On some generalizations of abelian power avoidability

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Abstract

We prove that 2-abelian-cubes are avoidable over a binary alphabet and that 3-abelian-squares are avoidable over a ternary alphabet, answering positively to two questions of Karhumäki *et al.*. We also show the existence of infinite additive-cube-free words on several ternary alphabets. To achieve this, we give sufficient conditions for a morphism to be k-abelian-n-power-free (resp. additive-n-power-free), and then we give several morphisms which respect these conditions.

Additionally, all our constructions show that the number of such words grows exponentially. As a corollary, we get a new lower bound of $3^{1/19} = 1.059526\ldots$ for the growth rate of abelian-cube-free words.

Keywords: Combinatorics on words, k-abelian equivalence, square-free, cube-free, morphism

1. Introduction

Avoidability of repetitions in words is one of the most studied topics in word combinatorics since the seminal papers of Thue [26, 27]. One famous example is Dejean's conjecture, recently solved by several authors (see [23]). The avoidability of abelian repetitions received a lot of interest since a question from Erdös in 1957 [9, 10].

Two words $u, v \in A^*$ are *abelian equivalent*, denoted $u \equiv_a v$, if for every $a \in A$, $|u|_a = |v|_a$. A word u is an *abelian-n-power*, where $n \geq 2$, if $u = u_1 u_2 \ldots u_n$ such that $u_i \equiv_a u_{i+1}$ for every $i \in \{1, \ldots, n-1\}$. An *abelian square* (resp. *abelian cube*) is an abelian-2-power (resp. abelian-3-power). It is not difficult to see that every ternary word of size at least 8 has an abelian square. Erdös [9, 10] raised the question whether they can be avoided in an infinite word on an alphabet of size 4. Evdokimov [11] showed that one can avoid them on an alphabet of size 25, which was later lowered to 5 by Pleasants [22]. Finally, Keränen [18] answered positively to Erdos's question in 1992. Furthermore, Dekking [7] showed that abelian cubes can be avoided in an infinite ternary word, and that abelian-4-powers can be avoided in an infinite binary word.

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We are here interested in two variations of the previous problem. The first one is the k-abelian-equivalence introduced by Karhumäki *et al.* [14, 16, 17]. Let $k \geq 1$. Two words u and v ($u, v \in A^*$) are *k-abelian-equivalent*, denoted $u \equiv_{a,k} v$, if for every $w \in A^*$ with $|w| \leq k$, $|u|_w = |v|_w$. A word u is a *k-abelian-n-power*, $n \geq 2$, if $u = u_1 u_2 \dots u_n$ such that $u_i \equiv_{a,k} u_{i+1}$ for every $i \in \{1, \dots, n-1\}$. A *k-abelian-square* (resp. *k-abelian-cube*) is a k-abelian-2-power (resp. k-abelian-3-power). This notion is between the abelian equivalence (which is the 1-abelian-equivalence) and the usual equality between words (which can be viewed as the ∞-abelian-equivalence). Since cubes are avoidable in the binary alphabet (*e.g.* in the Prouhet-Thue-Morse word), but are not avoidable in the abelian sense, it is natural to ask for the smallest k for which k-abelian-cubes are avoidable on a binary alphabet. In [14] authors showed that $k \leq 8$, and in [20] that $k \leq 5$. Finally, in [21], Mercaş and Saarela showed that $k \leq 3$. The same question can be asked for k-abelian-squares on a ternary alphabet: 2-abelian-squares cannot be avoided [15], but Huova showed that 64-abelian-squares can be avoided [12].

In Section 2, we give sufficient conditions for a morphism $h : A^* \to B^*$ to be k-abelian-n-power-free (for a fixed $n \geq 2$ and $k \geq 1$), that is for every abelian-n-power-free word $w \in A^*$, $h(w)$ is k-abelian-n-power-free. Then we give morphisms which respect the conditions, in order to construct 2-abelian-cube-free binary words and 3-abelian-square-free ternary words. This answers the two previous questions and also prove that the number of such words grows exponentially, as abelian-square-free on four letters [3], and abelian-cube-free ternary words ([1], see also Section 3).

The second notion is the additive-cube-avoidability. A word $w \in \mathbb{N}^*$ is an *additive cube* if $w = pqr$, where p, q and r are non-empty-word such that $|p| = |q| = |r|$ and $\sum(p) = \sum(q) = \sum(r)$. A word is *additive-cube-free* if it has no factor which is an additive cube. Clearly, such words are also abelian-cube-free. Recently Cassaigne *et al.* [5] showed that one can construct an infinite additive-cube-free word on the alphabet $\{0, 1, 3, 4\}$. The question of infinite additive-square-free word's existence on a finite alphabet is still open.

