ENS de Lyon - M1 **jeudi 3 mai 2018**

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examen de processus stochastiques et mouvement brownien (3 heures)

No document is allowed for this exam. The two exercices are independent. We admit the following generalization of Borel-Cantelli lemma, also known as Kochen-Stone lemma. (For your information, Kochen-Stone lemma is not hard to prove with the help of the Paley-Zygmund inequality)

Lemma. *If the events A_n satisfy $\sum P(A_n) = +\infty$ and*

$$\limsup_{n\to\infty} \frac{\left(\sum_{k=0}^n \mathbb{P}(A_k)\right)^2}{\sum_{k=0}^n \sum_{l=0}^n \mathbb{P}(A_k \cap A_l)} = c > 0,$$

then $\mathbb{P}(\limsup A_n) \geq c$.

Exercise 1. Speed of escape to infinity

In this exercice, we study the speed of escape to infinity of a Brownian motion in dimension 3 and more. We aim to prove the following theorem, which establishes that it escapes quicker than a deterministic function f if and only if this function passes an integrability test, called Dvoretzky-Erdös test.

Theorem. *Suppose $d \geq 3$ and B is a Brownian motion in $\mathbb{R}^d$ started from 0. Let $f : [1,\infty) \to (0,\infty)$ increasing. We say f satisfies the integrability condition (IC) if the integral $\int_1^\infty f(r)^{d-2} r^{d/2} \mathrm{d}r$ is finite. Then*

(a) If f satisfies the integrability condition (IC), then $\liminf \frac{|B_t|}{f(t)} = +\infty$ a.s.

(b) Otherwise, $\liminf \frac{|B_t|}{f(t)} = 0$ a.s.

In particular, Brownian motion a.s. satisfies $\liminf \frac{|B_t|}{t^{1/2}} = 0$, but $\liminf \frac{|B_t|}{t^\alpha} = +\infty$ for any $\alpha < 1/2$. By a simple series-integral comparison, the function f satisfies the integrability condition (IC) iff $\sum_{n\geq 0}(f(2^n)2^{-n/2})^{d-2} < +\infty$.

1. For $t \geq 0$, let $\mathcal{G}_t$ be the σ-field generated by the variables B_u for $u \geq t$, and let $\mathcal{G}_\infty = \cap_{t\geq 0}\mathcal{G}_t$. Prove $\mathcal{G}_\infty$ is trivial (ie contains only events of probability 0 or 1), and deduce that the law of $\liminf \frac{|B_t|}{f(t)}$ is a Dirac mass at some $x \in [0,+\infty]$.
2. Show that is suffices to prove, instead of the theorem, the apparently weaker results
   (a) If f satisfies (IC), then $\liminf \frac{|B_t|}{f(t)} \geq 1$ a.s.
   (b) Otherwise, $\mathbb{P}(\liminf \frac{|B_t|}{f(t)} \leq 1) > 0$.

3. In this question, we suppose the existence of a sequence $(t_n)_{n\geq 0}$ going to $+\infty$ such that $f(t_n) \geq \sqrt{t_n}$. In particular, f does not satisfy (IC). Prove the result 2.(b) in that case.

   In the following, we exclude this case and thus suppose $f(t) \leq \sqrt{t}$ for t large enough. By modifying f on a compact interval, we suppose, without loss of generality [1], that f satisfies $f(t) \leq \sqrt{t}$ for all $t \geq 1$.
4. Recall briefly why, if B is under $\mathbb{P}_x$ a BM started from $x \in \mathbb{R}^d$ (in particular, the notation $\mathbb{P}_0$ is somehow redundant with $\mathbb{P}$), then

$$\mathbb{P}_x(\inf |B_t| \leq r) = \left(\frac{r}{|x|}\right)^{d-2} \wedge 1.$$

5. We define the function $g_0 : x \mapsto |x|^{2-d}$, and, for $r > 0$, the function g_r by

$$g_r(x) = \frac{1}{|x|^{d-2}} \wedge \frac{1}{r^{d-2}}.$$

   Prove

$$\mathbb{P}_x(\inf\{|B_t|, t \geq 1\} \leq r) \leq \mathbb{P}_0(\inf\{|B_t|, t \geq 1\} \leq r) = r^{d-2}\mathbb{E}[g_r(B_1)] \leq ar^{d-2},$$

