Last Cases of Dejean's Conjecture

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Abstract

Dejean conjectured that the repetition threshold for a k-letter alphabet is $\frac{k}{k-1}$ when $k \geq 5$. Dejean's conjecture has already been proved for $k \leq 14$ and for $k \geq 27$. We present here a proof for $8 \leq k \leq 38$. The same technique is also applied to prove Ochem's stronger version of the conjecture for $9 \leq k \leq 38$.

Keywords: Infinite word, Power, Threshold repetition, Dejean's conjecture, Thue-Morse sequence, Ochem's conjecture

1. Introduction

A *square* in a word is a non-empty factor of the form uu. It is well known that every binary word of length at least 4 has a square, and that squares can be avoided in an infinite ternary word [1, 13, 14].

A *repetition* in a word w is a pair of words (p, q) such that pq is a factor of w, p is non-empty, and q is a prefix of pq. The *exponent* of a repetition (p, q) is $\frac{|pq|}{|p|}$. Squares are thus repetitions of exponent 2.

A word is said x-*free* (resp. x^+-free) if it does not contain a repetition of exponent y with $y \geq x$ (resp. $y > x$). For an integer $k \geq 2$, the *repetition threshold* for k letters, denoted by $\mathrm{RT}(k)$, is the infimum over the set of x such that there exists an infinite x-free word over a k-letter alphabet, or equivalently the smallest x such that there exists an infinite x^+-free word over a k-letter alphabet.

It is well known that the Thue-Morse sequence avoids factors of the form $uu\alpha$ where α is the first letter of u [1, 13, 14]. Thus repetitions of the Thue-Morse sequence have exponent at most 2, and $\mathrm{RT}(2) = 2$. Dejean [7] conjectured that for every $k \geq 2$, $\mathrm{RT}(k) = \mathrm{r}_k$, where:

$$\mathrm{r}_k = \begin{cases} \frac{7}{4} & \text{if } k = 3 \\ \frac{7}{5} & \text{if } k = 4 \\ \frac{k}{k-1} & \text{otherwise.} \end{cases}$$

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Dejean showed that $\mathrm{RT}(k) \geq \mathrm{r}_k$ for every $k \geq 2$. In order to prove the conjecture for a fixed k, it is thus sufficient to construct an infinite r_k^+-free word over a k-letter alphabet.

Dejean's conjecture has successively been proved for $k = 3$ [7], $k = 4$ [12], $5 \leq k \leq 11$ [9], $12 \leq k \leq 14$ [8], $k \geq 33$ [2], $k \geq 30$ [4] and $k \geq 27$ [5]. We present here a proof for $8 \leq k \leq 38$. We prove also Ochem's stronger version of the conjecture for $9 \leq k \leq 38$

2. Pansiot's Code and Kernel Repetitions

We denote by B the set $\{0, 1\}$, and by Σ_k the set $\{1, \ldots, k\}$ (where $k \geq 2$). Let Σ^* be the set of finite words over the alphabet Σ, and let $\Sigma^+ = \Sigma^* \setminus \{\epsilon\}$ (where ϵ is the empty word). Let Σ^ω be the set of infinite words over Σ. We denote by $w[i]$ the i-th letter of word w, and we denote by $w[i : j]$ (where $i \leq j$) the word $w[i]w[i+1] \ldots w[j]$. Let $\mathbb{S}_k$ be the group of permutations of Σ_k, and $\mathrm{Id}_k \in \mathbb{S}_k$ be the identity permutation.

Pansiot [12] noticed that if a word on the alphabet Σ_k is $\frac{k-1}{k-2}$-free, then it can be encoded by a binary word. Let $k \geq 3$ and let w be a (possibly infinite) $\frac{k-1}{k-2}$-free word over the alphabet Σ_k, of length at least $k-1$. Then every factor of length $k-1$ consists of $k-1$ different letters. The *Pansiot code* of w is the binary word $P_k(w)$ such that for all $i \in \{1, \ldots, |w| - k + 1\}$ (for all $i \geq 1$ if w is infinite):

$$P_k(w)[i] = \begin{cases} 0 & \text{if } w[i+k-1] = w[i] \\ 1 & \text{if } w[i+k-1] \notin \{w[i], \ldots, w[i+k-2]\}. \end{cases}$$

Note that w is uniquely defined by $P_k(w)$ and $w[1 : k-1]$. One can define an inverse operation: for a (possibly infinite) binary word w, $M_k(w)$ is the word on the alphabet Σ_k such that for all $i \in \{1, \ldots, |w| + k - 1\}$ (for all $i \geq 1$ if w is infinite):

$$M_k(w)[i] = \begin{cases} i & \text{if } i < k, \\ M_k(w)[i-k+1] & \text{if } i \geq k \text{ and } w[i-k+1] = 0, \\ \alpha & \text{otherwise,} \end{cases}$$

where $\{\alpha\} = \Sigma_k \setminus \{M_k(w)[i-k+1], \ldots, M_k(w)[i-1]\}$. Note that if $w[i] = i$ for all $i < k$, then $M_k(P_k(w)) = w$.

Moulin Ollagnier showed that Pansiot's coding can also be viewed by the way of an action on the symmetric group $\mathbb{S}_k$ [9]. Let Ψ be the morphism between the free monoid B^* and $\mathbb{S}_k$ such that $\Psi(0) = \sigma_0$ and $\Psi(1) = \sigma_1$, where:

$$\sigma_0 = \begin{bmatrix} 1 & 2 & \cdots & k-2 & k-1 & k \\ 2 & 3 & \cdots & k-1 & 1 & k \end{bmatrix} \quad \text{and} \quad \sigma_1 = \begin{bmatrix} 1 & 2 & \cdots & k-2 & k-1 & k \\ 2 & 3 & \cdots & k-1 & k & 1 \end{bmatrix}.$$

One can easily show that for all $i \geq 0$ and $1 \leq j \leq k-1$, $M_k(w)[i+j] = \Psi(w[1 : i])(j)$.

Suppose that an infinite word w validates Dejean's conjecture for k letters with $k \geq 3$. We can suppose without loss of generality that $w[i] = i$ for all $i < k$, and that w can be encoded by its Pansiot's code, since $\mathrm{r}_k < \frac{k-1}{k-2}$. Thus Dejean's conjecture is true for a $k \geq 3$ if and only if there exists an infinite binary word w such that $M_k(w)$ is r_k^+-free.

