-}6 \quad ..(-4).. \quad 24 \quad 20 \quad 17 \quad 14 \quad 12 \quad 9 \quad 7 \quad 3 \quad 2 \quad k{-}1) \\
I &= \{1, 6, 11, 16, 19, 21, ..(2).., k{-}5\} \quad K = \tfrac{k+14}{4}
\end{aligned}$$

u_a =
-1 -2 -3 -4 -5 -6 -7 -8 -9 -10 2 1 0
a **b** c d e f **g** h i j k **l** m n o p **q** r s **t** u **v** w **x** y **z** … s **t** u **v** w x y
-1 -2 -3 -4 -5 -6 -7 -8 -9 2 1 0
z **b** a c e d **g** f i h j **l** k n m o **q** p r **t** s **v** u **x** w **z** … q **t** s **v** u w y

u_b =
-1 -2 -3 -4 -5 -6 -7 -8 -9 -10 2 1 0
a **b** c d e f **g** h i j k **l** m n o p **q** r s **t** u **v** w **x** y **z** … s **t** u **v** w x y
-1 -2 -3 -4 -5 -6 -7 -8 -9 2 1 0
z **b** a c e d **g** f i h j **l** k n m o **q** p r **t** s **v** u **x** w **z** … q **t** s **v** u x w

Case $k \equiv 7 \mod 8$, $\Delta = -1$:

$$\begin{aligned}
\sigma_a &= (k{-}2 \quad 1 \quad 3 \quad 6 \quad ..(4).. \quad k{-}9 \quad k{-}5) \\
& (k{-}1 \quad 2 \quad 4 \quad 8 \quad ..(4).. \quad k{-}7 \quad k{-}3) \\
\sigma_b &= (k{-}1 \quad 1 \quad 3 \quad 6 \quad ..(4).. \quad k{-}9 \quad k{-}5) \\
& (k{-}2 \quad 2 \quad 4 \quad 8 \quad ..(4).. \quad k{-}7 \quad k{-}3) \\
I &= \{1, 6, 11, 16, 18, ..(2).., k{-}5\} \quad K = \tfrac{k+13}{4}
\end{aligned}$$

u_a =
-1 -2 -3 -4 -5 -6 -7 -8 -9 2 1 0
a **b** c d e f **g** h i j k **l** m n o p **q** r **s** t **u** v **w** … s **t** u **v** w x y
-1 -2 -3 -4 -5 -6 -7 -8 2 1 0
z **b** a c e d **g** f i h j **l** k n m o **q** p **s** r **u** t **w** … q **t** s **v** u w y

u_b =
-1 -2 -3 -4 -5 -6 -7 -8 -9 2 1 0
a **b** c d e f **g** h i j k **l** m n o p **q** r **s** t **u** v **w** … s **t** u **v** w x y
-1 -2 -3 -4 -5 -6 -7 -8 2 1 0
z **b** a c e d **g** f i h j **l** k n m o **q** p **s** r **u** t **w** … q **t** s **v** u x w

4.1 Avoiding short repetitions

In this section, we suppose that $\mathbf{w}' = h(\mathbf{u})$ for an infinite binary word $\mathbf{u}$, and that $\mathbf{w} = M_k(\mathbf{w}')$.

Lemma 2. $\mathbf{w}$ *has no forbidden repetition of excess at most* 5.

Proof. We use characterization of stabilizing words of Carpi, Shur and Gorbunova [4, 16]. Since $\mathbf{w}'$ is $(00, 111)$-free, $\mathbf{w}$ has no forbidden repetition of excess 1 and 2. $\mathbf{w}'$ does not contains a factor of the form $uB^{k-4}u$, for a $u \in B^4$. Thus by Carpi's characterization of 3-stabilizing words, $\mathbf{w}$ has no forbidden repetition of excess 3 [4]. Finally, Shur and Gorbunova showed that a word on alphabet $k \geq 8$ which has no forbidden repetitions of excess 1, 2 and 3 has no forbidden repetition of excess 4 and 5 [16]. ☐

Let $u_a = (M_k(h(0)))[0; s-1]$ and $u_b = (M_k(h(1)))[0; s-1]$. Let $\mathcal{E} = \{i \in \{0, \ldots, s-1\} : u_a[i] = u_b[i]\}$. One can show on Table 2 that $\{0, \ldots, k-5\} \subset \mathcal{E}$. The word $\mathbf{w}$ is a concatenation of words isomorphic to u_a and u_b. For every n, $\mathbf{w}[n\cdot s; n\cdot s + k - 1]$ is uniquely determined by $\varphi(h(\mathbf{u}[0; n-1])) = \varphi'(\mathbf{u}[0; n-1])$. More precisely:

Proposition 4. *For every* $n \geq 0$ *and* $i \in \{0, \ldots, k-2\}$*,* $\mathbf{w}[n\cdot s + i] = \varphi'(\mathbf{u}[0; n-1])[i]$.