In Section 3 we give sufficient conditions for a substitution $h : A^* \to 2^{B^*}$, A, $B \subseteq \mathbb{N}$, to be additive-cube-free. We present substitutions from the alphabet $\{0, 1, 3, 4\}$ to several ternary alphabets which respects these conditions. Moreover, the presented constructions show directly that the number of additive-cube-free words on these ternary alphabets grows exponentially. The lower bound of $3^{1/19} = 1.059526\dots$ we obtain for the growth rate for the alphabet $\{0, 1, 8\}$ is also a new lower bound for the number of abelian-cube-free words on a ternary alphabet.

2. *k*-abelian-*n*-power-free morphisms

2.1. Preliminaries

Let $|u|_w$ denote the number of occurrences of the factor w in u. The *Parikh vector* of a word $u \in A^*$, where $A = \{a_1, a_2, \dots, a_k\}$, is $\Psi(u) = (|u|_{a_1}, |u|_{a_2}, \dots, |u|_{a_k})$. For a set $S \subseteq A^*$, $\Psi_S(u)$ is the vector indexed by S such that

$\Psi_S(u)[w] = |u|_w$ for every $w \in S$. When the alphabet is clear in the context, we let $\Psi_k(u)$ be $\Psi_{A^k}(u)$, for $k \geq 1$.

Let $\mathrm{Pref}(u)$ be the set of prefixes of u, and $\mathrm{Suf}(u)$ be its set of suffixes. For $k \geq 0$, let $\mathrm{pref}_k(u)$ (resp. $\mathrm{suf}_k(u)$) be the prefix (resp. suffix) of u of size k.

There are several equivalent definitions for k-abelian-equivalence (see [17]). Two words u and v of size at most $k-1$ are k-abelian-equivalent if and only if they are equal. Otherwise, the following conditions are equivalent:

- u and v are k-abelian-equivalent (*i.e.* $u \equiv_{a,k} v$).
- For every $w \in A^*$ with $|w| \leq k$, $|u|_w = |v|_w$.
- For every $w \in A^k$, $|u|_w = |v|_w$, $\mathrm{pref}_{k-1}(u) = \mathrm{pref}_{k-1}(v)$ and $\mathrm{suf}_{k-1}(u) = \mathrm{suf}_{k-1}(v)$.
- For every $w \in A^k$, $|u|_w = |v|_w$, and $\mathrm{pref}_{k-1}(u) = \mathrm{pref}_{k-1}(v)$.

Given $k \geq 1$ and $n \geq 2$, a (possibly infinite) word w is *k-abelian-n-power-free* if no non-empty factor in w is a k-abelian-n-power. A word is *k-abelian-square-free* (resp. *k-abelian-cube-free*) if it is k-abelian-2-power-free (resp. k-abelian-3-power-free).

A morphism $h : A^* \to B^*$ is *k-abelian-n-power-free* if for every abelian-n-power-free word $u \in A^*$, $h(u)$ is k-abelian-n-power-free. Note that u has to be abelian-n-power-free, not only k-abelian-n-power-free; we explain in Section 2.4 why we use this weaker notion. A morphism $h : A^* \to B^*$ is *k-abelian-square-free* (resp. *k-abelian-cube-free*) if it is k-abelian-2-power-free (resp. k-abelian-3-power-free).

2.2. Testing k-abelian-n-power-freeness

In [2], Carpi gave a set of conditions which assures that a given morphism is abelian-n-power-free. We give in the following theorem a set of similar conditions which assures that a given morphism is k-abelian-n-power-free.

Theorem 1. *We fix $k \geq 1$ and $n \geq 2$, and two alphabets A and B. Let $h : A^* \to B^*$ be a morphism. Suppose that:*

(i) *For every abelian-n-power-free word $w \in A^*$ with $|w| \leq 2$ or $|h(w[2 : |w|-1])| \leq (k-2)n - 2$, $h(w)$ is k-abelian-n-power-free.*

(ii) *There are $p, s \in B^{k-1}$ such that for every $a \in A$, $p = \mathrm{pref}_{k-1}(h(a)p)$ and $s = \mathrm{suf}_{k-1}(sh(a))$.*

(iii) *The matrix N indexed by $B^k \times A$, with $N[w, x] = |h(x)p|_w$, has rank $|A|$.*

(iv) *Let $S \subseteq B^k$, with $|S| = |A|$, such that the matrix M indexed by $S \times A$, with $M[w, x] = |h(x)p|_w$, is invertible. Let*

$$\Psi_S(v, u) = \Psi_S(vp) + \Psi_S(su) - \Psi_S(sp)$$

and $\Psi_k(v, u) = \Psi_{B^k}(v, u)$. For every $a_i \in A$ and $u_i, v_i \in A^*$ with $u_i v_i = h(a_i)$; $0 \leq i \leq n$; such that:

(P) $|\{\mathrm{pref}_{k-1}(v_i p) : 0 \le i < n\}| = 1$,

(I) $M^{-1}(\Psi_S(v_{i-1}, u_i) - \Psi_S(v_i, u_{i+1}))$ *is an integer vector, for every* $1 \le i < n$,

(C) $\Psi_k(v_{i-1}, u_i) - \Psi_k(v_i, u_{i+1}) \in \mathrm{im}(N)$ *for every* $1 \le i < n$,

there is $(\alpha_0, \dots, \alpha_n) \in \{0,1\}^{n+1}$ *such that for every* $1 \le i < n$ *:*

$$M^{-1}\Psi_S(v_{i-1}, u_i) - (1-\alpha_{i-1})\Psi(a_{i-1}) - \alpha_i \Psi(a_i) \\ = M^{-1}\Psi_S(v_i, u_{i+1}) - (1-\alpha_i)\Psi(a_i) - \alpha_{i+1}\Psi(a_{i+1}). \quad (1)$$

Then h *is* k*-abelian-*n*-power-free.*

Proof. Suppose that $h(w)$ has a k-abelian-n-power $q_1 \dots q_n$. Let q_0 and q_{n+1} be such that $h(w) = q_0 q_1 \dots q_n q_{n+1}$. By condition (i), if $|q_1| < k-1$, then w has an abelian-n-power. So we have $|q_i| \ge k-1$ for every $1 \le i \le n$.

There are, for every $0 \le i \le n$, $a_i \in A$, $u_i \in \mathrm{Pref}(h(a_i))$ and $r_i \in A^*$ such that, for every $0 \le i \le n$, $r_0 \dots r_i a_i \in \mathrm{Pref}(w)$ and $q_0 \dots q_i = h(r_0 \dots r_i) u_i$. Note that, for a $1 \le i \le n$, r_i can be empty, but a_i is always the first letter of $r_{i+1}a_{i+1}$. Let v_i be such that $u_i v_i = h(a_i)$ for every $0 \le i \le n$. By condition (i), one can suppose w.l.o.g. that $|r_1 \dots r_n a_n| \ge 3$.

By condition (ii), for every $1 \le i \le n$, $\mathrm{pref}_{k-1}(q_i) = \mathrm{pref}_{k-1}(v_{i-1}p)$. Since $q_1 \dots q_n$ is a k-abelian-n-power, we have condition (P).

Claim 1. Let $r \in A^*$ and $u, v \in B^*$. Then:

- $N\Psi(r) = \Psi_k(h(r)p) = \Psi_k(sh(r)) = \Psi_k(sh(r)p) - \Psi_k(sp)$,
- $\Psi_k(vh(r)p) = \Psi_k(vp) + N\Psi(r)$,
- $\Psi_k(sh(r)u) = \Psi_k(su) + N\Psi(r)$.

Proof. If $\mathrm{pref}_{k-1}(u) = p$, then $\Psi_k(vu) = \Psi_k(vp) + \Psi_k(u)$. Similarly, if $\mathrm{suf}_{k-1}(v) = s$, then $\Psi_k(vu) = \Psi_k(v) + \Psi_k(su)$. All the equalities follow from the previous facts, and the definition of N. □

Claim 2. For every $1 \le i \le n$:

$$\Psi_k(q_i) = N(\Psi(r_i) - \Psi(a_{i-1})) + \Psi_k(v_{i-1}, u_i). \quad (2)$$

Proof. By double counting, we have :

$$\Psi_k(q_i) + \Psi_k(sh(r_i a_i)p) = \Psi_k(sh(r_i)u_i) + \Psi_k(v_{i-1}h(a_{i-1}^{-1} r_i a_i)p).$$

By Claim 1:

$$\Psi_k(q_i) + N\Psi(r_i a_i) + \Psi_k(sp) = \\ \Psi_k(su_i) + N\Psi(r_i) + \Psi_k(v_{i-1}p) + N\Psi(a_{i-1}^{-1} r_i a_i).$$

Thus: $\Psi_k(q_i) = \Psi_k(v_{i-1}, u_i) + N(\Psi(r_i) - \Psi(a_{i-1}))$. □

Since $\Psi_k(q_i) = \Psi_k(q_{i+1})$ for every $1 \leq i < n$, we have the condition (C). Now we have directly $\Psi_S(q_i) = M(\Psi(r_i) - \Psi(a_{i-1})) + \Psi_S(v_{i-1}, u_i)$. Since $\Psi_S(q_i) = \Psi_S(q_{i+1})$:

$$M^{-1}(\Psi_S(v_{i-1}, u_i) - \Psi_S(v_i, u_{i+1})) = \Psi(r_{i+1}) - \Psi(a_i) - \Psi(r_i) + \Psi(a_{i-1}).$$

The right part is an integer vector, so we have condition (I). Thus, by condition (iv), there is $(\alpha_0, \ldots, \alpha_n) \in \{0,1\}^{n+1}$ such that (1) is fulfilled.