Let w be a (possibly infinite) word on the alphabet Σ. A Φ-*kernel repetition* in w (where $\Phi : \Sigma^* \to \mathbb{S}_k$ is a morphism) is a pair (p, q) such that (p, q) is a repetition in w and $\Phi(p) = \mathrm{Id}_k$.

A repetition is *forbidden* if $\frac{|pq|}{|p|} > \mathrm{r}_k$. A repetition is a *short repetition* if $|q| < k - 1$, otherwise it is called a *kernel repetition*. Note that if (p, q) is a forbidden short repetition, then $|p| < \frac{k-2}{\mathrm{r}_k - 1}$. Moulin Ollagnier [9] showed that:

Proposition 1 ([9])**.** *Let $k \geq 3$ and w be a binary word. Then $M_k(w)$ has a kernel repetition (p, q) if and only if w has a Ψ-kernel repetition (p', q') with $|p'| = |p|$ and $p'q' = P_k(pq)$. (Note that $|q'| = |q| - k + 1$.)*

Moulin Ollagnier gives necessary and decidable conditions for a morphism h so that a fixed point of h has no forbidden Ψ-kernel repetition. He gives morphisms for $3 \leq k \leq 11$ which validate Dejean's conjecture. His ideas have then been adapted to morphic Sturmian words by Mohammad-Noori and Currie [8], proving conjecture for $7 \leq k \leq 14$. We extend here Moulin Ollagnier's ideas on words which are the image by a morphism of the Thue-Morse sequence.

3. Preliminary Results

We first introduce some notations and prove some technical results. They are simple adaptations of Moulin Ollagnier's ideas. Throughout this section, $f : \{x, y\}^* \to \{z, t\}^*$ denotes a morphism (possibly with $\{x, y\} = \{z, t\}$), such that:

(LL) the last letters of $f(x)$ and $f(y)$ differ, and

(PC) $\{f(x), f(y)\}$ is a prefix-code, *i.e.* $f(x)$ is not a prefix of $f(y)$, and $f(y)$ is not a prefix of $f(x)$.

Let L be the largest common prefix of $f(x)$ and $f(y)$, and let $\ell = |L|$.

Definition 2 (Interpretation, Markable)**.** Let w be an infinite word on the alphabet $\{x, y\}$. Let v be a non-empty factor of $f(w)$. An (f, w)-*interpretation* of v is a triplet (b, u, e) such that:

- u is a non-empty factor of w,
- the *beginning* b is a non-empty suffix of $f(u[1])$,
- the *end* e is a non-empty prefix of $f(u[|u|])$,
- $e'vb' = f(u)$, where $e'b = f(u[1])$ and $eb' = f(u[|u|])$.

A word u is (f,w)-*markable* if all its (f,w)-interpretations have the same beginning.

Proposition 3. *Let v be a factor of $f(w)$ and v' be a factor of v. If v' is (f,w)-markable, then v is (f,w)-markable.*

Proof. It suffices to show that if $v[1:|v|-1]$ or $v[2:|v|]$ is (f,w)-markable, then v is (f,w)-markable. Obviously, if $v[1:|v|-1]$ is (f,w)-markable, then v is (f,w)-markable.

Suppose that $v[2:|v|]$ is (f,w)-markable, and let b be the unique beginning of all its (f,w)-interpretations. Let $\gamma \in \{x,y\}$ such that b is a suffix of $f(\gamma)$. By definition, γ exists, and by condition (LL), γ is unique. Let e' be such that $e'b = f(\gamma)$. If e' is non-empty, then $v[1] = e'[|e'|]$, and $v[1]b$ is the only possible beginning of an (f,w)-interpretation of v. Otherwise, the only possible beginning of an (f,w)-interpretation of v is $v[1]$. □

Proposition 4. *Let v be an (f,w)-markable factor of $f(w)$. Then v has at most two (f,w)-interpretations. Moreover if v has two different (f,w)-interpretations (b,u_1,e_1) and (b,u_2,e_2), then $e_1 = e_2$, e_1 is a prefix of L, $|u_1| = |u_2|$ and u_1 and u_2 only differ by their last letter.*

Proof. Suppose that v has two different (f,w)-interpretations (b,u_1,e_1) and (b,u_2,e_2). By condition (LL), there is a unique $\alpha \in \{x,y\}$ such that b is a suffix of $f(\alpha)$. Thus $u_1[1] = u_2[1] = \alpha$. Let e' be such that $e'b = f(\alpha)$. Since $\{f(x), f(y)\}$ is a prefix-code (condition (PC)), there is a unique u' such that $f(u')$ is a prefix of $e'v$ and such that for every $\beta \in \{x,y\}$, $f(u'\beta)$ is not a prefix of $e'v$. Obviously, u' is a prefix of u_1 and u_2. Now let e'' be such that $f(u')e'' = e'v$.

If e'' is empty, then $u_1 = u_2 = u'$ and $e_1 = e_2 = f(u'[|u'|])$, contradiction. Thus $e_1 = e_2 = e''$ and $u_1 \neq u_2$. Moreover, since $e'v$ is a prefix of $f(u_1)$ and $f(u_2)$, u_1 and u_2 have length $|u'|+1$, thus e'' is a prefix of both $f(x)$ and $f(y)$, and e'' is a prefix of L. This implies also that there is no other (f,w)-interpretation of v, since every (f,w)-interpretation is of the form $(b, u'\gamma, e'')$ with $\gamma \in \{x,y\}$. □

A repetition (p,q) of w *extends* a repetition (p',q') if $|p| = |p'|$ and $p'q'$ is a proper factor of pq. A repetition (p,q) in w is *maximal* if there is no repetition (p',q') in w such that (p',q') extends (p,q). Note that if (p',q') extends (p,q), then (p,q) is a Φ-kernel repetition if and only if (p',q') is a Φ-kernel repetition (where $\Phi : \Sigma \to \mathbb{S}_k$ is a morphism), since that in this case p and p' are two conjugate words, and thus $\Phi(p) = \mathrm{Id}_k$ if and only if $\Phi(p') = \mathrm{Id}_k$.