   with $a = \mathbb{E}\left[g_0(B_1)\right] \in (0, +\infty)$. For the first inequality, you may want to use the strong Markov property of the Brownian motion started from 0.
6. We introduce, for $n \geq 0$, the event $A_n = \{\exists t \in (2^n, 2^{n+1}], |B_t| \leq f(t)\}$. If f satisfies (IC), prove only finitely many of the events A_n occur, a.s., and deduce 2.(a).
7. We now suppose, until the end of the exercice, that f does not satisfy (IC) (but still, $f(t) \leq \sqrt{t}$ for all $t \geq 1$). Prove

$$\mathbb{P}(\exists t \in [1, 2], |B_t| \leq r) \geq r^{d-2}\mathbb{E}\big[g_r(B_1) - g_r\big(\sqrt{2}B_1\big)\big].$$

   Writing $b = \mathbb{E}\big[g_1(B_1) - g_1\big(\sqrt{2}B_1\big)\big] \in (0, +\infty)$, deduce first :

$$\forall r \leq 1, \quad \mathbb{P}(\exists t \in [1, 2], |B_t| \leq r) \geq br^{d-2},$$

   and then $\sum \mathbb{P}(A_n) = +\infty$.
8. Show we always have $\mathbb{P}(A_n|\mathcal{F}_{2^{n-1}}) \leq a(f(2^{n+1})2^{-\frac{n-1}{2}})^{d-2}$. Prove

$$\limsup_{n\to\infty} \frac{\sum_{k=0}^{n}\sum_{l=0}^{n} \mathbb{P}(A_k \cap A_l)}{\left(\sum_{k=0}^{n} \mathbb{P}(A_k)\right)^2} \leq \frac{2^{d-1}a}{b},$$

   and deduce 2.(b).

1. Indeed, the modification does not change the integrability condition for f, nor does it change $\liminf |B_t|/f(t)$.

Exercise 2. Planar Brownian motion conditioned on avoiding the unit disk.

In this exercise, B is under $\mathbb{P}_x$ a planar Brownian motion started from $x \in \mathbb{R}^2$. We further suppose $|x| > 1$ and for $r > 0$, we define T_r the hitting time of the circle of radius r, namely $T_r = \inf\{t \geq 0, |B_t| = r\}$. In questions (1)-(4), we define the Brownian motion "conditioned on never hitting the unit disk". In questions (5)-(7), we prove it is transient but does not escape too quickly to infinity. In questions (8)-(9), we prove a striking result for the asymptotic probability that this process ever hits a far away disk of radius 1.

1. Recall briefly why, for $r > |x|$, we have $\ln|x| = \mathbb{E}_x[\ln|B_{T_1 \wedge T_r}|]$, and deduce :
$$\forall t \geq 0, \quad \ln|B_{t \wedge T_1 \wedge T_r}| = \mathbb{E}_x[\ln|B_{T_1 \wedge T_r}| \,\big|\mathcal{F}_t].$$
Hence $\ln|B_{t\wedge T_1 \wedge T_r}|$ is a closed martingale.
2. For $t \geq 0$, we let $M_t = \ln|B_{t\wedge T_1}| = \mathbb{1}_{t<T_1} \ln|B_t|$. Show M is a martingale.
3. In this question, we fix $t \geq 0$ and define $(C_s)_{0 \leq s \leq t}$ a process which, under $\mathbb{P}_x$, has law absolutely continuous with that of $(B_s)_{0\leq s\leq t}$, and with density the value of the martingale M at time t, divided by $M_0 = \ln|x|$. Equivalently, for f an arbitrary test function, we have
$$\mathbb{E}_x[f((C_s)_{0\leq s\leq t})] = \mathbb{E}_x\left[f((B_s)_{0\leq s\leq t})\frac{M_t}{M_0}\right].$$
   (a) Check that the law of $(C_s)_{0\leq s\leq t}$ is well-defined, and that, for any $s \leq t$ and test function f,
$$\mathbb{E}_x[f((C_r)_{0\leq r\leq s})] = \mathbb{E}_x\left[f((B_r)_{0\leq r\leq s})\frac{M_s}{M_0}\right].$$
   (b) Prove that the event $\{\exists s \leq t, |C_s| \leq 1\}$ has probability 0, and that for any $r, s \geq 0$ such that $r + s \leq t$ and any test functions f and g,
$$\mathbb{E}_x[f((C_q)_{0\leq q\leq r})g(C_{r+s})] = \mathbb{E}_x[f((C_q)_{0\leq q\leq r})g_s(C_r)],$$
   where we have written $g_s(y) = \mathbb{E}_y[g(C_s)]$, for any y satisfying $|y| > 1$.

