Master 1 — Spring 2019
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Solution of homework assignment : On the Brownian bridge.

Exercise 1 — *Absolute continuity.*
You have shown in the second exercise session that

$$\mathrm{Law}(B_{|[0,1]}|B_1 \in dx) = \mathrm{Law}(x\mathrm{Id} + \beta),$$

In other words, for every bounded measurable H,

$$\mathbb{E}[H(B_{|[0,1]}, B_1)] = \int_{\mathbb{R}} \mathbb{E}[H(x\mathrm{Id} + \beta, x)]\, \mathbb{P}_{B_1}(dx).$$

(1) For $\varepsilon > 0$, let $\nu_\epsilon = \mathrm{Law}(B_{|[0,1]}| - \varepsilon \leq B_1 \leq \varepsilon)$ be the (deterministic!) probability measure such that for every bounded measurable H,

$$\int_{\mathcal{C}([0,1])} H(\varphi)\nu_\varepsilon(d\varphi) = \frac{\mathbb{E}[H(B_{|[0,1]})\, \mathbb{1}_{|B_1|\leq\varepsilon}]}{\mathbb{P}(|B_1| \leq \varepsilon)}.$$

Show that it converges (in the weak topology of measures), as $\varepsilon \to 0$, to $\mathrm{Law}(\beta)$.

Let H be a bounded continuous functional.

$$\begin{aligned}\int_{\mathcal{C}([0,1])} H(\varphi)\nu_\varepsilon(d\varphi) = \frac{\mathbb{E}[H(B_{|[0,1]})\, \mathbb{1}_{|B_1|\leq\varepsilon}]}{\mathbb{P}(|B_1| \leq \varepsilon)} &= \frac{\int \mathbb{P}(B_1 \in dx)\, \mathbb{1}_{x\leq\varepsilon}\, \mathbb{E}[H(x\mathrm{Id} + \beta)]}{\int \mathbb{P}(B_1 \in dx)\, \mathbb{1}_{x\leq\varepsilon}} \\ &= \frac{\int_{-\varepsilon}^{\varepsilon} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, \mathbb{E}[H(x\mathrm{Id} + \beta)]dx}{\int \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx} \\ &\sim_{\varepsilon\to 0} \frac{2\varepsilon \frac{1}{\sqrt{2\pi}}\, \mathbb{E}[H(\beta)]}{2\varepsilon \frac{1}{\sqrt{2\pi}}} = \mathbb{E}[H(\beta)]\end{aligned}$$

because both $x \mapsto e^{-x^2/2}\, \mathbb{E}[H(x\mathrm{Id} + \beta)]$ and $x \mapsto e^{-x^2/2}$ are continuous at $x = 0$ (the first one thanks to the dominated convergence theorem). Hence the question.

(2) For $0 < a < 1$, what does the Markov property say about the joint distribution of $(B_{|[0,a]}, B_1)$? Deduce that, for H positive bounded continuous $\mathcal{C}([0, a]) \to \mathbb{R}$, the following quantity:

$$\mathbb{E}[H(B_{|[0,a]})||B_1| < \varepsilon] = \frac{\mathbb{E}[H(B_{|[0,a]})\, \mathbb{1}_{|B_1|\leq\varepsilon}]}{\mathbb{P}(|B_1| \leq \varepsilon)} = \int_{\mathcal{C}([0,1])} H(\varphi_{|[0,a]})\nu_\varepsilon(d\varphi).$$

converges, as $\varepsilon \to 0$, to

$$\int_{\mathcal{C}([0,a])} H(\phi) \frac{1}{\sqrt{1-a}} \exp\left(-\frac{\phi(a)^2}{2(1-a)}\right) \mathbb{P}_{B_{|[0,a]}}(d\phi).$$