Lemma 5. *Let $\Phi : \{z,t\}^* \to \mathbb{S}_k$ and $\Phi' : \{x,y\}^* \to \mathbb{S}_k$ be two morphisms such that:*

(CO) there is $\sigma \in \mathbb{S}_k$ such that for every $\alpha \in \{x,y\}$, $\Phi(f(\alpha)) = \sigma \cdot \Phi'(\alpha) \cdot \sigma^{-1}$.

Let (p,q) be a Φ-kernel repetition in $f(w)$ such that q is (f,w)-markable. Then w has a Φ'-kernel repetition (p',q') with $|p| = |f(p')|$ and $|q| \leq |f(q')| + \ell$.

Proof.

Claim 1. Suppose that (p, q) is not maximal, and suppose that there is no maximal Φ-kernel repetition extending (p, q). Then for every integer n, there is a Φ'-kernel repetition (p', q') of w such that $|p| = |f(p')|$ and the exponent of (p', q') is at least n.

Proof. There is an infinite sequence $((p, q) = (p_0, q_0), (p_1, q_1), \ldots)$ of repetitions of $f(w)$ such that (p_{i+1}, q_{i+1}) extends (p_i, q_i) for every $i \geq 0$. Clearly, for every $n \geq 1$, there is an $i_n \in \mathbb{N}$ such that p^n is a factor of $p_{i_n} q_{i_n}$, and *a fortiori* of $f(w)$.

Since q is (f, w)-markable, there is a $n_0 > 0$ such that p^{n_0} is (f, w)-markable. Note that for every conjugate word $\widetilde{p}$ of p, $\widetilde{p}^{n_0+1}$ is an (f, w)-markable factor of $f(w)$, and for every $n > 0$, $(\widetilde{p}, \widetilde{p}^n)$ is a Φ-kernel repetition of $f(w)$.

Let $\widetilde{p}$ be a conjugate word of p such that every (f, w)-interpretation of $\widetilde{p}^{n_0+1}$ has a beginning $b \in \{f(x), f(y)\}$. Let (b, u, e) be an (f, w)-interpretation of $\widetilde{p}^{n_0+2}$, and let p' be the prefix of u such that $f(p') = \widetilde{p}$. Such a prefix exists, since $\widetilde{p}^{n_0+1}$ is (f, w)-markable and has $b = f(u[1])$ as beginning.

By condition (CO), $\Phi'(p') = \Phi(\widetilde{p}) = \Phi(p) = \mathrm{Id}_k$. Thus for every $n > 0$, (p', p'^n) is a Φ'-kernel repetition of w, and $|f(p')| = |\widetilde{p}| = |p|$. □

By Claim 1, if (p, q) is not maximal and there is no maximal Φ-kernel repetition extending (p, q), then w has a Φ'-kernel repetition (p', q') with $|p| = |f(p')|$ and $|q| \leq |f(q')| + \ell$.

Thus we can now suppose without loss of generality that (p, q) is maximal (otherwise, replace (p, q) by a maximal repetition which extends (p, q)). Let (b, u, e) be an (f, w)-interpretation of q. By Proposition 4, all (f, w)-interpretations of q has beginning b and end e. By Proposition 3, pq is markable. Since q is a prefix and a suffix of pq, the beginning (resp. end) of an (f, w)-interpretation of pq is b (resp. e). Let (b, v, e) be a (f, w)-interpretation of pq. By maximality of (p, q), $b = f(v[1])$.

If q has only one (f, w)-interpretation, then u is a prefix and a suffix of v, and by maximality of (p, q), $e = f(v[|v|])$. Let p' be such that $p'u = v$, and let $q' = u$. Clearly $f(q') = q$, $f(p'q') = pq$ and $f(p') = p$. By condition (CO), $\Phi'(p') = \Phi(p) = \mathrm{Id}_k$, thus (p', q') is a Φ'-kernel repetition. (Note that in this case, $\ell = 0$ by maximality of (p, q).)

Otherwise, by Proposition 4, let (b, u_1, e) and (b, u_2, e) be the only two possible (f, w)-interpretations of q, and let $u' = u_1[1 : |u_1| - 1]$. Then u' is a prefix and a suffix of $v[1 : |v| - 1]$. Let p' be such that $p'u' = v[1 : |v| - 1]$. Clearly $f(p') = p$, and by Proposition 4, $|e| \leq \ell$, thus $|q| \leq |f(q')| + \ell$, with $q' = u'$. By condition (CO), $\Phi'(p') = \Phi(p) = \mathrm{Id}_k$, thus (p', q') is a Φ'-kernel repetition. □

4. Images of the Thue-Morse word

From now on, g denotes the Thue-Morse morphism on $\{a, b\}$, *i.e.* g is such that:

$$g : \begin{cases} a \to ab \\ b \to ba. \end{cases}$$

Let w_{TM} be a fixed point of g. We recall that w_{TM} is 2^+-free.

Let h be a morphism from $\{a, b\}^*$ into $\{0, 1\}^*$. Let $\sigma_a = \Psi(h(a))$ and $\sigma_b = \Psi(h(b))$ (we recall that Ψ is the morphism from the free monoid $\{0, 1\}^*$ into $\mathbb{S}_k$ such that $\Psi(0) = \sigma_0$ and $\Psi(1) = \sigma_1$). We suppose that h respects the following preliminary conditions:

(A) h is uniform (*i.e.* $|h(a)| = |h(b)|$),

(B) the last letters of $h(a)$ and $h(b)$ differ,

(C) $\{h(a), h(b)\}$ is *comma-free*, that is for every $(x, y, z) \in \{a, b\}^3$, there is no $w, w' \in \{0, 1\}^+$ such that $h(yz) = w \cdot h(x) \cdot w'$, and

(D) there is $\sigma \in \mathbb{S}_k$ such that $\sigma_a\sigma_b = \sigma \cdot \sigma_a \cdot \sigma^{-1}$ and $\sigma_b\sigma_a = \sigma \cdot \sigma_b \cdot \sigma^{-1}$.

Note that g and h respect conditions (LL) and (PC) of Section 3. By Proposition 1, $M_k(h(w_{TM}))$ has no forbidden repetition if and only if:

(S) $M_k(h(w_{TM}))$ has no forbidden short repetition, and

(K) for every Ψ-kernel repetition (p, q) in $h(w_{TM})$, we have $\frac{|pq|+k-1}{|p|} \leq \mathrm{r}_k$.

