Solutions for $\mathbb{E}$xercise sheet 11: Miscellanea

Solution 1 — *Capacity and Hausdorff dimension.*
Let f be a positive function on $\mathbb{R}^d$ called *potential*. The energy of a measure μ is $I_f(\mu) = \iint f(x-y)\mu(dx)\mu(dy)$. The capacity of some set A is

$$\mathrm{Cap}_f(A) = [\inf\{I_f(\mu) : \mu \text{ probability measure on } A\}]^{-1}$$

At some point you will see that a closed set is polar in dimension $d \geq 2$ if and only if it has zero capacity for the radial potential $f(\epsilon) = |\log(\epsilon)|$ if $d = 2$ and $f(\epsilon) = \epsilon^{2-d}$ if $d \geq 3$. We wish to show a connexion between the notion of capacity and Hausdorff dimension.

(1) Let $(U_i)_i \in \mathcal{P}(A)^{\mathbb{N}}$ be such that for all i, $\mathrm{diam}(U_i) \leq \delta$ and the $(U_i)_i$ forms a partition of A.

$$\begin{aligned}\iint_{|x-y|<\delta} \mu(dx)\mu(dy)|x-y|^{-\alpha} &\geq \iint_{|x-y|<\delta} \mu(dx)\mu(dy)(\sum_i \mathbb{1}_{x,y\in U_i})|x-y|^{-\alpha} \\ &\geq \sum_i \iint_{U_i^2} \mu(dx)\mu(dy)|x-y|^{-\alpha} \\ &\geq \sum_i \mu(U_i)^2 \,\mathrm{diam}(U_i)^{-\alpha}\end{aligned}$$

Hence

$$\begin{aligned}&\left(\iint_{|x-y|<\delta} \mu(dx)\mu(dy)|x-y|^{-\alpha}\right)\left(\sum_{i\in\mathbb{N}} \mathrm{diam}(U_i)^{\alpha}\right) \\ &\geq \left(\sum_i \mu(U_i)^2\,\mathrm{diam}(U_i)^{-\alpha}\right)\left(\sum_{i\in\mathbb{N}} \mathrm{diam}(U_i)^{\alpha}\right) \\ &\geq \left(\sum_i \mu(U_i)\,\mathrm{diam}(U_i)^{\alpha/2}\,\mathrm{diam}(U_i)^{-\alpha/2}\right)^2 = \left(\sum_i \mu(U_i)\right)^2 = \mu(A)^2\end{aligned}$$

by Cauchy-Schwarz, yielding the desired inequality. Taking the infimum on all $(U_i)_i$ then the limit $\delta \to 0$ yields

$$\mathcal{H}^{\alpha}(A) \geq \frac{\mu(A)^2}{\iint_A \mu(dx)\mu(dy)|x-y|^{-\alpha}}$$

Hence for a set of nonzero finite α-capacity, by definition there exists $\mu > 0$ such that $\iint_A \mu(dx)\mu(dy)|x-y|^{-\alpha} < \infty$, so the right-hand-side is bounded below away from 0. Hence $H^{\alpha}(A) > 0$ and the Hausdorff dimension is larger than α.

(2) Assume wlog that the segment is $[0,1]$. Let C be the α-Hölder constant. For $n \geq 1$ take $U_k = f([k/n,(k+1)/n])$ for $0 \leq k \leq n-1$. Then it is a cover of $f([0,1])$ and $\operatorname{diam}(U_k) \leq C(1/n)^\alpha$. Hence $\sum_k \operatorname{diam}(U_k)^{1/\alpha} \leq \sum_k C^{1/\alpha} 1/n \leq C^{1/\alpha}$. So we found arbitrarily fine covers with bounded α-sum. Hence $\mathcal{H}^\alpha(A) < \infty$ and $\dim_{\mathcal{H}}(A) \leq \epsilon$.

(3) If $d = 1$ $B([0,1])$ almost surely contains a ball so has Hausdorff dimension 1.
If $d \geq 2$, we use question 2 and the fact that B is almost surely $(1/2-\epsilon)$-Hölder on $[0,1]$ to show that $\dim_{\mathcal{H}}(B([0,1])) \leq 2$. For the lower bound we consider the (random) occupation measure $\mu = B_\star \operatorname{Leb}_{[0,1]}$. If we take $\alpha < 2$ and compute

$$\begin{aligned}
\mathbb{E}\left[\iint_{B([0,1])^2} \mu(dx)\mu(dy)(x-y)^{-\alpha}\right] &= \mathbb{E}\left[\iint_{[0,1]^2} dxdy(B(x)-B(y))^{-\alpha}\right] \\
&= \iint_{[0,1]^2} dxdy\,\mathbb{E}[(B(x)-B(y))^{-\alpha}] \\
&= \iint_{[0,1]^2} dxdy(x-y)^{-\alpha/2}\,\mathbb{E}[(B(1))^{-\alpha}]
\end{aligned}$$

This is a product of two integrals, the first one boils down to $\int_0^1 r^{-\alpha/2}dr < \infty$, the second one to $\int_0^\infty r^{d-1}r^{-\alpha}e^{-r^2/2}dr < \infty$, since $\alpha < 2$. Hence the random variable $\iint_{B([0,1])^2} \mu(dx)\mu(dy)(x-y)^{-\alpha}$ has finite expectation and is almost surely finite. Hence almost surely $\dim_{\mathcal{H}}(B([0,1])) > \alpha$. Hence $\dim_{\mathcal{H}}(B([0,1])) = 2$ almost surely.