Equation (1) together with equation (2) give:

$$\begin{aligned} &-\Psi(r_i) + \Psi(a_{i-1}) - (1-\alpha_{i-1})\Psi(a_{i-1}) - \alpha_i\Psi(a_i) \\ &\qquad = -\Psi(r_{i+1}) + \Psi(a_i) - (1-\alpha_i)\Psi(a_i) - \alpha_{i+1}\Psi(a_{i+1}) \end{aligned}$$

that is:

$$\Psi(r_i) - \alpha_{i-1}\Psi(a_{i-1}) + \alpha_i\Psi(a_i) = \Psi(r_{i+1}) - \alpha_i\Psi(a_i) + \alpha_{i+1}\Psi(a_{i+1}). \qquad (3)$$

In equation (3), either the left or the right part is a non-negative vector. Since equation (3) is fulfilled for every $1 \leq i < n$, $\Psi(r_i) - \alpha_{i-1}\Psi(a_{i-1}) + \alpha_i\Psi(a_i)$ is a non negative vector for every $1 \leq i \leq n$. Let $r'_i = a_{i-1}^{-\alpha_{i-1}} r_i a_i^{\alpha_i}$; $1 \leq i \leq n$. Since a_i is the first letter of $r_i a_{i+1}$, and $\Psi(r'_i) = \Psi(r_i) - \alpha_{i-1}\Psi(a_{i-1}) + \alpha_i\Psi(a_i)$ is a non-negative vector, r'_i is well defined in B^*. In one hand $r'_1 \ldots r'_n$ is a factor of w, and is non empty since $|r'_1 \ldots r'_n| \geq |r_1 \ldots r_n a_n| - 2$. On the other hand $\Psi(r'_i) = \Psi(r'_{i+1})$ (by equation 3), for every $1 \leq i < n$. Thus, w has an abelian-n-power $r'_1 \ldots r'_n$. □

We introduce $\Psi_S(v, u)$ in order to handle pairs (v, u) such that $|vu| < k - 1$ (otherwise we have $\Psi_k(v, u) = \Psi_k(vu)$). Theorem 1 gives a set of sufficient conditions, but are still far from a characterization, as Carpi partially done for abelian-n-power-free morphisms [2]. The key point is the condition (ii). One mentions that we can save up the suffix condition in (ii) by carefully handling the cases where u_i or v_i has size less than k. However, we still need either the prefix (or the suffix) condition in order to properly define N.

2.3. 2-abelian-cube-free and 3-abelian-square-free morphisms

Morphisms h_2 and h'_2 respect the conditions of Theorem 1 for $k = 2$ and $n = 3$, *i.e.* are 2-abelian-cube-free, while morphisms h_3 and h'_3 respect the conditions for $k = 3$ and $n = 2$, *i.e.* are 3-abelian-square-free. The checks were done by computer, and took only a few seconds. Thus, the infinite word $h_2(u)$ (resp. $h'_2(u)$) where u is an infinite abelian-cube-free word (for example a fixed point of Dekking's morphism $\mu : 0 \to 0012,\ 1 \to 112,\ 2 \to 022$ [7]) is a 2-abelian-cube-free binary word. Similarly, $h_3(v)$ (resp. $h'_3(v)$), where v is an infinite abelian-square-free word on an alphabet of size 4 (for example, a fixed point of Keränen's morphism g_{85} [18]), is an infinite 3-abelian-square-free ternary word.

Over all the 2-abelian-cube-free morphisms we found, h_2 is the smallest uniform morphism, while h'_2 is the one which minimize $|h(012)|$. If we are only

2-abelian-cube-free morphisms:

$$h_2 : \begin{cases} 0 \to 00100101001011001001010010011001001100101101011 \\ 1 \to 00100110010011001101100110110010011001101101011 \\ 2 \to 00110110101101001011010110100101001001101101011 \end{cases}$$

$$h_2' : \begin{cases} 0 \to 0010010100110010010100100110010011001101 1 \\ 1 \to 0101101100110110011001001100110 11 \\ 2 \to 010110100101001011001101 1 \end{cases}$$