Let L be the largest common prefix of $h(a)$ and $h(b)$, let $\ell = |L|$ and let $s = |h(a)| = |h(b)|$. Let Ψ' be the morphism between the free monoid $\{a, b\}^*$ and $\mathbb{S}_k$ such that $\Psi'(a) = \sigma_a$ and $\Psi'(b) = \sigma_b$.

Lemma 6. *Let u be a factor of $h(w_{TM})$ with $|u| \geq 2s-1$. Then u is (h, w_{TM})-markable.*

Proof. Obviously $h(a)$ and $h(b)$ are (h, w_{TM})-markable since $\{h(a), h(b)\}$ is comma-free. Thus Lemma 6 follows from Proposition 3 and from the fact that if $|u| \geq 2s - 1$, then u contains either $h(a)$ or $h(b)$ as factor. □

As corollary of Lemma 6 and Lemma 5 with $f = h$, $\{x, y\} = \{a, b\}$, $\{z, t\} = \{0, 1\}$, $\Phi = \Psi$ and $\Phi' = \Psi'$, we obtain (condition (CO) is trivially fulfilled with $\sigma = \mathrm{Id}_k$) :

Corollary 7. *Let (p, q) be a Ψ-kernel repetition in $h(w_{TM})$ such that $|q| \geq 2s-1$. Then there is a Ψ'-kernel repetition (p', q') in w_{TM} such that $|p| = |h(p')|$ and $|q| \leq |h(q')| + \ell$.*

Obviously, if (p', q') is a Ψ'-kernel repetition in w_{TM}, then $h(w_{TM})$ has a Ψ-kernel repetition (p, q) with $p = h(p')$ and $q = h(q')L$. Thus $M_k(h(w_{TM}))$ has no forbidden kernel repetition if and only if:

(SK) for every Ψ-kernel repetition (p,q) in $h(w_{TM})$ with $|q| < 2s-1$, we have $\frac{|pq|+k-1}{|p|} \leq \mathrm{r}_k$, and

(LK) for every Ψ'-kernel repetition (p',q') in w_{TM}, we have $\frac{s|p'q'|+\ell+k-1}{s|p'|} \leq \mathrm{r}_k$.

Lemma 8. *Let w be a factor of w_{TM} of length at least 4. Then w is (g, w_{TM})-markable.*

Proof. This follows from the fact that w_{TM} avoids aaa, bbb, $ababa$ and $babab$, and thus aa, bb, $abab$ and $baba$ have unique (g, w_{TM})-interpretations and are (g, w_{TM})-markable. □

As corollary of Lemma 8 and Lemma 5 with $f = g$, $\{x,y\} = \{z,t\} = \{a,b\}$, $\Phi = \Phi' = \Psi'$, we obtain (condition (CO) is fulfilled by condition (D)):

Corollary 9. *Let (p',q') be a Ψ'-kernel repetition in w_{TM} with $|q'| \geq 4$. Then there is a Ψ'-kernel repetition (p'',q'') in w_{TM} with $|p'| = 2\cdot|p''|$ and $|q'| \leq 2\cdot|q''|$.*

Note that in this case, $\frac{s|p''q''|+\ell+k-1}{s|p''|} \geq \frac{s|p'q'|+\ell+k-1}{s|p'|}$. Thus by Corollary 9, condition (LK) is equivalent to:

(LK') for every Ψ'-kernel repetition (p'',q'') in w_{TM} with $|q''| \leq 3$, we have $\frac{s|p''q''|+\ell+k-1}{s|p''|} \leq \mathrm{r}_k$.

5. Decidability and Results

Let $k \geq 3$, let $h : \{a,b\}^* \to \Sigma_k^*$ be a morphism which respects conditions (A-D), and let $\Psi' : \{a,b\}^* \to \mathbb{S}_k$ be the morphism such that $\Psi'(a) = \sigma_a = \Psi(h(a))$ and $\Psi'(b) = \sigma_b = \Psi(h(b))$. If Ψ' respects condition (LK'), then by Corollary 9, w_{TM} has no Ψ'-kernel repetition (p',q') with $\frac{s|p'q'|+\ell+k-1}{s|p'|} > \mathrm{r}_k$, where $s = |h(a)| = |h(b)|$ and ℓ is the size of the largest common prefix of $h(a)$ and $h(b)$. Moreover, if h respects (SK), then by Corollary 7, $h(w_{TM})$ has no Ψ-kernel repetition (p,q) with $\frac{|pq|+k-1}{|p|} > \mathrm{r}_k$, that is by Proposition 1, $M_k(h(w_{TM}))$ has no forbidden kernel repetition. Finally if h also respects (S), then $M_k(h(w_{TM}))$ has no forbidden repetition. To summarize, if h and Ψ' respect conditions (A-D), (S), (SK) and (LK'), then $M_k(h(w_{TM}))$ validates Dejean's conjecture for k letters.

Obviously, conditions (A-D) are decidable. Condition (S) is decidable since short forbidden repetitions have length less than $\frac{\mathrm{r}_k \times (k-2)}{\mathrm{r}_k - 1}$. It is sufficient to check every factor of size at most $\frac{\mathrm{r}_k \times (k-2)}{\mathrm{r}_k - 1}$ in $M_k(h(w_{TM}))$. Similarly, if (p,q) is a Ψ-kernel repetition of $h(w_{TM})$ with $|q| \leq 2s-2$ and $\frac{|pq|+k-1}{|p|} > \mathrm{r}_k$, then $|pq| < \frac{\mathrm{r}_k \times (2s-2)+k-1}{\mathrm{r}_k - 1}$. Condition (SK) is thus decidable. Finally, if (p',q') is a Ψ'-kernel repetition of w_{TM} with $\frac{s|p'q'|+\ell+k-1}{s|p'|} > \mathrm{r}_k$ and $|q'| \leq 3$, then $|p'| < \frac{k-1+\ell+3s}{s\times(\mathrm{r}_k - 1)}$. Thus condition (LK') is decidable.

