Correction du partiel de processus stochastiques et mouvement brownien

Exercice 1 : Volume of a brownian path

We consider $d \geq 2$ and $(B_t)_{t\geq 0} = (B_t^{(1)}, \dots, B_t^{(d)})_{t\geq 0}$ a brownian motion in $\mathbb{R}^d$ started from $0 = 0_{\mathbb{R}^d}$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $t \geq 0$, we let V_t be the volume of the beginning of the Brownian path $\{B_s, 0 \leq s \leq t\}$, namely

$$V_t = \lambda_d(\{B_s, 0 \leq s \leq t\}),$$

where λ_d is the Lebesgue measure on $\mathbb{R}^d$.

1. For $t \geq 0$, show that $A_t := \{(B_s(\omega), \omega), 0 \leq s \leq t, \omega \in \Omega\}$ is a measurable subset of $\mathbb{R}^d \times \Omega$, endowed with the product σ−field $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}$, and deduce that V_t is a well-defined random variable with values in $[0, +\infty]$.
   Answer : Writing $B(x, \varepsilon)$ the (open) ball of radius ε centered at $x \in \mathbb{R}^d$, we note that, for $t \geq 0$ and $\varepsilon > 0$, the set $\{B(B_t(\omega), \varepsilon) \times \{\omega\}, \omega \in \Omega\}$ is measurable. Indeed, this set is equal to $\phi^{-1}\big([0, \varepsilon)\big)$, where $\phi : \mathbb{R}^d \times \Omega \to \mathbb{R}_+$, defined by $\phi(x, \omega) = |B_t(\omega) - x|$, is a measurable function. Now, using the continuity of the brownian path, we have

   $$A_t = \bigcap_{n\in\mathbb{N}} \bigcup_{q\in\mathbb{Q}\cap[0,t]} B(B_q(\omega), \frac{1}{n}) \times \{\omega\},$$

   thus A_t is measurable. Now, as

   $$V_t = \int 1_{A_t}(x, \omega)\lambda_d(\mathrm{d}x),$$

   we immediately get that V_t is measurable and thus a well-defined random variable with values in $\mathbb{R}_+ \cup \{+\infty\}$.
2. Show V_t has finite expectation, and follows the same distribution as $t^{d/2}V_1$.
   Answer : Using the crude bound $V_t \leq \prod_{i=1}^d \Big(\max(B_s^{(i)}, 0 \leq s \leq t) - \min(B_s^{(i)}, 0 \leq s \leq t\Big)$, we get

   $$\begin{aligned}\mathbb{E}[V_t] &\leq \prod \mathbb{E}[\big(\max(B_s^{(i)}, 0 \leq s \leq t) - \min(B_s^{(i)}, 0 \leq s \leq t\big)] \\ &\leq \big(\mathbb{E}[\max(B_s^{(1)}, 0 \leq s \leq t)] - \mathbb{E}[\min(B_s^{(1)}, 0 \leq s \leq t]\big)^d \\ &\leq 2^d\mathbb{E}[|B_t^{(1)}|]^d < +\infty.\end{aligned}$$

   Moreover, using the scaling invariance property of brownian motion, we get

   $$V_t \overset{(d)}{=} \lambda_d(\{t^{1/2}B_{st^{-1}}, 0 \leq s \leq t\}) = t^{d/2}\lambda_d(\{B_s, 0 \leq s \leq 1\}) = t^{d/2}V_1.$$

3. Deduce that :
   (a) If $d \geq 3$, the brownian path has a.s. volume 0.
   (b) If $d = 2$, then $\lambda_2(\{B_s, 0 \leq s \leq 1\} \cap \{B_s, 1 \leq s \leq 2\}) = 0$ a.s.