3-abelian-square-free morphisms:

$$h_3 : \begin{cases} 0 \to 0102012021012010201210212 \\ 1 \to 0102101201021201210120212 \\ 2 \to 0102101210212021020120212 \\ 3 \to 0121020120210201210120212 \end{cases}$$

$$h_3' : \begin{cases} 0 \to 012010201202120121012010 21 \\ 1 \to 0120212012102120102 1 \\ 2 \to 0120210201021 \\ 3 \to 0121020121 \end{cases}$$

Morphisms such that $h(\mu^\infty(0))$ is 2-abelian-cube-free:

$$h_d : \begin{cases} 0 \to 001001100110110011001001100100101 \\ 1 \to 001011010110100101001001100100101 \\ 2 \to 001011010110110011001001101011011 \end{cases}$$

$$h_d' : \begin{cases} 0 \to 0101101001011 \\ 1 \to 010110110011011001100100110011011 \\ 2 \to 00100101001001100100110011011 \end{cases}$$

Table 1: Morphisms for k-abelian-n-power-free words.

interested in 2-abelian-cube-free infinite word, one can find simpler construction. The morphism $h_d \circ \mu$ is 2-abelian-cube-free so $h_d(\mu^\infty(0))$ is 2-abelian-cube-free.

We also claim that $h'_d(\mu^\infty(0))$ is 2-abelian-cube-free. One can modify the decision procedure of Theorem 1 to compute the set of "patterns" that u has to avoid to ensure that $h(u)$ is k-abelian-n-power-free. This notion of patterns was used by Carpi [3, 4] to prove that a substitution is abelian-square free, or by Keränen [19] to prove that a fixed point of g_{98} is abelian-square free, even though g_{98} is not abelian-square free. This was also used, under the name of *template*, by Aberkane *et al.* [1] to show the exponential growth rate of abelian-cube-free ternary words, and by Currie and Rampersad [6] for an algorithm which decide if a fixed point of a morphism is abelian-n-power-free. More recently, Mercaş and Saarela [20, 21] used this kind of patterns to show that a morphic word is k-abelian-cube-free.

Doing this, we are able to show that $h'_d \circ \mu^3(u)$ is 2-abelian-cube free if and only if u forbids factors of the form $F = \{pqr, 1p0q0r2 : \Psi(p) = \Psi(q) = \Psi(r)\} \cup \{0p1q0r2, 1p1q0r2 : \Psi(p1) = \Psi(q0) = \Psi(r0)\}$. Moreover, $\mu(u)$ forbids factors of the form F if and only if u forbids factors of the form F (in other words, μ is F-free). Thus, $h'_d(\mu^\infty(0))$ is 2-abelian-cube-free, but for every $n \geq 0$, $h'_d \circ \mu^n$ is not 2-abelian-cube-free (*e.g.* for every $n \geq 0$, $h'_d(\mu^n(1002))$ has a 2-abelian-cube).

2.4. Final remarks and questions

We finally shortly explain why we use this weak notion of k-abelian-n-power-freeness for morphisms. On one hand, k-abelian-squares cannot be avoided by a pure morphic word on a ternary alphabet [13]. So there is no morphism $h : \{0,1,2\} \to \{0,1,2\}$ such that for every k-abelian-square-free word u, $h(u)$ is k-abelian-square-free, except trivial ones. On the other hand, suppose that there is a morphism $h : A^* \to B^*$, with $|A| > |B|$, such that for every 2-abelian-cube-free word $u \in A^*$, $h(u)$ is 2-abelian-cube-free. Without lost of generality, there is $\{a,b\} \subseteq A$, such that the first letter of $h(a)$ and $h(b)$ is the same. Then *babbababb* is 2-abelian-cube-free, but $h(bab) \equiv_{a,2} h(abb)$ thus $h(babbababb)$ is an 2-abelian-cube. We have a contradiction, so such a morphism cannot exist. Nevertheless, we cannot conclude directly when $|A| = |B|$ and the first and last letters of the images differ. More specifically, the following question is still open.

Question 1. Is there a pure morphic binary word which avoids 2-abelian-cubes ?

Let us also raise some questions on avoidability of long repetitions. Every infinite binary word contains arbitrarily long abelian squares, while ones exist which avoid squares of period at least 3 [8, 25]. We recently showed that one can avoid 3-abelian-squares of period at least 3 over a binary alphabet [24]. It seems natural to ask the following:

Question 2. Is there a $p \in \mathbb{N}$ such that 2-abelian-squares of period at least p can be avoided over a binary alphabet ?

That reminds the questions suggested by Mäkelä (see [19]):

Question 3.

(1) Can we avoid abelian-squares of the form uv, with $|u| \geq 2$, over a ternary alphabet ?

(2) Can we avoid abelian-cubes of the form uvw, with $|u| \geq 2$, over a binary alphabet ?

In [24], we answer negatively to (1). Then we modify this question to the following one:

Question 4. Is there a $p \in \mathbb{N}$ such that one can avoid abelian cubes of period at least p over a binary alphabet ?