For every $i \in I$ and $n \geq 0$, we have $\mathbf{w}[i + n\cdot s] = i$ (note that $k - 1 \neq I$). Since $s = 2k + 2$ or $s = 2k - 2$, there is an other occurrence of i exactly in mid-position, and we have $\mathbf{w}[i + n\cdot s + \frac{s}{2}] = i$. Thus every letter at position $\mathcal{I} = I + \frac{s}{2}\cdot\mathbb{N}$ is independent of $\mathbf{u}$.

Let $\mathcal{J} = (\mathcal{E} \setminus \mathcal{I}) + s\cdot\mathbb{N}$, and let $\gamma : \mathcal{J} \to \{0, \ldots K-1\}$ such that $\mathbf{w}[j] \in \{a_{\gamma(j)}, b_{\gamma(j)}\}$. Note that γ is well defined, and is independent of $\mathbf{u}$, by the properties of ODCC.

Proposition 5. $\gamma(x + s) = 1 + \gamma(x)$ *(modulo K).*

Looking at the permutations in Table 2, we see:

Fact 2. *For every* $x \in \mathbb{N}$*,* $\{x, \ldots, x + 5\} \cap \mathcal{I} \neq \emptyset$.

Suppose that $\mathbf{w}$ has a forbidden repetition. Let l be the least integer such that $\mathbf{w}$ has a forbidden repetition of period l. Let (p, e) be an forbidden repetition of period l, and let t be an integer such that $\mathbf{w}[t; \infty]$ has pe as prefix. Suppose w.l.o.g than $t \geq s\cdot K$ (since $h(w_{TM})$ is recurrent, if $\mathbf{w}$ has a forbidden repetition of period l, it has infinitely many forbidden repetitions of period l).

Lemma 3. *Let* $i \in I$ *and* $j \notin I$.

- *The distance in* $\mathbf{w}$ *of two consecutive occurrences of* i *is* $\frac{s}{2} \in \{k-1, k+1\}$.
- *The distance in* $\mathbf{w}$ *of two consecutive occurrences of* j *is* $k-1$*,* k *or* $k+1$.
- *The distance in* $\mathbf{w}$ *between two occurrences of* ij *is at least* $\frac{(k+1)(k-1)}{2}$.
- *The distance in* $\mathbf{w}$ *between two occurrences of* ji *is at least* $\frac{(k+1)(k-1)}{2}$.

Proof. The distance between two consecutive occurrences of the same letter is $k-1$, k or $k+1$, by definition of M_k. Moreover, the distance between two consecutive occurrences of i is $\frac{s}{2}$ by the previous remark on $\mathcal{I}$. Now suppose $\Delta = 1$ (the case $\Delta = -1$ is similar). Suppose that ij appears in $\mathbf{w}$ at position x, and let $y > x$ be the next occurrence of ij in $\mathbf{w}$. The next occurrence of j cannot be at position $x+k+2$ otherwise $\mathbf{w}$ would have an forbidden repetition of excess 2. So the next occurrence of j is at position $x+k$. The gap between the q-th next occurrence of i after x and the q-th occurrence of j after x cannot decrease, since the is are spaced by $k+1$. Suppose that there are n occurrences of i between x and y. Then $y-x=(k+1)(n-1)$ and $y-x \geq (k-1)\cdot n$. So $n \geq \frac{k+1}{2}$, and $y-x \geq \frac{(k+1)(k-1)}{2}$. The distance in $\mathbf{w}$ between two occurrences of ij is also at least $\frac{(k+1)(k-1)}{2}$, by symmetry. □

