The bi-join decomposition

Fabien de Montgolfier [a,1], Michaël Rao [b,2],

[a] *LIAFA, Université Paris 7, France*

[b] *LITA, Université de Metz, France*

Abstract

We introduce a new generalization of the modular decomposition called the *bi-join decomposition*. We characterize the completely decomposable graphs and we give a linear-time decomposition algorithm.

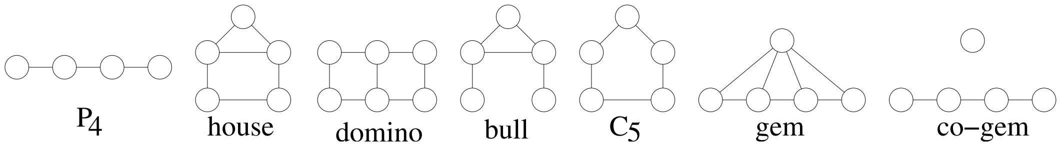
Keywords: decomposition, modular decomposition, graph classes.

1 Introduction

Graph decompositions are widely used in graph algorithms and graph theory. Well-known examples are the modular decomposition, and its generalization the split decomposition. Both correspond to a decomposition tree, whose leaves are the graph vertices and whose internal nodes correspond to decomposition operations. When the degree of the tree is bounded, some NP-hard problems can be solved in linear-time using recursivity on the tree [8]. Modular decomposition has been studied for a long time. Some properties of the modular decomposition extend to the split decomposition, including unicity [4] and linear-time computation of the decomposition tree [5].

For a decomposition $\mathcal{D}$ we say that a graph G is *completely decomposable* if every subgraph large enough (with at least four or five vertices, depending on the decomposition) admits a non-trivial decomposition, and on the other side G is *prime* if it admits no non-trivial decomposition (the meaning of "trivial" of course depends on the decomposition).

[1] Email: `fm@liafa.jussieu.fr`
[2] Email: `rao@univ-metz.fr`



A *module* M of a graph $G = (V, E)$ is a set of vertices such that a vertex $v \notin M$ is adjacent either to all vertices of M or to none of them. The modular decomposition is the set of all modules of a graph [8]. We do not discuss this notion here but just recall the following famous theorem from [2].

Theorem 1.1 *The following propositions are equivalent:*
(1) G is completely decomposable w.r.t. modular decomposition.
(2) G is P_4-free.
(3) For every subgraph H of G, either H or $\overline{H}$ are not connected.
(4) G can be obtained from a single vertex by a sequence of extensions by a true twin or a false twin.

These graphs are called *cographs*. *Splits* are a generalization of modules. A *distance hereditary graph* is a graph in which all chordless paths between any two vertices have the same length. There is also a nice characterization theorem from [1,6].

Theorem 1.2 *The following propositions are equivalent:*
(1) G is completely decomposable w.r.t. split decomposition.
(2) G is a distance hereditary graph.
(3) G is (house,hole,domino,gem)-free.
(4) G can be obtained from a single vertex by a sequence of extensions by a true twin or by a false twin or a pendant vertex.

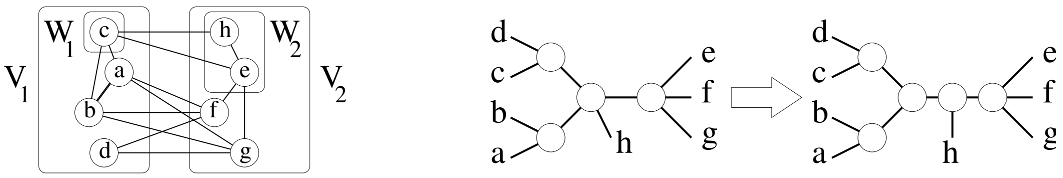
In this paper, we introduce a new generalization of the modular decomposition called the *bi-join decomposition*. We show that some properties of the modular and the split decomposition are preserved, we characterize the completely decomposable graphs and we give a linear-time decomposition algorithm.

2 Bi-joins and the bi-join decomposition

A *bipartition* of a set V is an (unordered) pair $\{V_1, V_2\}$ such that $V_1 \neq \emptyset$, $V_2 \neq \emptyset$, $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$. Two bipartitions of V $\{V_1, V_2\}$ and $\{V_3, V_4\}$ *overlap* if the four sets $V_1 \cap V_3$, $V_2 \cap V_3$, $V_1 \cap V_4$ and $V_2 \cap V_4$ are nonempty. In a graph, there is a *complete join* between $A \subseteq V$ and $B \subseteq V$ if every vertex of A is linked with every vertex of B.

A *bi-join* in a graph $G = (V, E)$ is a bipartition $\{V_1, V_2\}$ of V such that there exists $W_1 \subseteq V_1$ and $W_2 \subseteq V_2$ with a complete join between W_1 and W_2, a complete join between $V_1 \setminus W_1$ and $V_2 \setminus W_2$, and no other edges between V_1 and V_2. A bi-join $\{V_1, V_2\}$ is *trivial* if $|V_1| = 1$ or $|V_2| = 1$, and it is *strong* if there is no other bi-join of G which overlaps $\{V_1, V_2\}$. Notice that $\forall v \in V$, $\{\{v\}, V \setminus \{v\}\}$ is a strong trivial bi-join. A unique unrooted tree can be associated with the set of strong bi-joins. Its leaves are the vertices of the graph, and its edges are the strong bi-joins. For a node α of such a tree T, if α is adjacent with k edges, its *components* $C_1, \ldots, C_k \subset V$ are the leaf-sets of the connected components of $T - \alpha$.