3. Ternary words avoiding additive cubes

3.1. Testing additive-n-power-freeness

Let Σ be the morphism from the free monoid on the alphabet $\mathbb{N}$ to the additive group $(\mathbb{Z}, +)$ such that $\Sigma(x) = x$ for every $x \in \mathbb{N}$. A word $w \in \mathbb{N}^*$ is an *additive-n-power*, with $n \geq 2$, if $w = p_1 \dots p_n$, such that for every $1 \leq i < n$, $|p_i| = |p_{i+1}|$ and $\Sigma(p_i) = \Sigma(p_{i+1})$. A word is an additive-cube (resp. additive-square) if it is an additive-3-power (additive-2-power). A (possibly infinite) word w is *additive-n-power-free* if no non-empty factor of w is an additive-n-power. Clearly, such words are also abelian-n-power-free. In [5], authors prove that the fixed point of the morphism $0 \to 03, 1 \to 43, 3 \to 1, 4 \to 01$ is additive-cube-free.

A *substitution* is a morphism $s : A^* \to 2^{B^*}$ between the free monoid A^* and the power monoid of B^*, that is the monoid of subsets of B^*, with the operation $U \cdot V = \{uv : (u, v) \in U \times V\}$. A morphism $h : A^* \to B^*$ can be viewed as a substitution $s : A^* \to 2^{B^*}$ such that $s(w) = \{h(w)\}$. A substitution $s : A^* \to 2^{B^*}$, where $A, B \subseteq \mathbb{N}$, is *additive-n-power-free* if for every additive-n-power-free word $u \in A^*$, every $v \in s(u)$ is additive-n-power-free.

We give sufficient conditions for a substitution to be additive-n-power-free in the following theorem.

Theorem 2. *We fix $n \geq 2$ and $A, B \subseteq \mathbb{N}$. Let $s : A^* \to 2^{B^*}$ be a substitution. Suppose that:*

(i) For every additive-n-power-free word $w' \in A^*$ with $|w'| \leq 2$, every $w \in s(w')$ is additive-n-power-free.

(ii) There is $(l, \gamma, \beta) \in \mathbb{N} \times \mathbb{Z} \times \mathbb{Z}$, with $\beta \neq 0$, such that for every $a \in A$ and $w \in s(a)$, we have $|w| = l$ and $\Sigma(w) = \gamma + a\beta$.

(iii) For every $a_i \in A$, $w_i \in s(a_i)$, and $u_i, v_i \in A^*$ with $u_i v_i = w_i$; $0 \leq i \leq n$; such that for every $1 \leq i < n$:

(L) $|v_{i-1} u_i| \equiv |v_i u_{i+1}| \pmod{l}$,

$$s_{015}:\begin{cases}0\to\{005015100100115010115, 005015100100115100115\}\\ 1\to\{005015100100105055115, 050015100100105055115\}\\ 3\to\{005015101155155055115, 050015101155155055115\}\\ 4\to\{005015155055155055115, 050015155055155055115\}\end{cases}$$

$$s_{016}:\begin{cases}0\to\{001011601010006001016, 001011601010006001106\}\\ 1\to\{001660601010006001016, 001660601010006001106\}\\ 3\to\{001661661101606661106, 001661661101660661106\}\\ 4\to\{001661660661606661106, 001661660661660661106\}\end{cases}$$

$$s_{017}:\begin{cases}0\to\{001700100117110010071, 001700100117110010701\}\\ 1\to\{001700177070010010071, 001700177070010010701\}\\ 3\to\{001700177110171770077, 010700177110171770077\}\\ 4\to\{001700177070771770077, 010700177070771770077\}\end{cases}$$

$$s_{027}:\begin{cases}0\to\{002072022020722007, 002072022027022007\}\\ 1\to\{722072022020722007, 722072022027022007\}\\ 3\to\{707720077070722007, 707720077072702200 7\}\\ 4\to\{707727277070722007, 707727277072702200 7\}\end{cases}$$

$$s_{037}:\begin{cases}0\to\{003003073030377073 07, 003003073030377007 37, 003003073030377070 37\}\\ 1\to\{003003007077377007 37, 003003007077377070 37, 003003007077377073 07\}\\ 3\to\{003377303377377007 37, 003377303377377070 37, 003377303377377073 07\}\\ 4\to\{003377377077377007 37, 003377377077377073 07, 003377377077377070 37\}\end{cases}$$

$$s_{018}:\begin{cases}0\to\{008100100801181101 1, 008101008001181101 1, 008100108001181101 1\}\\ 1\to\{008100100801181800 8, 008101008001181800 8, 008100108001181800 8\}\\ 3\to\{008101881880881181 1, 008110881880881181 1, 008181081880881181 1\}\\ 4\to\{008101881880880818 8, 008110881880880818 8, 008118801880880818 8\}\end{cases}$$