For every $k \in \{4, 8, \dots, 38\}$, we found a morphism h_k which respects conditions (A-D), (S), (SK) and (LK'). This proves that Dejean's conjecture holds for $8 \le k \le 38$. Following, we give an example for $k = 18$. Morphisms for $k \in \{4, 8\}$ can be found at the end of this section, and morphisms for $9 \le k \le 38$ are presented in the next section for the stronger case of Ochem's conjecture. For every $k \in \{2, 3, 5, 6, 7\}$ and for every $\sigma_a, \sigma_b \in \mathbb{S}_k$ such that $\sigma_a \cdot \sigma_b = \sigma \cdot \sigma_a \cdot \sigma^{-1}$ and $\sigma_b \cdot \sigma_a = \sigma \cdot \sigma_b \cdot \sigma^{-1}$ for a $\sigma \in \mathbb{S}_k$, the word w_{TM} has a Ψ'-kernel repetition (p, q) with $\frac{|pq|}{|p|} \ge \mathrm{r}_k$, thus the technique presented here is not applicable in these cases.

Example: $k = 18$. Let h_{18} be the morphism such that:

$$h_{18} : \begin{cases} a \to 10101101010110101101010110110101011010110 \\ b \to 10101011010110101101011010110101011010101. \end{cases}$$

Then $s = |h_{18}(a)| = |h_{18}(b)| = 41$, $L = 10101$, $\ell = 5$, and we have:

$$\begin{cases} \sigma_a = \Psi(h_{18}(a)) = \left[\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ 7 & 9 & 6 & 8 & 10 & 11 & 14 & 12 & 16 & 13 & 15 & 2 & 17 & 4 & 18 & 1 & 3 & 5 \end{smallmatrix}\right] \\ \sigma_b = \Psi(h_{18}(b)) = \left[\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ 6 & 9 & 7 & 8 & 10 & 11 & 14 & 12 & 16 & 13 & 15 & 2 & 17 & 4 & 18 & 3 & 1 & 5 \end{smallmatrix}\right]. \end{cases}$$

Obviously, h_{18} respects (A) and (B). Moreover, it is not hard to show that h_{18} respects condition (C). Finally, h_{18} respects (D) with:

$$\sigma = \left[\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ 1 & 4 & 3 & 10 & 6 & 14 & 11 & 17 & 12 & 15 & 8 & 7 & 5 & 18 & 2 & 9 & 13 & 16 \end{smallmatrix}\right].$$

Now if (p, q) is a forbidden short repetition in $M_{18}(h_{18}(w_{TM}))$ (*i.e.* $|q| \le 16$), then $|pq| < 288$. If (p, q) is a Ψ-kernel repetition in $h_{18}(w_{TM})$ with $|q| < 2 \times 41 - 1$ and $\frac{|pq|+k-1}{|p|} > \frac{18}{17}$, then $|pq| < 1729$, and thus $M_{18}(h_{18}(w_{TM}))$ has a forbidden repetition of length less than 1746. Thus to check conditions (S) and (SK), it is sufficient to check whether $M_{18}(h_{18}(g^8(a)))$ contains no forbidden repetition, since $g^8(a)$ contains all factors of w_{TM} of length at most 65. Finally, if (p', q') is a Ψ'-kernel repetition in w_{TM} with $|q'| \le 3$ and $\frac{s|p'q'|+\ell+k-1}{s|p'|} \le \frac{18}{17}$, then $|p'| \le 60$. To check (LK'), it is thus sufficient to check every Ψ'-kernel repetition in $g^8(a)$.

Morphisms for $k \in \{4, 8\}$.

$$h_4 : \begin{cases} a \to 1011010101101101011011010101101101010110 \\ 1101010110101011011010110110101011011010 \\ b \to 1011010101101101011011010101101010110110 \\ 1010110110101011011010110110101011010101 \end{cases}$$

$$h_8 : \begin{cases} a \to 101101101101010101011010101101 \\ b \to 101101010101011011011011011010 \end{cases}$$

6. Ochem's Conjecture

The *frequency* of a letter $a \in \Sigma$ in an infinite word $w \in \Sigma^\omega$ is $\lim_{n\to\infty} \frac{|w[1:n]|_a}{n}$ (where $|v|_a$ is the number of occurrences of a in v), if the limit exists. Obviously, in a Dejean word on a k-letter alphabet with $k \geq 5$ (*i.e.* an infinite word with maximal repetition exponent $\frac{k}{k-1}$), each letter has frequency at most $\frac{1}{k-1}$, since every factor of size $k-1$ consists of $k-1$ different letters. On the other hand, the frequency of each letter is at least $\frac{1}{k+1}$, otherwise the word would have a factor of size $k+1$ with $k-1$ different letters, that is 00 would be a factor of the Pansiot's code, and the word would have a repetition of exponent $\frac{k+1}{k-1} > \mathrm{r}_k$.

Ochem proposed the following stronger version of Dejean's conjecture.

Conjecture ([10])**.** (1) For every $k \geq 5$, there exists an infinite ${\frac{k}{k-1}}^+$-free word over k-letter with letter frequency $\frac{1}{k+1}$.

(2) For every $k \geq 6$, there exists an infinite ${\frac{k}{k-1}}^+$-free word over k-letter with letter frequency $\frac{1}{k-1}$.

This conjecture has already been proved for several cases when $k < 9$ [3, 10, 11]. If the Pansiot's code of a Dejean word w has a 0 at every position $i \pmod{k-1}$ for an $i \in \{0, \dots k-1\}$, then w has the same letter at position $i \pmod{k-1}$, thus w has a letter with frequency $\frac{1}{k-1}$. Such a code can be constructed as the image of the Thue-Morse word by a uniform morphism of size $0 \pmod{k-1}$, with a 0 at position $i \pmod{k-1}$ in $h(a)$ and $h(b)$. Similarly if the Pansiot's code of w has a 0 at every position $i \pmod{k+1}$ for an $i \in \{0, \dots k+1\}$, then w has a letter with frequency $\frac{1}{k+1}$. Using the technique presented in previous sections, we were able prove Ochem's conjecture for $9 \leq k \leq 38$. Note that the technique is not applicable in the case $k = 8$.

