Leftover exercises from 2018

Exercise 1 — *Limit in distribution of Gaussian vectors.*
Let $(X_n)_{n\geq 0}$ be a sequence of Gaussian variables $(X_n)_{n\geq 0}$. Give a necessary and sufficient condition for convergence in distribution, show that the limit is always Gaussian, and determine its parameters.

Solution 1 — *Limit in distribution of Gaussian vectors.*
We restrict ourselves to gaussian **variables**. It is rather easy to lift this up to vectors afterwards. Let μ_n and σ_n be the parameters of X_n If we have convergence in distribution, then we have convergence of the characteristic functions to the one of the limit. So there exists a characteristic function $f : \mathbb{R} \to \mathbb{R}$ such that for all $t \in \mathbb{R}$, $f_n(t) = e^{i\mu_n t - \frac{\sigma_n^2}{2}t^2} \to f(t)$. Now taking the modulus then the log yields $\sigma_n^2 \to -\frac{2}{t^2}\log(|f(t)|) = \sigma^2 \geq 0$. We deduce that $|f(t)| = e^{-\frac{\sigma^2}{2}t^2}$. Now $e^{i\mu_n t} = e^{\frac{\sigma_n^2}{2}t^2} f_n(t) \to e^{\frac{\sigma^2}{2}t^2} f(t) = u(t)$, which is a continuous function in $\mathbb{C}$ of modulus 1 (with $u(0) =: 1$). So it can be lifted up to a continuous real function, *i.e.* there exists h continuous with $h(0) = 0$ such that $u(t) = e^{ih(t)}$ for all t. We have

$$e^{i(\mu_n t - h(t))} \to 0.$$

We shall now show that $(\mu_n)_n$ is bounded. This important step is treated with a probabilistic proof: we use the fact that the distribution of X_n is symmetric about its mean[1]. Suppose there is an increasing subsequence $\mu_{k_n} \to \infty$. Then $\mathbb{P}(X_{k_n} \geq \mu_{k_n}) = 1/2$ for all n, and $\mathbb{P}(X_{k_n} \geq \mu_{k_p}) \geq 1/2$ for all $n \geq p$. So by taking $n \to \infty$ with fixed p we get $\mathbb{P}(X \geq \mu_{k_p}) \geq 1/2$ for all p, which is absurd as $\mu_{k_p} \to \infty$.
So $(\mu_n)_n$ is bounded above and the symmetric argument allows to show that it is bounded below.
Back to our problem, we shall now show that $A = \{t \in \mathbb{R} : \mu_n t \to h(t)\}$ is the whole of $\mathbb{R}$.

- It is nonempty as it contains 0.
- It is closed because of the uniform control of μ_n in n.
- It is open: let $t \in A$. For $s \in \mathbb{R}$ we have $e^{i(\mu_n t - h(t) - \mu_n s + h(s)}) \to 0$. By the bound on μ_n and continuity of h we can find $\epsilon > 0$ such that for all $s \in (t-\epsilon, t+\epsilon)$ and all n, $|\mu_n t - h(t) - \mu_n s + h(s)| < \pi/2$. But for $|\theta| < \pi/2$, $\theta \mapsto e^{i\theta}$ is an homeomorphism. We deduce $\mu_n t - h(t) - \mu_n s + h(s) \to 0$ and hence $s \in A$.

We conclude by connectedness of $\mathbb{R}$. We get that for every $t \neq 0$, $\mu_n \to h(t)/t$, so μ_n converges to some μ and $h(t) = \mu t$. This proves that $f(t) = e^{i\mu t - \frac{\sigma^2}{2}t^2}$, so X is a Gaussian with parameters $\mu = \lim \mu_n$ and $\sigma^2 = \lim \sigma_n^2$. Conversely these convergences directly imply convergence in distribution.

[1] Since we know that σ_n is bounded, we could as well use the fact that X_n concentrates around its mean

Exercise 2 — *The precise constant (Lévy, 1937).*
We want to show that with probability one,

$$\limsup_{h\downarrow 0} \frac{m_B(h,[0,1])}{\sqrt{2h\log(1/h)}} = 1.$$

(1) Show that if X is standard Gaussian and $x>0$, then

$$\frac{1}{\sqrt{2\pi}(x+1/x)}e^{-x^2/2} \le \mathbb{P}(X\ge x) \le \frac{1}{\sqrt{2\pi}x}e^{-x^2/2}.$$

(2) For $c<\sqrt{2}$, show that almost surely for all $\epsilon>0$ there exists $s,t\in[0,1]$ with $|t-s|\le\epsilon$ and $|B(t)-B(s)|\ge c\sqrt{|t-s|\log(1/|t-s|)}$. (Hint: divide $[0,1]$ in intervals of length 2^{-n}).