   Answer : We introduce $\tilde{V}_1 = \lambda_d(\{B_s, 1 \leq s \leq 2\}) = \lambda_d(\{B_{1+s} - B_1, 0 \leq s \leq 1\})$, which has the same law as V_1 (and is independent from V_1). We also introduce $\hat{V}_1 = \lambda_d(\{B_s, 0 \leq s \leq 1\} \cap \{B_s, 1 \leq s \leq 2\})$. Now,

$$\begin{aligned} V_2 &= \lambda_d(\{B_s, 0 \leq s \leq 1\} \cup \{B_s, 1 \leq s \leq 2\}) \\ &= V_1 + \tilde{V}_1 - \hat{V}_1 \\ &\leq V_1 + \tilde{V}_1, \end{aligned}$$

   with equality iff $\hat{V}_1 = 0$. Taking expectations and using question 2, we get

$$2^{d/2}\mathbb{E}[V_1] \leq 2\mathbb{E}[V_1],$$

   and thus $\mathbb{E}[V_1] = 0$ if $d \geq 3$. In particular $V_1 = 0$ a.s. By scaling, we also have $V_t = 0$ a.s., and even a.s., $\forall t \geq 0, V_t = 0$. Thus the brownian path has volume 0. In the case $d = 2$, we deduce $V_2 = V_1 + \tilde{V}_1$ a.s., and thus $\hat{V}_1 = 0$ a.s.

4. Prove again the result of 3.(a) by using the Hölder continuity property of the brownian paths.

   Answer : Fix $\alpha \in (1/3, 1/2)$. As the brownian path is a.s. $\alpha-$hölder, we have that C_α is a.s. finite, where

$$C_\alpha := \sup\left\{\frac{|B_s - B_t|}{|s-t|^\alpha}, 0 \leq s < t \leq 1\right\}.$$

   For any n integer, we get

$$\begin{aligned} V_1 &\leq \sum_{i=1}^{n} \lambda_d(\{B_t, (i-1)/n \leq t \leq i/n\}) \\ &\leq \sum_{i=1}^{n} \lambda_d\big(\overline{B}(B_{i/n}, C(i/n)^\alpha)\big) \\ &\leq nC^d \left(\frac{i}{n}\right)^{\alpha d} \lambda_d\big(\overline{B}(0,1)\big) = cn^{1-\alpha d} \end{aligned}$$

   where $\overline{B}(x, \varepsilon)$ is the closed ball of radius ε centered at x, and c is a.s. finite and independent from n. As $\alpha > 1/3$ and $d \geq 3$, this converges to 0 as $n \to \infty$, and shows V_1 is 0 a.s. (more precisely, it is 0 as soon as the brownian path is $\alpha-$hölder for some $\alpha > 1/3$).

   We may also note that this approach does not say anything in the case $d = 2$. Indeed, in the plane, the path of a α-hölder function must have area zero if $\alpha > 1/2$, but may well have positive area if $\alpha = 1/2$. And the brownian paths are even not 1/2-hölder...

5. We now suppose $d = 2$. For $z \in \mathbb{R}^2$, we write $T_z := \inf\{t \geq 0, B_t = z\} \in [0, +\infty]$.

   (a) Show $\mathbb{E}[V_1] = \int \mathbb{P}(T_z \leq 1)\lambda_2(\mathrm{d}z)$.
   Answer : We noticed in question 1. that $V_1 = \int 1_{A_1}(x, \omega)\lambda_2(\mathrm{d}x)$, *so the expectation of* V_1 *is also the* $\lambda_2 \otimes \mathbb{P}-$*measure of the set* A_1. *By Fubini theorem, we can compute this measure by first integrating over* ω, *so that we get*

$$\mathbb{E}[V_1] = \int \mathbb{P}\big(z \in \{B_s, 0 \leq s \leq 1\}\big)\lambda_2(\mathrm{d}z) = \int \mathbb{P}(T_z \leq 1)\lambda_2(\mathrm{d}z).$$

   (b) Prove $T_z \stackrel{(d)}{=} |z|^2 T_{z_0}$, where $z_0 = (1, 0)$, and deduce $\mathbb{E}[V_1] = \pi\mathbb{E}[T_{z_0}^{-1}]$.
   Answer : Write $z = rz_1$, *with* $r = |z|$ *and* $|z_1| = 1$. *By the scaling invariance property of brownian motion, we get*

$$\begin{aligned} T_z = \inf\{t \geq 0, B_t = z\} &\stackrel{(d)}{=} \inf\{t \geq 0, rB_{r^{-2}t} = z\} \\ &= r^2 \inf\{t \geq 0, rB_t = z_1\} = r^2 T_{z_1}. \end{aligned}$$