Lemma 4. *l is a multiple of $\frac{s}{2}$.*

Proof. By Fact 2, e contains a $i \in I$, which are spaced by $\frac{s}{2}$. □

Lemma 5. *$|e| \geq \frac{k+1}{2}$.*

Proof. Since $|e| \geq 5$, e contains a factor ij or ji for a $i \in I$ and $j \notin I$. By Lemma 3, $|p| \geq \frac{(k-1)(k+1)}{2}$. Thus $|e| > \frac{k+1}{2}$. □

Fact 3. *For every $x, y \in \mathcal{J} \cap \in \{0, \frac{s}{2}-1\}$ such that $x \neq y$ and $\gamma(x)=\gamma(y)$, we have $\gamma(x+\frac{s}{2}) \neq \gamma(y+\frac{s}{2})$.*

Lemma 6. *l is a multiple of s.*

Proof. We know that l is a multiple of $\frac{s}{2}$. Suppose that it is not a multiple of s. Since $|e| > \frac{k+1}{2} \geq 16$, there are $x, y \in \{t, \dots t+|e|-1\}$ such that $x, y \in \mathcal{J}$, $y-x \in \{1,2\}$, $\gamma(x)=\gamma(y)$ and $\left\lfloor \frac{x}{s} \right\rfloor = \left\lfloor \frac{y}{s} \right\rfloor$. Thus, we have a contradiction with Fact 3. □

Lemma 7. *l is a multiple of $s{\cdot}K$.*

Proof. Suppose that $l=n{\cdot}s$. Again, e contains an occurrence $x \in \mathcal{J}$ of a $j \notin I$. We have $\gamma(x) \equiv \gamma(x+s{\cdot}n) \mod K$. Thus by Proposition 5, $n \equiv 0 \mod K$. □

Lemma 8. *If $\mathbf{u}$ is $(000,111)$-free, then $|p| > s{\cdot}K$.*

Proof. Suppose that $|p| = s{\cdot}K$. Since $|e| \geq \frac{k}{2}$, e contains two occurrences x and y such that $x, y \in \mathcal{J}$, $\left\lfloor \frac{x}{s} \right\rfloor = \left\lfloor \frac{y}{s} \right\rfloor$, and $\gamma(y) \equiv \gamma(x)+1 \mod K$. Let $v = \mathbf{u}\left[\left\lfloor \frac{x}{s} \right\rfloor - K; \left\lfloor \frac{x}{s} \right\rfloor - 1\right]$. Couples i and $i+1$ are good on the same time on v, which is impossible by Proposition 3 since u is $(000,111)$-free. □

4.2 Avoiding short kernel repetition

From now on, $\mathbf{w}' = h(w_{TM})$ and $\mathbf{w} = M_k(\mathbf{w}')$. If $\mathbf{w}$ has a forbidden repetition, then by results in the previous Section, its period is a multiple of $s{\cdot}K$ and is at least $2{\cdot}s{\cdot}K$. In this case, its excess is at least $4{\cdot}K \geq k$, since $\frac{s}{2} \geq k-1$ and $K \geq \frac{k}{4}$. Thus this repetitions is a kernel repetition, and $\mathbf{w}'$ has a forbidden φ-kernel repetition. Let l be the least integer such that $\mathbf{w}'$ has a φ-kernel repetition (p', e') of period l and with $\frac{|e'|+k-1}{l} > \frac{1}{k-1}$. Let t be the least integer such that $\mathbf{w}'[t;\infty]$ has $p'e'$ as prefix. We suppose w.l.o.g. that the excess is maximal, that is $(p', \mathbf{w}'[t; t+|e'|])$ is not a repetition.

Note that t is a multiple of $\frac{s}{2}$, since the last letters of $h(0)$ and $h(1)$ differs, and by minimality of t.

We now denote by w'_{TM} the infinite binary word defined as the fixed point of the following morphism :

$$\mu : \begin{cases} 0 & \mapsto 1010 \\ 1 & \mapsto 1011. \end{cases}$$

This word is the derivative word of w_{TM}, and it can also be constructed as a Toeplitz word for the pattern $101\bullet$, that is $w'_{TM} = \lim_{n\to\infty} T_k$ with $T_0 = \bullet^\omega$, and $T_{i+1} = F(T_i)$, where $F(w)$ is the word obtained from $(101\bullet)^\omega$ by replacing the sequence of all occurrences of $\bullet$ by w [1].