(3) Fix $m\ge 1$ and define the following families of intervals:

$$\Lambda_n(m) = \left\{[(k/m-1)2^{-n/m},(k/m)2^{-n/m}],\quad m\le k\le m2^{n/m}\right\},\quad n\ge 1.$$

For $c>\sqrt{2}$, show that almost surely, for n large enough and any interval $[s,t]$ in the family $\Lambda_n(m)$, $|B(t)-B(s)|\le c\sqrt{|t-s|\log(1/|t-s|)}$.

(4) Fix $\epsilon>0$, show that there exists $m\ge 1$ such that any interval $[s,t]\subset[0,1]$ can be approximated with an interval $[s',t']\in\Lambda(m)=\cup_{n\ge1}\Lambda_n(m)$, with $|t-t'|,|s-s'|\le \epsilon|t-s|$, and $|t'-s'|\le|t-s|$.

(5) Deduce that almost surely, for h small enough, $m_B(h,[0,1])\le C\sqrt{h\log(1/h)}$, for a constant C that can be brought arbitrarily close to $\sqrt{2}$. Conclude.

Solution 2 — *The precise constant (Lévy, 1937).* (1) The upper bound comes from the inequality $\int_x^\infty e^{-t^2/2}dt \le \int_x^\infty \frac{t}{x}e^{-t^2/2}dt$. The lower bound can be obtained by differentiating the difference.

(2) First of all, $\mathbb{P}(E_{k,n}) = \mathbb{P}(B_{(k+1)2^{-n}}-B_{k2^{-n}} \ge c\sqrt{2^{-n}\log(2^n)}) = \mathbb{P}(B_1\ge c\sqrt{n\log 2}) \ge \frac{1}{1000c\sqrt{n}}2^{-c^2n/2}$. Then

$$\mathbb{P}(\forall 0\le k\le 2^-n, B_{(k+1)2^{-n}}-B_{k2^{-n}} < c\sqrt{2^{-n}\log(2^n)}) = \mathbb{P}(\bigcap_k E_{k,n}^{\complement})$$
$$=\prod_k(1-\mathbb{P}(E_{k,n})) \le (1-\frac{1}{1000c\sqrt{n}}2^{-c^2n/2})^{2^n} \le \exp(-2^n\frac{1}{1000c\sqrt{n}}2^{-c^2n/2})$$
$$=\exp(-\frac{1}{1000c\sqrt{n}}2^{(1-c^2/2)n}) = \text{summable in} n.$$

So by Borel-Cantelli, we get that infinitely often in n, there is an increment of length 2^{-n} that exceeds $c\sqrt{2^{-n}\log(2^n)}$. This implies the claim.

(3)

$$\begin{aligned}
\mathbb{P}(\exists[s,t] \in \Lambda_n(m), |B(t)-B(s)| > c\sqrt{|t-s|\log(1/|t-s|)})\\
&\leq m2^{n/m}\,\mathbb{P}(|B_1| \geq c\sqrt{n/m\log 2})\\
&\leq m2^{n/m}\frac{1}{\sqrt{2\pi}c\sqrt{n/m\log 2}}2^{-(c^2/2)n/m} = \text{summable in } n
\end{aligned}$$

So almost surely, for n large enough, any interval in $\Lambda_n(m)$ has the required growth bound.

(4) Take m to be determined later in terms of ϵ. Then given t and s, we can find n so that $1 \leq |t-s|/2^{-n/m} \leq (2^{1/m}) \leq 1+\epsilon/3$. We can now find k so that $|s - \frac{k}{m}2^{-n/m}| \leq \frac{1}{m}2^{-n/m} \leq \frac{1}{m}|t-s|$. Set $s' = \frac{k}{m}2^{-n/m}$, $t' = (\frac{k}{m}+1)2^{-n/m}$. Then $|s'-s| \leq \frac{1}{m}|t-s|$ and $|t'-t| \leq |t'-s'| + |s'-s| \leq (2^{1/m}-1)|t-s| + \frac{1}{m}|t-s|$. Now choose retrospectively m so that $2^{1/m} - 1 + \frac{1}{m} < \epsilon$ and $\frac{1}{m} < \epsilon$ makes everything work. Remark that we additionaly get $|t'-s'| \leq |t-s|$ which eases the solution of the next question.