   By the invariance of the law of the brownian motion under an isometry of $\mathbb{R}^2$, *we also get that* T_{z_1} *has the same law as* T_{z_0}. *Further,*

$$\begin{aligned} \mathbb{E}[V_1] = \int \mathbb{P}(T_{z_0} \leq |z|^{-2})\lambda_2(\mathrm{d}z) &= 2\pi \int_{\mathbb{R}_+} r\mathbb{P}(T_{z_0} \leq r^{-2})\mathrm{d}r \\ &= \pi \int_{\mathbb{R}_+} \mathbb{P}(T_{z_0}^{-1} \geq s)\mathrm{d}s = \pi\mathbb{E}[T_{z_0}^{-1}], \end{aligned}$$

   where in the first line we used the polar coordinates change of variable, and in the second line the change of variable $s = r^2$.

   (c) Prove similarly $\mathbb{E}\left[\lambda_2(\{B_s, 0 \leq s \leq 1\} \cap \{B_t, 1 \leq t \leq 2\})\right] = \pi\mathbb{E}\left[\max\left(T_{z_0}, \widetilde{T}_{z_0}\right)^{-1}\right]$, where $\widetilde{T}_{z_0}$ is an independent copy of T_{z_0}.

Hint : Observe that $\lambda_2(\{B_s, 0 \leq s \leq 1\} \cap \{B_t, 1 \leq t \leq 2\})$ *can be rewritten as*

$$\lambda_2(\{B_{1-s} - B_1, 0 \leq s \leq 1\} \cap \{B_{1+t} - B_1, 0 \leq t \leq 1\}).$$

   Answer : We follow the hint, and observe that $(B_{1-s} - B_1)_{0 \leq s \leq 1}$ *and* $(B_{1+t} - B_1)_{t \geq 0}$ *are two independent brownian motions. Indeed, the first one is a brownian motion by time reversal, and is* $\sigma(B_s, 0 \leq s \leq 1)-$*measurable, while the second one is a brownian motion independent of* $\sigma(B_s, 0 \leq s \leq 1)$, *by the simple Markov property. Therefore the hitting times of* z_0 *for these two processes are independent copies of* T_{z_0}. *Now, the same reasoning as in last question leads to*

$$\begin{aligned} \mathbb{E}\left[\lambda_2(\{B_s, 0 \leq s \leq 1\} \cap \{B_t, 1 \leq t \leq 2\})\right] &= \pi \int_{\mathbb{R}_+} \mathbb{P}(T_{z_0}^{-1} \geq s, \tilde{T}_{z_0}^{-1} \geq s)\mathrm{d}s \\ &= \pi\mathbb{E}\left[\max\left(T_{z_0}, \widetilde{T}_{z_0}\right)^{-1}\right]. \end{aligned}$$

(d) Deduce the planar brownian motion path also has a.s. volume (or area) 0.

Answer : By question 3.(b), the expectation computed in 5.(c) is actually 0, and therefore $\max(T_{z_0}, \widetilde{T}_{z_0})$ *is almost surely equal to* $+\infty$. *This in turn implies that* T_{z_0} *is itself infinite almost surely. Thus, by 5.(b), the expectation of* V_1 *is zero, and we finish the proof just like the case* $d \geq 3$.

Exercice 2 : Langevin process and recurrence

Suppose $(B_t)_{t\geq 0}$ is a 1-dimensional Brownian motion started from 0, and $(\mathcal{F}_t)_{t\geq 0}$ is its canonical filtration. We define the *integrated Brownian motion* or *Langevin process* $(A_t)_{t\geq 0}$ by $A_t = \int_0^t B_s \mathrm{d}s$.

1. (a) Show the Langevin process is continuous and adapted. Show its one dimensional marginal, the distribution of A_t, is a centered gaussian with variance $t^3/3$.

   Hint : Approximate A_t *by a linear combination of the coordinates of the brownian motion* $(B_s)_{s\geq 0}$.