Solution 2 — *Some more boundary value problems.*
In this exercise we admit that for $x, y \in \mathbb{R}^d$, $t > 0$, we have $\partial_t p_t(x,y) = \frac{1}{2}\Delta_y p_t(x,y)$. (Fokker-Planck equation)

(1) This process has clearly independent increments, so we need only show that it is centered.

$$\begin{aligned}
\mathbb{E}_x[X_t] &= \mathbb{E}_x\left[f(B_t) - \frac{1}{2}\int_0^t \Delta f(B_s)ds\right] \\
&= \int_y f(y)p_t(x,y)dy - \frac{1}{2}\int_0^t \left(\int_y p_s(x,y)\Delta f(y)dy\right)ds \\
\frac{\partial}{\partial t}\mathbb{E}_x[X_t] &= \int_y f(y)\frac{\partial}{\partial t}p_t(x,y)dy - \frac{1}{2}\int_y p_t(x,y)\Delta f(y)dy \\
&= \int_y f(y)\frac{\partial}{\partial t}p_t(x,y)dy - \frac{1}{2}\int_y \Delta p_t(x,y)f(y)dy \\
&= \int_y \left(\frac{\partial}{\partial t}p_t(x,y) - \frac{1}{2}\Delta p_t(x,y)\right)f(y)dy = 0
\end{aligned}$$

Where we used Fubini, Lebesgue's differentiation theorem, and integration by part (the fact that f has compact support makes the boundary term vanish). Hence $\mathbb{E}_x[X_t] = \mathbb{E}_x[X_0]$ and we are done.

(2) Once again we need only show that the increments are centered. We want to reuse question 1. Let $\epsilon > 0$ and ϕ_ϵ a $\mathcal{C}^\infty$ approximation of unity with support contained in $B(0, \epsilon)$. Let also $D_\epsilon = \mathbb{R}^d \setminus B(D^\complement, \epsilon)$. Set $f_\epsilon = (\mathbb{1}_{D_{\epsilon/2}} * \phi_{\epsilon/4})f$. Then f_ϵ verifies the hypotheses of question 1. Hence, setting T_ϵ to be the hitting time of $D_\epsilon^\complement$, and using the optional stopping theorem for $f_\epsilon(B_t) - \int_0^t \Delta f_\epsilon(B_s)ds$ at stopping time $t \wedge T_\epsilon$, we get

$$\begin{aligned} f(x) &= \mathbb{E}_x \left[f_\epsilon(B_{t\wedge T_\epsilon}) - \frac{1}{2}\int_0^{t\wedge T_\epsilon} \Delta f_\epsilon(B_s)ds \right] \\ &= \mathbb{E}_x \left[f(B_{t\wedge T_\epsilon}) - \frac{1}{2}\int_0^{t\wedge T_\epsilon} \Delta f(B_s)ds \right] \\ &\xrightarrow{\epsilon\to 0} \mathbb{E}_x \left[f(B_{t\wedge T}) - \frac{1}{2}\int_0^{t\wedge T} \Delta f(B_s)ds \right] \end{aligned}$$

where we used the fact that f and f_ϵ coincide on D_ϵ at the second line, and the continuity of paths with the dominated convergence theorem at the last line (this uses the boundedness of f and its derivatives, along with integrability of the first exit time of bounded domains). This finishes the question.

(3) Show that in the sense of distributions, we have $\Delta G(x, \cdot) = \delta_x$, where G is the Green function of the Brownian motion in the whole of $\mathbb{R}^3$ or in a bounded domain of $\mathbb{R}^2$.

Let D be the domain in which we are working, possibly $\mathbb{R}^d$ for $d \geq 3$. We need to show that for ϕ $\mathcal{C}^\infty$ and compactly supported (in particular ϕ vanishes at the boundary of D), we have

$$\int \Delta\phi(y)G(x,y)dy = -2\int \delta_x(y)\phi(y)dy = -2\phi(x).$$

But by definition of G, for all θ, $\int \theta(y)G(x,y)dy = \mathbb{E}_x[\int_0^T \theta(B_s)ds]$. Hence if we go back to the result of question 2, we have

$$\begin{aligned} \phi(x) &= \mathbb{E}_x \left[\phi(B_{t\wedge T}) - \frac{1}{2}\int_0^{t\wedge T} \Delta\phi(B_s)ds \right] \\ &\xrightarrow{t\to\infty} \mathbb{E}_x \left[\phi(B_T) - \frac{1}{2}\int_0^{T} \Delta\phi(B_s)ds \right] \\ &= 0 - \frac{1}{2}\mathbb{E}_x \left[\int_0^T \Delta\phi(B_s)ds \right] = -\frac{1}{2}\int_y \Delta\phi(y)G(x,y)dy. \end{aligned}$$

which is what we wanted. We used a dominated convergence theorem at line 2:

- when D is bounded the almost sure convergence is immediate, and when D is unbounded, in dimension ≥ 3, it comes from the transience of Brownian motion and compactness of $\mathrm{supp}(\phi)$.
- the domination is by $\|\phi\|_\infty + \|\Delta\phi\|_\infty \int_0^T \mathbb{1}_{B_s\in\mathrm{supp}(\phi)}\, ds$, whose expectation is bounded by $C\int_{\mathrm{supp}(\phi)} G(x,y)dy < \infty$.

We also used the fact that ϕ vanishes at the boundary of D at line 3.

(4) This is only a matter of applying question 2 to u and once again the dominated convergence theorem as $t \to \infty$.

(5) The fact that u is continuous at the boundary follows from the same proof as for the Laplace problem, using the Poincaré cone condition.
To show that $\Delta u = f$, it is a simple matter from the previous question that this holds in the *weak sense*, using Fubini. For the strong sense, see the book mentioned in the remark...

Solution 3 — *Transition probabilities and Green's function on the disc.*
To be updated when I know how to do this exercise!