(5) Fix ϵ and m accordingly. Now almost surely, there is n_0 such that for $n \geq n_0$, all intervals in $\Lambda_n(m)$ have the growth bound with the constant c. Moreover, from the lecture, almost surely there is a h_0 such that all intervals of length $< h_0$ have the growth bound with the constant C from the lecture. Now take s, t such that $|s-t| \leq \epsilon$, $\epsilon|s-t| \leq h_0$ and $|s-t| \leq 2^{-n_0/m}$. Then consider s', t' as in the previous question. It comes that $|t'-t|, |s'-s| \leq h_0$ and that $|s'-t'| \in \Lambda_n(m)$ with $n \geq n_0$. Hence

$$\begin{aligned}
|B_t - B_s| &\leq |B_t - B'_t| + |B_s - B'_s| + |B_t - B_s|\\
&\leq C\sqrt{|s'-s|\log(1/|s'-s|)} + C\sqrt{|t'-t|\log(1/|t'-t|)} + c\sqrt{|t'-s'|\log(1/|t'-s'|)}\\
&\leq 2C\sqrt{\epsilon|t-s|\log(1/(\epsilon|t-s|))} + c\sqrt{|t-s|\log(1/|t-s|)}\\
&\leq (2C\sqrt{\epsilon(1+1)} + c)\sqrt{|t-s|\log(1/|t-s|)}.
\end{aligned}$$

Where at the second inequality we used the increasing character (close to 0) of $x \mapsto \sqrt{x\log(1/x)}$ and at the last one we used $\log(1/\epsilon) \leq \log(1/|s-t|)$. The constant obtained can be brought arbitrarily close to $\sqrt{2}$ as c was arbitrary $> \sqrt{2}$, ϵ arbitrary > 0 and C fixed.

Exercise 3 — *A bit more on differentiability.*
We know that almost surely, B is nowhere differentiable. Set $D^*B(t) = \limsup_{h\downarrow 0}\frac{1}{t}(B_{t+h} - B_t)$ and $D_*B(t) = \liminf_{h\downarrow 0}\frac{1}{t}(B_{t+h} - B_t)$.

(1) Show that B almost surely not bounded above nor below. Deduce that $D^*B(0) = +\infty$ a.s. and $D_*B(0) = -\infty$ a.s.

(2) Deduce that almost surely, the Lebesgue measure of times t such that $D^*B(t) \neq +\infty$ or $D_*B(t) \neq -\infty$ is 0.

(3) Show that with probability one a fixed point t is not a one-sided local maximum of B. Deduce that with probability one there exists a density of exceptional random times where $D^*B(t) \leq 0$.
(4) Show that there almost surely exists an uncountable density of points t where $D^*(t) = 0$. (Hint : consider $\tau(x) = \inf\{t \geq 0, B_t = x\}$. Show that this is almost surely a strictly increasing function whose discontinuity points are dense and deduce that $V_n = \{x \geq 0, \exists h \in (0, 1/n), \tau(x-h) < \tau(x) - nh\}$ is open and dense. What can be said about $\bigcap_{n\geq 1} V_n$?)

Solution 3 — *A bit more on differentiability.*
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(1) We showed earlier that almost surely, $\limsup B_t = +\infty$ and $\liminf B_t = -\infty$ almost surely (actually we showed that the rate stricly more than $\sqrt{t}$) Hence the claim by time inversion.
(2) $\mathbb{E}[\mathrm{Leb}\{t \geq 0, D^*B(t) \neq +\infty \text{ or } D_*B(t) \neq -\infty\}] = \int_{\mathbb{R}} dt\, \mathbb{P}(D^*B(t) \neq +\infty \text{ or } D_*B(t) \neq -\infty) = \int_R 0 = 0$, where we used Fubini and Markov.
(3) We know that 0 is almost surely not a local extremum at its right because there is an accumulation of instants where B is strictly positive and negative near 0. For a fixed point t, we treat the right side by Markov and the left side by time reversal. Now for fixed $p, q \in \mathbb{Q}_+$ almost surely p, q are not one-sided local extrema. Hence the maximum of B on $[p, q]$ is reached somewhere in the interior, and that is a point inside (p, q) where $D^*B \leq 0$. We get the claim by countable union.
(4) We consider $\tau(x) = \inf\{t \geq 0, B_t = x\}$. This is by definition strictly increasing function, and if it were continuous on some open interval, then B would be monotonous on some open interval, which it is almost surely not. Now if we consider $V_n = \{x \geq 0, \exists h \in (0, 1/n), \tau(x-h) < \tau(x) - nh\}$, it is open because τ is càglàd strictly increasing. It is dense because otherwise we found an open interval of x where $\forall h \in (0, 1/n)$, $\tau(x) - nh \leq \tau(x-h) \leq \tau(x)$, implying continuity on some open interval. Then by the Baire category theorem, $\bigcap_{n\geq 1} V_n$ is uncountable and dense. Let x be in this set, and $t = \tau(x)$. Then there exists a sequence $t_n \uparrow t$, $B^*(t_n) > t - 1/n$, $t_n < t - nB^*(t_n)$. Hence the lower left derivative of B at t is 0. The upper left derivative is 0 too by definition. We get the claim by time reversal.