   Answer : The Langevin process is clearly continuous (its paths are even differentiable). The approximation of the integral of a continuous function by a Riemann sum gives us that A_t *is the (pointwise) limit when* $n \to \infty$ *of the sum* $A_t^n := \sum_{i=1}^n \frac{t}{n} B_{it/n}$. *The rv* A_t^n *is clearly* $\mathcal{F}_t$*−measurable, thus so is* A_t, *and the process is thus adapted. Moreover,* A_t^n *is a centered gaussian variable, with variance*

$$\begin{aligned} Var(A_t^n) = \frac{t^2}{n^2}\sum_{i,j=1}^n \mathrm{Cov}(B_{it/n}, B_{jt/n}) &= \frac{t^2}{n^2}\sum_{i,j=1}^n \min(\frac{it}{n}, \frac{jt}{n}) \\ &\underset{n\to\infty}{\to} \int_{[0,t]^2} \min(r,s)\mathrm{d}r\mathrm{d}s = t^3/3. \end{aligned}$$

   Thus A_t^n *converges in law to a centered gaussian with variance* $t^3/3$ *(we recall the convergence in law for gaussian random variables is characterized by the convergence of the first two moments). In particular,* A_t *is a centered gaussian with variance* $t^3/3$.

   (b) Prove that the processes $(-A_t)_{t\geq 0}$ and $(\lambda^{3/2}A_{\lambda^{-1}t})_{t\geq 0}$, for any given $\lambda > 0$, have the same law (as random variables taking values in the Wiener space) as the Langevin process $(A_t)_{t\geq 0}$.

   Answer : It suffices to write $-A_t = \int_0^t (-B_s)\mathrm{d}s$ *and*

$$\lambda^{3/2}A_{\lambda^{-1}t} = \int_0^{\lambda^{-1}t} \lambda^{3/2}B_s\mathrm{d}s = \int_0^t \lambda^{1/2}B_{\lambda^{-1}u}\mathrm{d}u,$$

   and to observe that the processes $(-B_t)_{t\geq 0}$ *and* $(\lambda^{1/2}B_{\lambda^{-1}t})_{t\geq 0}$ *are brownian motions.*

(c) Show the Langevin process takes almost surely positive as well as negative values at arbitrary small times.

Answer : We just prove that the Langevin process takes almost surely positive values at arbitrary small times (then we deduce the result for example because $(-A_t)_{t\geq 0}$ *is also a Langevin process). In other words, we prove* $\mathbb{P}(\forall \varepsilon > 0, \sup\{A_s, 0 \leq s \leq \varepsilon\} > 0) = 1$*. Observe this event is in the* $\sigma-$*field* $\mathcal{F}_{0+}$*, thus by Blumenthal 0-1 law, it must have probability 0 or 1. But it also has probability at least 1/2, because it is the decreasing limit, when* ε *decreases to 0, of the event* $\sup\{A_s, 0 \leq s \leq \varepsilon\} > 0$*, which contains the event* $\{A_\varepsilon > 0\}$*, itself of probability 1/2. Thus we get result.*

(d) Show the Langevin process is recurrent, namely takes almost surely every real value at arbitrary large times.

Hint : It suffices to show that we almost surely have

$$\limsup_{t\to+\infty} A_t = +\infty, \qquad \liminf_{t\to+\infty} A_t = -\infty.$$

Anwser : It suffices to prove, for fixed $n > 0$*, that the event* $\sup_{t\geq 0} A_t \geq n$ *is almost sure. Indeed, we then a.s. have* $\sup_{t\geq 0} A_t = +\infty$*, as well as* $\inf_{t\geq 0} A_t = -\infty$ *(again by a simple symmetry argument), which proves the Langevin process is a.s. recurrent.*

Now, for fixed $n > 0$*, by the scaling invariance property of the Langevin process, we get that for any* $\lambda > 0$*,*

$$\mathbb{P}(\sup_{t\geq 0} A_t \geq n) = \mathbb{P}(\sup_{t\geq 0} \lambda^{3/2} A_{\lambda^{-1}t} \geq n) = \mathbb{P}(\sup_{t\geq 0} A_t \geq \lambda^{-3/2} n).$$