Morphisms h_k^+ *for* $9 \leq k \leq 38$.

$$h_9^+: \begin{cases} a \to 1\underline{0}1011011\underline{0}1101011\underline{0}1101101\underline{0}1011011\underline{0}101010 \\ b \to 1\underline{0}1101101\underline{0}1010101\underline{0}1011011\underline{0}1011011\underline{0}110101 \end{cases}$$

$$h_{10}^+: \begin{cases} a \to 1\underline{0}10101011\underline{0}10101101\underline{0}11011011\underline{0}11011011\underline{0}10101011\underline{0}1010110 \\ b \to 1\underline{0}10110101\underline{0}10101011\underline{0}11011011\underline{0}11011011\underline{0}11011011\underline{0}1101101 \end{cases}$$

$$h_{11}^+: \begin{cases} a \to 10101\underline{0}101101101\underline{0}110101011\underline{0}101101101\underline{0}101010101\underline{0}101010101\underline{0}1101 \\ b \to 10101\underline{0}101101101\underline{0}110101011\underline{0}101011011\underline{0}101011011\underline{0}101010101\underline{0}1010 \end{cases}$$

$$h_{12}^+: \begin{cases} a \to 1\underline{0}1101101101\underline{0}1101010101\underline{0}1101011011\underline{0}1010110101\underline{0}101010110 \\ b \to 1\underline{0}1010101101\underline{0}1101101101\underline{0}1011010101\underline{0}1010110101\underline{0}110110101 \end{cases}$$

$$h_{13}^+: \begin{cases} a \to 1\underline{0}11010101101\underline{0}10110110101\underline{0}1011011010 \\ \quad\;\; 1\underline{0}11010101101\underline{0}10110101011\underline{0}1011010101 \\ b \to 1\underline{0}11010101011\underline{0}11010101011\underline{0}1011010110 \\ \quad\;\; 1\underline{0}11010101101\underline{0}10110110101\underline{0}1011011010 \end{cases}$$

$$h_{14}^+: \begin{cases} a \to 1\underline{0}110101101011\underline{0}110101101101\underline{0}110110101011\underline{0}110101011011\underline{0}11010110101 \\ b \to 1\underline{0}110110101101\underline{0}110110101011\underline{0}110101101101\underline{0}110110101101\underline{0}11011011010 \end{cases}$$

$$h_{15}^{+}:\ \begin{cases} a \to 1\underline{0}1010101101011\underline{0}1010101101101\underline{0}101011010101 \\ \quad\quad 1\underline{0}1010101011011\underline{0}101010110101 \\ b \to 1\underline{0}1010110101101\underline{0}1010110101011\underline{0}101010110110 \\ \quad\quad 1\underline{0}1010110101101\underline{0}101011011010 \end{cases}$$

$$h_{16}^{+}:\ \begin{cases} a \to 1\underline{0}11010101101011\underline{0}10101101011011\underline{0}11011010110101\underline{0}1101010110101 \\ b \to 1\underline{0}11010101011011\underline{0}10101101101011\underline{0}11011010101101\underline{0}1101010110110 \end{cases}$$

$$h_{17}^{+}:\ \begin{cases} a \to 1\underline{0}101101101011011\underline{0}110101011010101\underline{0}101101011011011\underline{0}11010110101010 \\ b \to 1\underline{0}110101011011011\underline{0}101101101010101\underline{0}110101101010101\underline{0}11010101101101 \end{cases}$$

$$h_{18}^{+}:\ \begin{cases} a \to 1\underline{0}1011011011011011\underline{0}1010101011010101\underline{0}1010101010101101\underline{0}110110110110110 \\ b \to 1\underline{0}1011011011011011\underline{0}1010101010110101\underline{0}1010101011010101\underline{0}110110110101101 \end{cases}$$

$$h_{19}^{+}:\ \begin{cases} a \to 1\underline{0}10110101101011011\underline{0}11010101101010101\underline{0}1011011010101101 \\ b \to 1\underline{0}10110110101011011\underline{0}11010101101011011\underline{0}1101011010101010 \end{cases}$$

$$h_{20}^{+}:\ \begin{cases} a \to 1\underline{0}110110110101101101\underline{0}11011010110110110 \\ \quad\quad 1\underline{0}110110101010101011\underline{0}11010101101010110 \\ b \to 1\underline{0}110110110101101011\underline{0}11011010110110101 \\ \quad\quad 1\underline{0}110110101010101101\underline{0}11010101101010101 \end{cases}$$

$$h_{21}^{+}:\ \begin{cases} a \to 1\underline{0}1101010110110101101\underline{0}1010110101010101011\underline{0}101101010101010110 \\ b \to 1\underline{0}1101010110110101011\underline{0}1010110101010101101\underline{0}101101010101010101 \end{cases}$$

$$h_{22}^{+}:\ \begin{cases} a \to 1\underline{0}11011010101101101011\underline{0}11011010110101101011\underline{0}1010101101011010110 \\ b \to 1\underline{0}11011010101101101101\underline{0}11011010110101101101\underline{0}1010101101011010101 \end{cases}$$

$$h_{23}^{+}:\ \begin{cases} a \to 1\underline{0}110110110101010101011\underline{0}101101010110110101011\underline{0}10101101010101010110 \\ b \to 1\underline{0}110110110101010101101\underline{0}101101010110110101101\underline{0}10101101010101010101 \end{cases}$$

$$h_{24}^{+}:\ \begin{cases} a \to 1\underline{0}1011010110110101101011\underline{0}101011011011011010110 \\ b \to 1\underline{0}1011010110110101101101\underline{0}101011011011011010101 \end{cases}$$

$$h_{25}^{+}:\ \begin{cases} a \to 1\underline{0}10110101101101010101011\underline{0}1101011010110101010110 \\ b \to 1\underline{0}10110101101101010101101\underline{0}1101011010110101010101 \end{cases}$$

$$h_{26}^{+}:\ \begin{cases} a \to 1\underline{0}101101101011011010101011\underline{0}11010110101101101010110 \\ b \to 1\underline{0}101101101011011010101101\underline{0}11010110101101101010101 \end{cases}$$

$$h_{27}^{+}:\ \begin{cases} a \to 1\underline{0}1011010110101101010101011\underline{0}110101011010110101010110 \\ b \to 1\underline{0}1011010110101101010101101\underline{0}110101011010110101010101 \end{cases}$$

$$h_{28}^{+}:\ \begin{cases} a \to 1\underline{0}10110101011010101010101011\underline{0}1101010101101010101010110 \\ b \to 1\underline{0}10110101011010101010101101\underline{0}1101010101101010101010101 \end{cases}$$

$$h_{29}^{+}:\ \begin{cases} a \to 1\underline{0}101010110101010101010101011\underline{0}10101010110101010101010110 \\ b \to 1\underline{0}101010110101010101010101101\underline{0}10101010110101010101010101 \end{cases}$$