For $n \geq 1$ and $\delta \geq 1$, let $\chi_{(n,\delta)}$ be the infinite word such that for every $x \in \mathbb{N}$ (the sum is over GF(2)) :

$$\chi_{(n,\delta)}[x] = \sum_{i=0}^{n-1} w'_{TM}[x + i{\cdot}\delta]$$

Let $f_0(n)$ be the least r such that for every odd δ, $\chi_{(n,\delta)}$ is 0^r-free, and let $f_1(n)$ be the least r such that for every odd δ, $\chi_{(n,\delta)}$ is 1^r-free. The following comes form easy observations.

Proposition 6. *Let δ be odd and $x > 0$. Then*

- $f_0(1) = 2$ *and* $f_1(1) = 4$,
- $f_0(2) \leq 3$,
- $f_0(3) \leq 8$ *and* $f_1(3) \leq 4$,
- $f_0(4x) \leq 4f_0(x)$ *and* $f_1(4x) \leq 4f_1(x)$,
- $f_0(4x+1) \leq 4f_0(x)$,
- $f_0(4x+2) \leq 4f_1(x+1)$ *and* $f_0(4x+2) \leq 4f_1(x)$,
- $f_0(4x+3) \leq 4f_0(x)$.

Proof. Since w'_{TM} is $(00, 1111)$-free and contains 0 and 111, and $\chi_{(2,\delta)} \in (11BB)^\omega$, we have $f_0(1) = 2$, $f_1(1) = 4$ and $f_0(2) \leq 3$.

Suppose now that $\delta \equiv 1 \mod 4$. The case $\delta \equiv 3 \mod 4$ is proved similarly. We use the Toeplitz definition of w'_{TM}. Let $w_{(n,d)}[i] = \chi_{(n,\delta)}[4i + d]$ for every $d \in \{0, 1, 2, 3\}$, $i \geq 0$ and $n > 0$. We have $w_{(3,3)} = w_{(4,3)} = w'_{TM}$, and 0^8 cannot appear in $\chi_{(3,\delta)}$ (resp. 1^{16} cannot appear in $\chi_{(4,\delta)}$) because w'_{TM} is 0^2-free (resp 1^4-free). Moreover $w_{(3,0)} = 0^\omega$, so $\chi_{(3,\delta)}$ is 1^4-free. Similarly, other results follow from facts that:

- $w_{(4n,1)} = w_{(4n+1,1)} = \chi_{(n,\delta)}[\delta + 1; \infty]$.
- $w_{(4n+2,0)} = \chi_{(n,\delta)}$.
- $w_{(4n,2)} = \overline{\chi_{(n,\delta)}[\frac{\delta+1}{2}; \infty]}$.
- $w_{(4n+2,0)} = \overline{\chi_{(n,\delta)}}$.

$\square$

By previous observations, we have:

Proposition 7. *For every $x \in \{1, \dots 17\}$, $f_0(x) \leq 32$.*

One can note than more generally, we have $\max(f_0(x), f_1(x)) \leq 4x$ for every $x \geq 1$. By Proposition 7, we have:

Proposition 8. *For every $n \leq 17$ and every odd $\delta \geq 33$, $\chi_{(n,\delta)}$ has no factor $0^{\delta-1}$.*

The least integer r such that $\chi_{(n,\delta)}$ is 0^r-free, for a fixed n and δ, can be computed by computer : $\mu^i(1)$ contains every factor of w'_{TM} of size at most $4^{(i-1)}+1$, so every factor of size n of w'_{TM} appear in its prefix of size $4n$. Thus the prefix of $\chi_{(n,\delta)}$ of size $4(l+(n-1)\delta)$ contains every prefix of $\chi_{(n,\delta)}$ of size at most l. Computer checks show that:

Proposition 9.

(i) For every $n \leq 17$ and every odd $11 \leq \delta \leq 31$, if $\chi_{(n,\delta)}$ has a factor $0^{\delta-1}$, then $(n,\delta) \in \{15,17\} \times \{11,15,17\}$.

(ii) w_{TM} has no factor w of the size $11{\cdot}n$ such that $\varphi'(w) = \mathrm{Id}_k$, for $n \in \{15,17\}$.

(iii) The maximal excess of a φ'-kernel repetition of period $n{\cdot}K$ in w_{TM}, for $n \leq 17$ and $11 \leq K$, is one.