In particular, taking λ *to infinity, this is also equal to the probability of the event* $\{\sup_{t\geq 0} A_t > 0\}$*, which is 1 by question 1.(c).*

2. We aim to show that the bidimensional process $(A_t, B_t)_{t\geq 0}$ (also called *Kolmogorov process*) is transient, in the sense that we almost surely have

$$\liminf_{t\to+\infty} (|A_t| + |B_t|) = +\infty.$$

(a) Show that, looking at integers n, we a.s. have

$$\liminf_{n\to+\infty, n\in\mathbb{N}} |A_n| = +\infty.$$

Hint : Use question 1.(a)

Answer : By question 1.(a), A_n *has the same law as* $t^{3/2}N/3^{1/2}$*, where* N *is a centered standard gaussian. But the law of* N *has density bounded by* $1/\sqrt{2\pi}$*, thus, for any* $\varepsilon > 0$*, we have* $\mathbb{P}(|N| \leq \varepsilon) \leq \varepsilon\sqrt{2/\pi}$*.*

Hence, for $c > 0$ fixed, we have

$$\sum_{n\in\mathbb{N}} \mathbb{P}(|A_n| \leq c) \leq c\sqrt{6/\pi} \sum_n n^{-3/2} < \infty,$$

and by Borel-Cantelli lemma, the event $\liminf |A_n| \geq c$ is almost sure. We conclude by taking c to $+\infty$.

(b) Suppose $K \subset \mathbb{R}^2$ is compact, and T is a stopping time such that the event $\{T < +\infty\}$ has positive probability, and we have $(A_T, B_T) \in K$ on this event. Show we can find a compact set $\tilde{K}$, depending only on K, such that conditionally on $\{T < +\infty\}$, the process (A_t, B_t) stays in $\tilde{K}$ on the whole time interval $[T, T+1]$ with probability at least $1/2$.

Answer : Fix a compact set K and a stopping time T as in the statement. We also let $M := \max\{|y|, (x,y) \in K\} < +\infty$, and argue in this argument conditionally on $\{T < +\infty\}$. Using the strong Markov property of brownian motion, we get that the process $B^{(T)}$ is a brownian motion independent from $\mathcal{F}_T$. In particular, we can choose a finite constant $c > 0$ (not depending on K or T) such that the probability of the event $\sup\{|B_t^{(T)}|, 0 \leq t \leq 1\} \leq c$ is at least $1/2$. Now, define the compact set $\tilde{K}$ by

$$\tilde{K} := \{(x,y), \exists (x',y') \in K, |y - y'| \leq c, |x - x'| \leq M + c\}.$$

The event "The process (A_t, B_t) stays in $\tilde{K}$ on the whole time interval $[T, T+1]$" contains the event $\sup\{|B_t^{(T)}|, 0 \leq t \leq 1\} \leq c$, and thus has (conditional) probability at least $1/2$.

(c) Conclude.

Answer : We argue by the absurd and suppose that the probability of the event $\{\liminf(|A_t| + |B_t|) < +\infty\}$ is positive. Then there exists a finite constant $c > 0$ and $\varepsilon > 0$ such that

$$\mathbb{P}\left(\bigcap_{s>0} \{\inf_{t\geq s}(|A_t| + |B_t|) < c\}\right) \geq \varepsilon.$$

Define $K = \{(x,y), |x| + |y| \leq c\}$ and $\tilde{K}$ given by last question. For n integer, introduce the stopping time $T_n := \inf\{t \geq n, (A_t, B_t) \in K\}$. The probability of the event $\{T_n < \infty\}$ is at least ε, and conditionally on this, the Kolmogorov process stays in $\tilde{K}$ on the whole time interval $[T_n, T_n+1]$ with proability at least $1/2$.

In particular, the probability that there exists an integer k larger than n such that $|A_n| \leq C$ is at least $\varepsilon/2$, where C is the finite constant $\max\{|x|, (x,y) \in \tilde{K}\}$. Taking the intersection over n integer, we deduce that the probability of the event $\liminf |A_n| \leq C$ is at least $\varepsilon/2$, contradicting question 2.(a).

Finally, we deduce the transience of the Kolmogorov process.