$$h_{30}^{+}:\ \begin{cases} a \to 1\underline{0}1010101101101101101010101011\underline{0}110110110101010110101010110 \\ b \to 1\underline{0}1010101101101101101010101101\underline{0}110110110101010110101010101 \end{cases}$$

$$h_{31}^{+}:\ \begin{cases} a \to 1\underline{0}10101011010101010101010101011\underline{0}1101101101010101010110110110 \\ b \to 1\underline{0}10101011010101010101010101101\underline{0}1101101101010101010110110101 \end{cases}$$

$$h_{32}^{+}:\ \begin{cases} a \to 1\underline{0}110101011010110101101010101011\underline{0}1101010101101011011010101010110 \\ b \to 1\underline{0}110101011010110101101010101101\underline{0}1101010101101011011010101010101 \end{cases}$$

$$h_{33}^{+}:\ \begin{cases} a \to 1\underline{0}10110101101101010101010101011\underline{0}1101011010110101010101010110 \\ b \to 1\underline{0}10110101101101010101010101101\underline{0}1101011010110101010101010101 \end{cases}$$

$$h_{34}^{+}:\ \begin{cases} a \to 1\underline{0}101101011010101010101010101011\underline{0}11010110101010101010101010110 \\ b \to 1\underline{0}101101011010101010101010101101\underline{0}11010110101010101010101010101 \end{cases}$$

$$h_{35}^{+}\colon \begin{cases} a \to 1\underline{0}1011011011010101010101010101010101011\underline{0}110101101101010101010101010101010110 \\ b \to 1\underline{0}1011011011010101010101010101010101101\underline{0}110101101101010101010101010101010101 \end{cases}$$

$$h_{36}^{+}\colon \begin{cases} a \to & 1\underline{0}101101010110101010101010101010101 \\ & 1\underline{0}110101010110101010101010101010110 \\ b \to & 1\underline{0}101101010110101010101010101010110 \\ & 1\underline{0}110101010110101010101010101010101 \end{cases}$$

$$h_{37}^{+}\colon \begin{cases} a \to & 1\underline{0}1011010101010101010101010101010101 \\ & 1\underline{0}1010110101010101010101010101010110 \\ b \to & 1\underline{0}1011010101010101010101010101010110 \\ & 1\underline{0}1010110101010101010101010101010101 \end{cases}$$

$$h_{38}^{+}\colon \begin{cases} a \to & 1\underline{0}1010101101010101101101010110101010101 \\ & 1\underline{0}1101101101010101010101010110101010110 \\ b \to & 1\underline{0}1010101101010101101101010110101010110 \\ & 1\underline{0}1101101101010101010101010110101010101 \end{cases}$$

Morphisms h_k^- *for* $9 \leq k \leq 38$.

$$h_{9}^{-}\colon \begin{cases} a \to 1\underline{0}101101101\underline{0}110110101\underline{0}110101101\underline{0}10110101 \\ b \to 1\underline{0}101101011\underline{0}101011011\underline{0}101101011\underline{0}11010110 \end{cases}$$

$$h_{10}^{-}\colon \begin{cases} a \to 1\underline{0}1101011011\underline{0}1011010101\underline{0}1011011011\underline{0}101101010 \\ b \to 1\underline{0}1101010101\underline{0}1011010101\underline{0}1011011011\underline{0}110101101 \end{cases}$$

$$h_{11}^{-}\colon \begin{cases} a \to 1\underline{0}10110101011\underline{0}10110101101\underline{0}10110101101\underline{0}1010110101 \\ b \to 1\underline{0}10101101011\underline{0}10101101011\underline{0}10110101101\underline{0}1010110110 \end{cases}$$

$$h_{12}^{-}\colon \begin{cases} a \to 1\underline{0}101010101011\underline{0}110110101011\underline{0}101010110101\underline{0}101011011011\underline{0}10110110101 \\ b \to 1\underline{0}101010101101\underline{0}101010101011\underline{0}101011011011\underline{0}110101101011\underline{0}10110110110 \end{cases}$$

$$h_{13}^{-}\colon \begin{cases} a \to 1011011\underline{0}1101101011011\underline{0}1010110101011\underline{0}1011010110101\underline{0}110110 \\ b \to 1011011\underline{0}1101101010101\underline{0}1010110110101\underline{0}1011011011011\underline{0}110101 \end{cases}$$

$$h_{14}^{-}\colon \begin{cases} a \to 1\underline{0}10101011011011\underline{0}10101010101011\underline{0}11011011011011\underline{0}1101101101010 \\ b \to 1\underline{0}10101011011011\underline{0}11011011011011\underline{0}10101010101101\underline{0}1101101010101 \end{cases}$$

$$h_{15}^{-}\colon \begin{cases} a \to 1\underline{0}110101101010101\underline{0}110101011011011\underline{0}101101011011011\underline{0}10110110101010 \\ b \to 1\underline{0}101101101010101\underline{0}110101011011011\underline{0}101101101010101\underline{0}10110101101101 \end{cases}$$

$$h_{16}^{-}\colon \begin{cases} a \to 1\underline{0}1101011010101101\underline{0}1101010110101011\underline{0}1011010101101101\underline{0}101011011010101 \\ b \to 1\underline{0}1101010110101011\underline{0}1101101010101101\underline{0}1010101101101011\underline{0}101101011010110 \end{cases}$$

$$h_{17}^{-}\colon \begin{cases} a \to & 1\underline{0}10110110101101011\underline{0}1011011010110110 \\ & 1\underline{0}11010110101101101\underline{0}1011011010110101 \\ b \to & 1\underline{0}11010110101101011\underline{0}1011011010110101 \\ & 1\underline{0}10110110101101011\underline{0}1101011010110110 \end{cases}$$

$$h_{18}^{-}\colon \begin{cases} a \to & 1\underline{0}101101010101101011\underline{0}11010101010110110 \\ & 1\underline{0}110101010101101101\underline{0}11010101010110101 \\ b \to & 1\underline{0}110101010101101101\underline{0}10110101010110101 \\ & 1\underline{0}101101010101101101\underline{0}11010101010110110 \end{cases}$$

$$h_{19}^{-}\colon \begin{cases} a \to 1\underline{0}1011011010110101101\underline{0}1101010110101101011\underline{0}101101101010110110 \\ b \to 1\underline{0}1101010110110101101\underline{0}1101010110101101101\underline{0}101101101011010101 \end{cases}$$