Lemma 9. *If $K \geq 11$, then $l \geq 18{\cdot}s{\cdot}K$.*

Proof. Suppose that $K \geq 11$ and $l = n{\cdot}s{\cdot}K$ with $n \leq 17$.

Case 1: s divides t. Since $s{\cdot}K$ divides l, w_{TM} has a φ'-kernel repetition (p'', e'') with $p' = h(p'')$ and $|e''| \geq \left\lfloor \frac{|e'|}{s} \right\rfloor$. By Proposition 3 $\chi_{(n,K)}$ contains the factor 0^{K-1}. By Proposition 9 (i), $K \in \{11,15,17\}$, and by (ii), $K \in \{15,17\}$. On the other hand, we have $|e'| \geq 3$, and we get the contradiction with (iii).

Case 2: s does not divide t. Then $t \equiv \frac{s}{2} \mod s$. Note that in this case $n \in \{2,3\}$, otherwise the excess of (p', e') would be more than $\frac{s}{2}$, and t would not be minimal. We have $p' = \kappa_a h(w) \iota_b$ or $p' = \kappa_b h(w) \iota_a$, where w has size $K-1$. We suppose $p' = \kappa_a h(w) \iota_b$ (the other case is similar). Then w is followed by b in w_{TM}. Let $p_1 = h(w)\iota_b\kappa_a$ and $p_2 = h(w)\iota_b\kappa_b = h(wb)$. Then $\varphi(p_1) = \mathrm{Id}_k$ and $\varphi'(wb) = \varphi(p_2) = \varphi(p_1)\varphi(10)^{-1}\varphi(01) = (k{-}3 \quad k{-}1 \quad k{-}2)$. Thus for every $i \in \{0, \dots k{-}4\}$, $\varphi'(wb)[i] = i$. $\chi_{(n,K)}$ has a factor 0^{K-1}, which is impossible by Proposition 9 (i). ☐

4.3 Avoiding long kernel repetitions

We know by the previous lemmas that $l \geq 18{\cdot}s{\cdot}K$, and thus $|e'| \geq 8(k+1) \geq 4s$.

Since l and t are multiples of s, and by maximality of e', w_{TM} has a φ'-kernel repetition (p_1, e_1) such that $p' = h(p_1)$ and $e' = h(e_1)L$. Thus:

$$\frac{|e_1|{\cdot}s + \ell + k - 1}{|p_1|{\cdot}s} = \frac{|e'| + k - 1}{l} > \frac{1}{k-1}$$

Note that $|e_1| \geq 4$. The definition of markable words are those of [15, Section 3]. By [15, Corollary 9], and since every factor of w_{TM} of size at least 4 is (ν, w_{TM})-markable, $h(w_{TM})$ has a φ'-kernel repetition (p_2, e_2) with $p_1 = \nu(p_2)$ and $e_1 = \nu(e_2)$.

Thus **w** has a repetition of exponent E, with :

$$E = \frac{|e_2| \cdot s + \ell + k - 1}{|p_2| \cdot s} > \frac{|e_1| \cdot s + \ell + k - 1}{|p_1| \cdot s} > \frac{1}{k-1}$$

which is a forbidden repetition of period $\frac{l}{2}$. We have a contradiction.

5 Alphabets with less than 9 letters

Ochem's conjecture is already proved for several cases. Chalopin and Ochem proved the first case of the conjecture for 5 letters, and the second case for 6 letters [5]. Rao proved it for $9 \leq k \leq 38$ [15]. Theorem 1 prove it for $k \geq 24$. We give here constructions for the last cases.

All the words follow the same construction. Their Pansiot code is the morphic word $h(g^{\infty}(a))$, where h and g are given in the following table ($\Delta = 1$ for first case of the conjecture, and $\Delta = -1$ for the second case).