$$s_{038}:\begin{cases}0\to\{003800303830033833003, 003800308330033833003\}\\ 1\to\{003800303830080038388, 003800308330080083838\}\\ 3\to\{003808833833038838838, 003808838330338838838\}\\ 4\to\{003808838388088388388, 083008838388088388388\}\end{cases}$$

$$s_{019}:\begin{cases}0\to\{0090110191001009, 0090110911001009\}\\ 1\to\{0090119110110199, 0900119110110199\}\\ 3\to\{0090190090099199, 0900190090099199\}\\ 4\to\{0090119199099199, 0900119199099199\}\end{cases}$$

$$s_{029}:\begin{cases}0\to\{002900200200900220 29, 002900200200900202 29\}\\ 1\to\{002900992200900220 29, 002900992200900202 29\}\\ 3\to\{002202922990992990 99, 229202200990992990 99\}\\ 4\to\{229209922990992990 99, 229902922990992990 99\}\end{cases}$$

$$s_{049}:\begin{cases}0\to\{00400400900499009, 00400400900949009\}\\ 1\to\{00400449440099409, 00400449440499009\}\\ 3\to\{00409909909499409, 00409909949099409\}\\ 4\to\{44944909949499009, 44944909949909409\}\end{cases}$$

Table 2: Additive-cube-free substitutions.

(M) $\Sigma(v_{i-1}u_i) \equiv \Sigma(v_i u_{i+1}) + x_i\gamma \pmod{\beta}$,

(where $x_i = (|v_{i-1}u_i| - |v_i u_{i+1}|)/l$ *for every* $1 \le i < n$*)*
there is $(\alpha_0, \ldots, \alpha_n) \in \{0,1\}^{n+1}$ *such that for every* $1 \le i < n$*:*

(a) $\alpha_i - \alpha_{i-1} = x_i + \alpha_{i+1} - \alpha_i$,

(b) $\Sigma(v_{i-1}u_i) + \beta[(\alpha_{i-1} - 1)a_{i-1} - \alpha_i a_i]$
$= \Sigma(v_i u_{i+1}) + \gamma x_i + \beta[(\alpha_i - 1)a_i - \alpha_{i+1}a_{i+1}]$.

Then s *is additive-*n*-power-free.*

Proof. Suppose that $w \in s(w')$ has an additive-n-power $q_1 \ldots q_n$. Let q_0 and q_{n+1} be such that $w = q_0 q_1 \ldots q_n q_{n+1}$.

For every $0 \le i \le n$, there is $a_i \in A$, $w_i \in s(a_i)$, $u_i \in \mathrm{Pref}(w_i)$, and $r_i \in A^*$ such that $r_0 \ldots r_i a_i \in \mathrm{Pref}(w')$ and $q_0 \ldots q_i \in s(r_0 \ldots r_i) \cdot \{u_i\}$. Let v_i be such that $u_i v_i = w_i$ for every $0 \le i \le n$. By condition (i), one can suppose w.l.o.g. that $|r_1 \ldots r_n a_n| \ge 3$.

By condition (ii), for every $p \in s(p')$, we have $\Sigma(p) = \gamma|p'| + \beta\Sigma(p')$.

For every $1 \le i \le n$, we have $u_{i-1}q_i \in s(r_i) \cdot \{u_i\}$. Thus, by condition (ii), and by the fact that $u_{i-1}v_{i-1} = w_{i-1}$ we have:

$$|q_i| = |v_{i-1}u_i| + l(|r_i| - 1) \tag{4}$$

and

$$\Sigma(q_i) = \gamma(|r_i| - 1) + \beta(\Sigma(r_i) - a_{i-1}) + \Sigma(v_{i-1}u_i). \tag{5}$$

By equation (4) and by the fact that for every $1 \le i < n$, $|q_i| = |q_{i-1}|$, we have the condition (L), and:

$$|r_{i+1}| - |r_i| = (|v_{i-1}u_i| - |v_i u_{i+1}|)/l = x_i.$$

Since for every $1 \le i < n$, $\Sigma(q_i) = \Sigma(q_{i-1})$, we have:

$$\gamma(|r_i| - 1) + \beta(\Sigma(r_i) - a_{i-1}) + \Sigma(v_{i-1}u_i) \\ = \gamma(|r_{i+1}| - 1) + \beta(\Sigma(r_{i+1}) - a_i) + \Sigma(v_i u_{i+1}). \tag{6}$$

Thus

$$\Sigma(v_{i-1}u_i) = \Sigma(v_i u_{i+1}) + \gamma x_i + \beta(\Sigma(r_{i+1}) - a_i - \Sigma(r_i) + a_{i-1}),$$

and equation (M) is fulfilled.