   It follows that $(C_s)_{0\leq s\leq t}$ is a time-homogeneous Markov process, which we can now extend [2] to $\mathbb{R}_+$. We admit the process $(C_s)_{s\geq 0}$ satisfies the strong Markov property, as well as verifies the equation
$$\mathbb{E}_x[f((C_s)_{0\leq s\leq t})] = \mathbb{E}_x\left[f((B_s)_{0\leq s\leq t})\frac{M_t}{M_0}\right],$$
   for any $t \geq 0$ and test function f.

2. We can proceed to this extension by a gluing procedure. Alternatively, we may also invoke a Kolmogorov extension lemma.

4. Prove that, for $r \geq |x|$, the process $(C_{t \wedge T_r})_{t\geq 0}$ has the same law as the process $(B_{t\wedge T_r})_{t\geq 0}$ conditionally on the event $T_r < T_1$.

   It is now natural (although slightly abusive) to call the process C "Brownian motion conditioned on never hitting the unit disk", though this is not a well-defined conditioning.
5. For $1 < r < |x|$, prove
$$\mathbb{P}_x(\exists t \geq 0, |C_t| = r) = \frac{\ln r}{\ln |x|},$$
   and deduce that the process C is transient.
6. For $1 < |x| < r$ and $t > 0$, prove
$$\mathbb{P}_x\left(\sup_{0\leq s\leq t} |C_s| \geq r\right) \leq 4\frac{\ln r}{\ln |x|}\mathbb{P}\left(|N| \geq \frac{r - |x|}{\sqrt{2t}}\right),$$
   where N is a centered reduced normal variable. Deduce that we almost surely have
$$\forall \alpha > 1/2, \quad \limsup \frac{|C_t|}{t^\alpha} = 0.$$
7. Prove that we almost surely have
$$\forall \varepsilon > 0, \quad \liminf \frac{|C_t|}{t^\varepsilon} = 0.$$
   Hint : For $\varepsilon \in (0,1)$, consider the family of events E_n, where E_n is the event that the process C, after hitting the circle of radius 2^n, hits the circle of radius $2^{\varepsilon n}$ before hitting the circle of radius 2^{n+1}.

   For $y \in \mathbb{R}^2$, we call $E_y := \{\exists t \geq 0, |C_t - y| = 1\}$ the event that the process C ever hits the closed disk $\overline{B}(y,1)$. We now seek an estimate of $\mathbb{P}_x(E_y)$ when $|y| \to \infty$. For $|y| > |x| + 1$, writing $r = |y| - 1$ and using the strong Markov property at time T_r, this probability equals the expectation of $\mathbb{P}_{C_{T_r}}(E_y)$. We admit that C_{T_r} is asymptotically uniform on the sphere, and that this expectation is equivalent to $\mathbb{P}_{\nu_r}(E_y)$, where ν_r is the uniform measure on $\partial B(0,r)$.
8. Prove
$$\mathbb{P}_{\nu_r}(E_y) = \mathbb{E}_{\nu_r}\left[\frac{\ln |B_{T_{\partial B(y,1)}}|}{\ln r}\mathbb{1}_{T_{\partial B(y,1)} < T_1}\right],$$
   with $T_{\partial B(y,1)} = \inf\{t \geq 0, |B_t - y| = 1\}$.
9. Prove $\mathbb{P}_{\nu_r}(T_{\partial B(y,1)} < T_1) \underset{|y|\to\infty}{\to} 1/2$, and deduce $\mathbb{P}_{\nu_r}(E_y) \to 1/2$.

   Hint : Argue that it suffices to show $\mathbb{P}_{\nu_r}(T_{H_y} < T_{\partial B(y,1)} \wedge T_1) \underset{|y|\to\infty}{\to} 1$, where $H_y = \{z \in \mathbb{R}^2, |z| = |z - y|\}$ is the mediator of the segment between y and the origin, and use the scale-invariance property of Brownian motion.