Markov's property says the distribution of $B_1 - B_a$ is a centered Gaussian of variance $1-a$, independent of $B_{|[0,a]}$. Hence

$$\begin{aligned}
&\mathbb{E}[H(B_{|[0,a]})\,\mathbb{1}_{|B_1|<\varepsilon}] \\
&= \mathbb{E}[H(B_{|[0,a]})\,\mathbb{1}_{|B_1-B_a+B_a|<\varepsilon}] \\
&= \int_{\mathcal{C}([0,a])} \mathbb{P}(B_{|[0,a]} \in d\phi) \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi(1-a)}} \exp\left(-\frac{x^2}{2(1-a)}\right) H(\phi)\,\mathbb{1}_{|x+\phi(a)|<\varepsilon} \\
&= \int_{\mathcal{C}([0,a])} \mathbb{P}(B_{|[0,a]} \in d\phi) H(\phi) \int_{\phi(a)-\varepsilon}^{\phi(a)+\varepsilon} \frac{dx}{\sqrt{2\pi(1-a)}} \exp\left(-\frac{x^2}{2(1-a)}\right) \\
&\sim 2\varepsilon \int_{\mathcal{C}([0,a])} \mathbb{P}(B_{|[0,a]} \in d\phi) H(\phi) \frac{1}{\sqrt{2\pi(1-a)}} \exp\left(-\frac{\phi(a)^2}{2(1-a)}\right),
\end{aligned}$$

by dominated convergence. As $\mathbb{P}(|B_1| \leq \varepsilon) \sim \frac{2\varepsilon}{\sqrt{2\pi}}$, dividing these two asymptotics yields the result.

(3) Deduce that the distribution of $\beta_{|[0,a]}$ is absolutely continuous with regard to that of $B_{|[0,a]}$ when $a < 1$. Is it the case when $a = 1$?

By question 1, since the restriction $B_{|[0,1]} \mapsto B_{|[0,a]}$ is a continuous map, we have as $\varepsilon \to 0$

$$\mathbb{E}[H(B_{|[0,a]}) \mid |B_1| \leq \varepsilon] \to \mathbb{E}[H(\beta_{|[0,a]})]$$

Equating this with question 2 gets

$$\mathbb{E}[H(\beta_{|[0,a]})] = \int_{\mathcal{C}([0,a])} \mathbb{P}(B_{|[0,a]} \in d\phi) H(\phi) \frac{1}{\sqrt{1-a}} \exp\left(-\frac{\phi(a)^2}{2(1-a)}\right)$$

Hence (because equality against all bounded continuous functions characterizes equality of measures) the law of $\beta_{|[0,a]}$ has density $\phi \mapsto \frac{1}{\sqrt{1-a}} \exp\left(-\frac{\phi(a)^2}{2(1-a)}\right)$ w.r.t. the law of $B_{|[0,a]}$.
When $a = 1$ this is not the case anymore: look at the set $\{\phi, \phi(1) = 0\}$. β belongs to it with probability 1, and B with probability 0.

Exercise 2 — *Location of the minimum.*
We want to compute the distribution of $T = \inf\{t \geq 0, \beta_t = \min_{[0,1]} \beta\}$.

(1) Show that the global minimum of β is almost surely reached exactly once. You may use the fact that for every $a < b < c < d \in \mathbb{Q} \cap (0,1)$, the global minimum of B on $[a,b]$ and $[c,d]$ are almost surely different (4th exercise session)

For every $a < b < c < d < 1$ this almost sure property of $B_{|[0,d]}$ is also true for $\beta_{|[0,d]}$ by absolute continuity. Hence by countable union, almost surely for every $a < b < c < d \in \mathbb{Q} \cap (0,1)$, the global minimum of β on $[a,b]$ and $[c,d]$ are different.

Moreover we know that almost surely the global minimum of β on $[0,1]$ is not 0 (otherwise since $B = \beta + B_1 \mathrm{Id}$ then $\liminf_{t\to 0} B_t/t > -\infty$ which is almost surely not the case.) These two properties imply that almost surely the global minimum of β is reached only once.

(2) Show that the Brownian bridge is cyclically exchangeable, i.e. that for every $x \in [0,1)$, the process $t \mapsto \beta_{(x+t) \bmod 1} - \beta_x$ is still distributed like β. (You may start by reasoning on the Brownian motion.)

