partiel de processus stochastiques et mouvement brownien (2 heures)

Your lecture notes are allowed, but other documents are not.

We expect an accurate and concise writing.

Finally, it is advised not to spend too much time on the very first question of Exercise 1. Skipping question 4 could also be a reasonable choice, as this question is independent from all others.

Exercise 1 : Volume of a brownian path

We consider $d \geq 2$ and $(B_t)_{t\geq 0} = (B_t^{(1)}, \dots, B_t^{(d)})_{t\geq 0}$ a brownian motion in $\mathbb{R}^d$ started from $0 = 0_{\mathbb{R}^d}$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $t \geq 0$, we let V_t be the volume of the beginning of the Brownian path $\{B_s, 0 \leq s \leq t\}$, namely

$$V_t = \lambda_d(\{B_s, 0 \leq s \leq t\}),$$

where λ_d is the Lebesgue measure on $\mathbb{R}^d$.

1. For $t \geq 0$, show that $A_t := \{(B_s(\omega), \omega), 0 \leq s \leq t, \omega \in \Omega\}$ is a measurable subset of $\mathbb{R}^d \times \Omega$, endowed with the product $\sigma-$field $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}$, and deduce that V_t is a well-defined random variable with values in $[0, +\infty]$.
2. Show V_t has finite expectation, and follows the same distribution as $t^{d/2} V_1$.
3. Deduce that :
   (a) If $d \geq 3$, the brownian path has a.s. volume 0.
   (b) If $d = 2$, then $\lambda_2(\{B_s, 0 \leq s \leq 1\} \cap \{B_s, 1 \leq s \leq 2\}) = 0$ a.s.
4. Prove again the result of 3.(a) by using the Hölder continuity property of the brownian paths.
5. We now suppose $d = 2$. For $z \in \mathbb{R}^2$, we write $T_z := \inf\{t \geq 0, B_t = z\} \in [0, +\infty]$.
   (a) Show $\mathbb{E}[V_1] = \int \mathbb{P}(T_z \leq 1) \lambda_2(\mathrm{d}z)$.
   (b) Prove $T_z \overset{(d)}{=} |z|^2 T_{z_0}$, where $z_0 = (1, 0)$, and deduce $\mathbb{E}[V_1] = \pi \mathbb{E}[T_{z_0}^{-1}]$.
   (c) Prove similarly
   $$\mathbb{E}\left[\lambda_2(\{B_s, 0 \leq s \leq 1\} \cap \{B_t, 1 \leq t \leq 2\})\right] = \pi \mathbb{E}\left[\max\left(T_{z_0}, \widetilde{T}_{z_0}\right)^{-1}\right],$$
   where $\widetilde{T}_{z_0}$ is an independent copy of T_{z_0}.

   Hint : Observe that $\lambda_2(\{B_s, 0 \leq s \leq 1\} \cap \{B_t, 1 \leq t \leq 2\})$ *can be rewritten as*
   $$\lambda_2(\{B_{1-s} - B_1, 0 \leq s \leq 1\} \cap \{B_{1+t} - B_1, 0 \leq t \leq 1\}).$$
   (d) Deduce the planar brownian motion path also has a.s. volume (or area) 0.

Exercise 2 : Langevin process and recurrence

Suppose $(B_t)_{t\geq 0}$ is a 1-dimensional Brownian motion started from 0, and $(\mathcal{F}_t)_{t\geq 0}$ is its canonical filtration. We define the *integrated Brownian motion* or *Langevin process* $(A_t)_{t\geq 0}$ by $A_t = \int_0^t B_s \mathrm{d}s$.

1. (a) Show the Langevin process is continuous and adapted. Show its one dimensional marginal, the distribution of A_t, is a centered gaussian with variance $t^3/3$.

   Hint : Approximate A_t by a linear combination of the coordinates of the brownian motion $(B_s)_{s\geq 0}$.

   (b) Prove that the processes $(-A_t)_{t\geq 0}$ and $(\lambda^{3/2} A_{\lambda^{-1}t})_{t\geq 0}$, for any given $\lambda > 0$, have the same law (as random variables taking values in the Wiener space) as the Langevin process $(A_t)_{t\geq 0}$.

   (c) Show the Langevin process takes almost surely positive as well as negative values at arbitrary small times.

   (d) Show the Langevin process is *recurrent*, namely takes almost surely every real value at arbitrary large times.

   Hint : It suffices to show that we almost surely have

$$\limsup_{t\to+\infty} A_t = +\infty, \qquad \liminf_{t\to+\infty} A_t = -\infty.$$

2. We aim to show that the bidimensional process $(A_t, B_t)_{t\geq 0}$ (also called *Kolmogorov process*) is *transient*, in the sense that we almost surely have

$$\liminf_{t\to+\infty} (|A_t| + |B_t|) = +\infty.$$

   (a) Show that, looking at integers n, we a.s. have

$$\liminf_{n\to+\infty, n\in\mathbb{N}} |A_n| = +\infty.$$

   Hint : Use question 1.(a)

   (b) Suppose $K \subset \mathbb{R}^2$ is compact, and T is a stopping time such that the event $\{T < +\infty\}$ has positive probability, and we have $(A_T, B_T) \in K$ on this event. Show we can find a compact set $\tilde{K}$, depending only on K, such that conditionally on $\{T < +\infty\}$, the process (A_t, B_t) stays in $\tilde{K}$ on the whole time interval $[T, T+1]$ with probability at least $1/2$.

   (c) Conclude.