T can be recursively constructed, starting from the star with n leaves. Let non-trivial strong bi-joins be $\{V_1^1, V_2^1\}, \ldots, \{V_1^k, V_2^k\}$. At step i there is a unique node α of the tree T_i such that neither V_1^i nor V_2^i is strictly included in a component of α. This node α is split: in T_{i+1} an edge separates the components of α included in V_1^i from the connected components included in V_2^i. This algorithm can be applied to any family of strong bipartitions. (For more details see [9].) This $O(n)$-sized tree T, called the *bi-join decomposition tree*, is a good representation of all the bi-joins of G.



Left: a non-trivial strong bi-join $\{\{a, b, c, d\}, \{e, f, g, h\}\}$ of a graph. Right : this bi-join is added to the tree; the obtained tree is the bi-join decomposition tree of the graph.

Theorem 2.1 *[9] The nodes of T can be labeled* complete *or* prime. *For every bi-join $\{V_1, V_2\}$ there is a node α in T such that V_1 is the union of components of α. Conversely if α has components $C_1, \ldots, C_k$, if α is labeled* complete *then any union of components is a bi-join (for all $I \subset \{1, \ldots, k\}$, $\{\cup_{i \in I} C_i, \cup_{i \notin I} C_i\}$ is a bi-join) and if α is labeled* prime *then for every $a \in \{1, \ldots, k\}$, $\{C_a, \cup_{i \neq a} C_i\}$ is a bi-join, and there is no other bi-join.*

The proof is based on the fact that two non-strong bipartitions overlap at one node. Then we use the property that if $\{V_1, V_2\}$ overlaps $\{V_3, V_4\}$ then $\{V_1, V \setminus V_1\}$,$\{V_2, V \setminus V_2\}$,$\{V_3, V \setminus V_3\}$,$\{V_4, V \setminus V_4\}$, and $\{V_1 \cup V_3, V_2 \cup V_4\}$ are bi-joins of G. Finally it can be proved that any union of components is a bi-join. A *quotient* G_α can be computed for each node α of the tree: it is $G[\{v_1, \ldots, v_k\}]$ where v_i is any vertex of component C_i.

Theorem 2.2 *[9] If a node is labeled prime its quotient is prime w.r.t. bi-join decomposition. If it is labeled complete its quotient is either the complete bipartite graph $K_{a,b}$ or the union of two complete graphs $K_a \oplus K_b$.*

The *neighborhood-nonneighborhood complement* of a graph G at vertex v, denoted $\widetilde{G_v}$ is a graph with vertex-set $V \setminus \{v\}$ and an edge between x and y if either $xy \in E(G)$ and $x, y \in N(v)$, or $xy \in E(G)$ and $x, y \notin N(v)$, or $xy \notin E(G)$ and $x \in N(v) \not\ni y$, or $xy \notin E(G)$ and $x \notin N(v) \ni y$.

Lemma 2.3 (Fundamental Lemma) *Let $\{V_1, V_2\}$ be a bipartition of V and $v \in V_1$. Then $\{V_1, V_2\}$ is a bi-join of G iff V_2 is a module of $\widetilde{G_v}$.*

Proof. For $w \in V_1$ such that $w \neq v$, $w \in N(v)$ in G iff $V_2 \subset N(w)$ in $\widetilde{G_v}$ and $w \notin N(v)$ in G iff $V_2 \cap N(w) = \emptyset$ in $\widetilde{G_v}$. □

3 Characterization theorem

In the introduction a characterization of graphs completely decomposable w.r.t. modular decomposition (cographs) and w.r.t. split decomposition (distance hereditary graphs) was given. Now we present a similar characterization for the completely decomposable graphs w.r.t. bi-join decomposition.

Let u be a vertex of G. u is a *false twin* of v if $N(u) = N(v)$, a *true twin* of u if $N(u) \cup \{u\} = N(v) \cup \{v\}$, a *false anti-twin* of u if $N(u) = V \setminus (N(v) \cup \{u, v\})$ and a *true anti-twin* of u if $N(u) = V \setminus N(v)$. An *(twin,anti-twin)-extension* of the graph is to select a vertex u and add another vertex that can be a true or false twin or anti-twin of u.

Theorem 3.1 *Let G be a graph. The following conditions are equivalent:*
(1) G is completely decomposable w.r.t. bi-join decomposition.
(2) G is (C_5,bull,gem,co-gem)-free.
(3) G can be obtained from a single vertex by a sequence of (twin,anti-twin)-extensions.

Proof. A subgraph H of G has a non-trivial bi-join iff for any $v \in V(H)$, $\widetilde{H_v}$ has a module (Fundamental Lemma). $\widetilde{H_v}$ is therefore a cograph, and contains a vertex x with a twin y (Theorem 1.1). If H has at least four vertices then w.l.o.g. $x \neq v \neq y$ and the neighborhood-nonneighborhood complement at v transforms true (resp. false) twins into true (resp. false) twins if $\{x, y\}$ does not overlap $N(v)$, and into true (resp. false) anti-twins otherwise. This proves equivalence between (1) and (3).

For equivalence between (1) and (2) one has to check that G is a C_5, a bull, a gem or a co-gem iff for every $v \in G$, $\widetilde{G_v} = P_4$. $\square$

4 Linear-time decomposition algorithm

Theorem 4.1 *The bi-join decomposition of a graph can be computed in linear time.*

Proof. It is a consequence of the Fundamental Lemma and of the existence of linear-time modular decomposition algorithms [3,7]. The strong modules of $\widetilde{G_v}$, computed in linear time, give the strong bi-joins of G in an obvious way. A minimum degree vertex v must be chosen, so that the number of edges of $\widetilde{G_v}$ is proportional to the number of edges of G. $\square$

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