k	Δ	g	h	x_h	x_g
5	1	$a \to abaababaabababaabaab$ $aababababaababaabaabab$ $b \to abaabaabaababaababaa$ $baabaababababaababaaba$	$a \to 101101$ $b \to 101010$	13	40
6	-1	$a \to aaab$ $b \to bbba$	$a \to 1010110110$ $b \to 1011010101$	39	5
6	1	$a \to abaabb$ $b \to abbaba$	$a \to 10101011010101010110101$ $b \to 1010101101011010101110$	46	12
7	-1	$a \to ababa$ $b \to bbabb$	$a \to 1010101011011010101010101010$ $b \to 10101010110110110110110$	24	7
7	1	$a \to aaab$ $b \to bbba$	$a \to 1011010110110110$ $b \to 1011011010110101$	63	5
8	-1	$a \to abbba$ $b \to baaab$	$a \to 10101010110101010110101$ $b \to 1010101101011010101110$	30	5
8	1	$a \to aab$ $b \to baa$	$a \to 1011011011010101101010101$ $b \to 10110110110110101010101011010$	27	7

Table 3: Construction for cases $k \leq 8$.

The proof follows ideas of [15] and [3]. Let $\mathbf{w}' = g^{\infty}(a)$ and $\mathbf{w} = h(\mathbf{w}')$ for one case in Table 3. By Remark 1, $M_k(w)$ has at least one letter of frequency $\frac{1}{k+\Delta}$. We show that $M_k(w)$ is a Dejean word. Let $s = |h(a)| = |h(b)|$ and $s' = |g(a)| = |g(b)|$. Let ℓ (resp. ℓ') be the size of the largest common prefix of $h(a)$ and $h(b)$ (resp. $g(a)$ and

$g(b))$. Let $\sigma_a = \varphi(h(a))$ and $\sigma_b = \varphi(h(b))$. Let $\varphi' : \{a,b\}^* \to \mathbb{S}_k$ be the morphism such that $\varphi'(a) = \sigma_a$ and $\varphi'(b) = \sigma(b)$. One can easily check by computer that:

Fact 4. *There is a $\sigma \in \mathbb{S}_k$ such that for every $x \in \{a,b\}$, $\varphi'(g(x)) = \sigma \cdot \varphi'(x) \cdot \sigma^{-1}$.*

Let x_h and x_h defined in Table 3. One can check by computer that:

Fact 5.

- *Every factor w of $\mathbf{w}$ such that $|w| \geq x_h$ is $(h, \mathbf{w}')$-markable.*
- *Every factor w of $\mathbf{w}'$ such that $|w'| \geq x_g$ is $(g, \mathbf{w}')$-markable.*

A Ψ-kernel repetition (p,e) is *weak* if $\frac{|e|+\frac{6}{5}}{|p|} \geq \frac{1}{k-1}$. The following corollary follow from [15, Lemma 5], Fact 4 and Fact 5.

Corollary 1.

- *If $\mathbf{w}$ has a φ-kernel-repetition (p,e) with $|e| \geq x_h$, then $\mathbf{w}'$ has a φ'-kernel-repetition (p',e') with $s \cdot |e'| + \ell \geq |e|$ and $s \cdot |p'| = |p|$.*
- *If $\mathbf{w}'$ has a φ'-kernel-repetition (p,e) with $|e| \geq x_g$, then $\mathbf{w}'$ has a φ'-kernel-repetition (p',e') with $s' \cdot |e'| + \ell' \geq |e|$ and $s' \cdot |p'| = |p|$.*

Therefore if $M_k(\mathbf{w})$ has a forbidden kernel repetition of excess at least $k-1+x_h$, then $\mathbf{w}'$ has a φ'-kernel repetition (p',e') such that

$$\frac{s \cdot |e'| + k - 1 + \ell}{s \cdot |p'|} > \frac{1}{k-1}.$$

Note that (p',e') is a weak repetition since $\frac{k-1+\ell}{s} \leq \frac{6}{5}$. Similarly, if $\mathbf{w}'$ has a weak φ'-kernel repetition (p',e') of excess at least x_g, then $\mathbf{w}'$ has a φ'-kernel repetition (p'',e'') such that

$$\frac{s' \cdot |e''| + \ell' + \frac{6}{5}}{s' \cdot |p''|} > \frac{1}{k-1}.$$

Then (p'',e'') is also a weak repetition since $\frac{\ell' + \frac{6}{5}}{s'} \leq \frac{6}{5}$. The fact that $M_k(\mathbf{w})$ is a Dejean word follow from the computer checked facts that:

- $M_k(\mathbf{w})$ has no forbidden repetition of excess at most $k-1+x_h$.
- $\mathbf{w}'$ has no weak repetition of excess at most x_g.

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