So, by condition (iii), there is $(\alpha_0, \ldots, \alpha_n) \in \{0,1\}^{n+1}$ such that (a) and (b) are fulfilled.

By equation (a), we have, for every $1 \le i < n$;

$$|r_i| + \alpha_i - \alpha_{i-1} = |r_{i-1}| + \alpha_{i+1} - \alpha_i. \tag{7}$$

If r_i is empty, $a_i = a_{i+1}$ otherwise the first letter of r_i is a_i. In equation (7), the right side or the left side must be non-negative. Thus, for every $1 \le i \le n$,

$|r_i| + \alpha_i - \alpha_{i-1} \geq 0$, and $r'_i = a_{i-1}^{-\alpha_{i-1}} r_i a_i^{\alpha_i}$; $1 \leq i \leq n$; is well defined. We have $|r'_i| = |r_i| + \alpha_i - \alpha_{i-1}$ and $\Sigma(r'_i) = \Sigma(r_i) + \alpha_i a_i - \alpha_{i-1} a_{i-1}$. By equation (7), for every $1 \leq i < n$, $|r'_i| = |r'_{i+1}|$. Moreover $r'_1 \dots r'_n$ is a factor of w', and is non empty since $|r'_1 \dots r'_n| \geq |r_1 \dots r_n a_n| - 2$.

When we subtract (b) to (6), we get $\beta\Sigma(r'_i) = \beta\Sigma(r'_{i+1})$. Thus, $\Sigma(r'_i) = \Sigma(r'_{i+1})$ for every $1 \leq i < n$, and w' has an additive-n-power $r'_1 \dots r'_n$. □

Theorem 2 can be used to find additive-square-free, additive-cube-free and additive-4-power-free substitutions. However, we have few hopes to find an additive-square-free substitution, while additive-4-powers are equivalent to abelian-4-powers on binary words.

3.2. Additive-cube-free substitutions

We have checked by computer that every substitution in Table 2 respects the conditions of Theorem 2. Since there is an infinite additive-cube-free word on the alphabet $\{0, 1, 3, 4\}$, one can construct infinite additive-cube-free words on the alphabets $\{0, 1, 5\}$, $\{0, 1, 6\}$, $\{0, 1, 7\}$, $\{0, 2, 7\}$, $\{0, 3, 7\}$, $\{0, 1, 8\}$, $\{0, 3, 8\}$, $\{0, 1, 9\}$, $\{0, 2, 9\}$ and $\{0, 4, 9\}$. In our substitutions, each letter has at least two images. This clearly shows that number of additive-cube-free words on these alphabets grows exponentially. For the alphabet $\{0, 1, 8\}$, we got 3 images of size 19 for each letter, giving the lower bound of $3^{1/19} = 1.059526\dots$ for the growth rate. This bound is also a new lower bound for the growth rate of abelian-cube-free words on ternary alphabet. (The previous known bound was $2^{1/24} = 1.029302\dots$ in [1].)

We conjecture that for every alphabet $A = \{0, i, j\}$ such that i and j are coprime and $j \geq 6$, there exists an infinite additive-cube-free word on the alphabet A. The cases $\{0, 1, 2\}$, $\{0, 1, 3\}$, $\{0, 1, 4\}$ and $\{0, 2, 5\}$ are left open. Furthermore, it seems difficult to construct a very long word on the alphabet $\{0, 1, 2, 3\}$ avoiding additive cubes (the longest we got has size $\sim 1.4 \times 10^5$).

Question 5. Are there infinite additive-cube-free words on the following alphabets : $\{0, 1, 2, 3\}$, $\{0, 1, 4\}$ and $\{0, 2, 5\}$?

The substitutions in Table 3 also respect the conditions of Theorem 2, thus the existence of an infinite additive-cube-free word on the alphabet $\{0, 1, 2, 3\}$ imply the existence of infinite additive-cube-free words on alphabets $\{0, 1, 4\}$ and $\{0, 2, 5\}$.

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$$s_{014} : \begin{cases} 0 \to \{004114104011011004011\} \\ 1 \to \{004114104011011014144\} \\ 2 \to \{004114104010044044144\} \\ 3 \to \{004114104044144044144\} \end{cases}$$

$$s_{025} : \begin{cases} 0 \to \{02200520220250552\} \\ 1 \to \{02252520220250552\} \\ 2 \to \{02255055200550552\} \\ 3 \to \{02255055252550552\} \end{cases}$$

Table 3: Additive-cube-free substitutions from $\{0, 1, 2, 3\}^*$.

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