$$h_{20}^{-}\colon \begin{cases} a \to 1\underline{0}10110101101101101011\underline{0}11010110110110101101\underline{0}1011010101101010101 \\ b \to 1\underline{0}11010101101101101011\underline{0}11010110110110101011\underline{0}1011010101101010110 \end{cases}$$

$$h_{21}^-\colon \begin{cases} a \to 1\underline{0}101010101011010101101\underline{0}110110101010110101011\underline{0}1101101010101011010110 \\ b \to 1\underline{0}11011010101011010101101\underline{0}110110101010110101101\underline{0}1010101010101101010101 \end{cases}$$

$$h_{22}^-\colon \begin{cases} a \to 1\underline{0}101010101011010101101011\underline{0}1011011011010101101101\underline{0}10101010101011011010101 \\ b \to 1\underline{0}101010101011010101101101\underline{0}1011011011010101101011\underline{0}10101010101011011010110 \end{cases}$$

$$h_{23}^-\colon \begin{cases} a \to 1\underline{0}10101011010110110101011\underline{0}1101101011010101010110 \\ b \to 1\underline{0}10101011010110110101101\underline{0}1101101011010101010101 \end{cases}$$

$$h_{24}^-\colon \begin{cases} a \to 1\underline{0}101101011010101101101011\underline{0}1101011010101011010110110 \\ b \to 1\underline{0}101101011010110101101101\underline{0}1101011010101010110110101 \end{cases}$$

$$h_{25}^-\colon \begin{cases} a \to 1\underline{0}1010101101101011010101011\underline{0}110110101010101101010110 \\ b \to 1\underline{0}1010101101101011010101101\underline{0}110110101010101101010101 \end{cases}$$

$$h_{26}^-\colon \begin{cases} a \to 1\underline{0}10101010101010101010101011\underline{0}1101101011011010101010110 \\ b \to 1\underline{0}10101010101010101010101101\underline{0}1101101011011010101010101 \end{cases}$$

$$h_{27}^-\colon \begin{cases} a \to 1\underline{0}110101011010101101010101101\underline{0}11010110101010101101010101 \\ b \to 1\underline{0}110101101010101101010101011\underline{0}11010101101010101101010110 \end{cases}$$

$$h_{28}^-\colon \begin{cases} a \to 1\underline{0}1011011011011010101010101011\underline{0}11010110110110101010101010110 \\ b \to 1\underline{0}1011011011011010101010101101\underline{0}11010110110110101010101010101 \end{cases}$$

$$h_{29}^-\colon \begin{cases} a \to 1\underline{0}10110101101101010101010101011\underline{0}1010110110110101010101010110 \\ b \to 1\underline{0}10110101101101010101010101101\underline{0}1010110110110101010101010101 \end{cases}$$

$$h_{30}^-\colon \begin{cases} a \to 1\underline{0}101101011010110101101010101011\underline{0}10101101011010110101101010110 \\ b \to 1\underline{0}101101011010110101101010101101\underline{0}10101101011010110101101010101 \end{cases}$$

$$h_{31}^-\colon \begin{cases} a \to 1\underline{0}1011010101010101010101010101011\underline{0}110101010101010101010101010110 \\ b \to 1\underline{0}1011010101010101010101010101101\underline{0}110101010101010101010101010101 \end{cases}$$

$$h_{32}^-\colon \begin{cases} a \to 1\underline{0}11011010110101101010101010101011\underline{0}11010110101101011010101010110 \\ b \to 1\underline{0}11011010110101101010101010101101\underline{0}11010110101101011010101010101 \end{cases}$$

$$h_{33}^-\colon \begin{cases} a \to 1\underline{0}101010110110110101010101010101011\underline{0}11011011010101010101010101010110 \\ b \to 1\underline{0}101010110110110101010101010101101\underline{0}11011011010101010101010101010101 \end{cases}$$

$$h_{34}^-\colon \begin{cases} a \to 1\underline{0}1011010110101101101010101010101 \\ \phantom{a \to {}} 1\underline{0}1010110101101101101010101010110 \\ b \to 1\underline{0}1011010110101101101010101010110 \\ \phantom{b \to {}} 1\underline{0}1010110101101101101010101010101 \end{cases}$$

$$h_{35}^-\colon \begin{cases} a \to 1\underline{0}1010101101011010110110110101010101 \\ \phantom{a \to {}} 1\underline{0}1101101011010110101010110101010110 \\ b \to 1\underline{0}1010101101011010110110110101010110 \\ \phantom{b \to {}} 1\underline{0}1101101011010110101010110101010101 \end{cases}$$

$$h_{36}^-\colon \begin{cases} a \to 1\underline{0}11011010110110101010101010101010101 \\ \phantom{a \to {}} 1\underline{0}11010110110110101010101010101010110 \\ b \to 1\underline{0}11011010110110101010101010101010110 \\ \phantom{b \to {}} 1\underline{0}11010110110110101010101010101010101 \end{cases}$$

$$h_{37}^-\colon \begin{cases} a \to 1\underline{0}1101101010110101010101010101010101 \\ \phantom{a \to {}} 1\underline{0}1101011010110101010101010101010110 \\ b \to 1\underline{0}1101101010110101010101010101010110 \\ \phantom{b \to {}} 1\underline{0}1101011010110101010101010101010101 \end{cases}$$

$$h_{38}^{-}\colon \begin{cases} a \rightarrow 1\underline{0}101010101010110101010101010101011010101 \\ 1\underline{0}101010101010101101010101010101011010110 \\ b \rightarrow 1\underline{0}101010101010110101010101010101011010110 \\ 1\underline{0}101010101010101101010101010101011010101 \end{cases}$$

7. Conclusion

In conclusion, we propose the following conjecture:

Conjecture. For every $k \geq 9$, there are morphisms h_k^+ and h_k^- such that $M_k(h_k^+(w_{TM}))$ and $M_k(h_k^-(w_{TM}))$ are r_k^+-free, and $M_k(h_k^+(w_{TM}))$ (resp. $M_k(h_k^-(w_{TM}))$) has a letter with frequency $\frac{1}{k-1}$ (resp. $\frac{1}{k+1}$).

After the submission of the present article to the conference WORDS 2009, Currie and Rampersad announced an independent proof for the last cases of Dejean's conjecture [6].

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