A first idea is to define $X_t = B_{(x+t) \bmod 1} - B_x$. However this has a jump discontinuity at $t = 1 - x$. Hence we rather consider the continuous process

$$X_t = B_{(x+t) \bmod 1} - B_x + B_1 \mathbb{1}_{x+t>1} .$$

This process is continuous and we may rewrite it as such:

$$\begin{aligned} X_t &= (B_{x+t} - B_x) \mathbb{1}_{x+t \le 1} + (B_{x+t-1} + B_1 - B_x) \mathbb{1}_{x+t>1} \\ &= \widetilde{B}_t \mathbb{1}_{t \le 1-x} + (B_{t-(1-x)} + \widetilde{B}_{1-x}) \mathbb{1}_{t>1-x} \end{aligned}$$

after setting $\widetilde{B}_u = B_{x+u} - B_u$. By Markov's property $\widetilde{B}$ is independent from $B_{[0,x]}$. Hence we have a process that follows a Brownian motion up to time $1-x$, then follows an independent Brownian motion afterwards. This is exactly the description of a Brownian motion given by Markov's property at $t = 1-x$.
Now we look at the Brownian bridge derived from X. Remark that we have $B_1 = X_1$. Hence

$$\begin{aligned} X_t - tX_1 &= B_{(x+t) \bmod 1} - B_x + B_1 \mathbb{1}_{x+t>1} - tB_1 \\ &= B_{(x+t) \bmod 1} - B_x - B_1(t - \mathbb{1}_{x+t>1}) \\ &= B_{(x+t) \bmod 1} - (B_x - xB_1) - B_1(t + x - \mathbb{1}_{x+t>1}) \\ &= B_{(x+t) \bmod 1} - B_1((x+t) \bmod 1) - (B_x - xB_1) = \beta_{(x+t) \bmod 1} - \beta_x. \end{aligned}$$

As a result $t \mapsto \beta_{(x+t) \bmod 1} - \beta_x$ is a Brownian bridge.

(3) Deduce the law of T.

Since the minimum is reached only once, if T is the argmin of β, then $U = T - x \mod 1$ is the argmin of $t \mapsto \beta_{(x+t) \bmod 1} - \beta_x$. Hence $T - x \mod 1$ is distributed like U. An immediate consequence is that $\mathbb{E}[e^{2i\pi nT}] = 0$ for $n \neq 0$, hence the bounded random variable $e^{2i\pi T}$ has the same moments as $e^{2i\pi U}$ where U is uniform. So $e^{2i\pi T} \overset{d}{=} e^{2i\pi U}$ and $T \overset{d}{=} U$.

Exercise 3 — *Maximum of* $|\beta|$.

(1) (a) Show that

$$\mathbb{P}(S > a, |B_1| < \varepsilon) = 2\,\mathbb{P}(T_a < T_{-a}, |B_1 - 2a| < \varepsilon)$$

In what follows, $\widetilde{B}$ will be the Brownian motion B reflected at T_a, and $\widetilde{T}_\cdot$ the associated hitting times.

$$\begin{aligned}\mathbb{P}(S > a, |B_1| < \varepsilon) &= \mathbb{P}(T_a < 1 \wedge T_{-a}, |B_1| < \varepsilon) + \mathbb{P}(T_{-a} < 1 \wedge T_a, |B_1| < \varepsilon)\\ &= 2\,\mathbb{P}(T_a < 1 \wedge T_{-a}, |B_1| < \varepsilon)\\ &= 2\,\mathbb{P}(\widetilde{T}_a < \widetilde{T}_{-a}, |\widetilde{B}_1 - 2a| < \varepsilon)\end{aligned}$$

Hence the result since B and $\widetilde{B}$ are likewise distributed.

(b) Show that

$$\mathbb{P}(T_a < T_{-a}, |B_1 - 2a| < \varepsilon) = \mathbb{P}(|B_1 - 2a| < \varepsilon) - \mathbb{P}(T_a < T_{-a}, |B_1 - 4a| < \varepsilon)$$

We have

$$\begin{aligned}\mathbb{P}(|B_1 - 2a| < \varepsilon) &= \mathbb{P}(T_a < T_{-a}, |B_1 - 2a| < \varepsilon) + \mathbb{P}(T_{-a} < T_a, |B_1 - 2a| < \varepsilon)\\ &= \mathbb{P}(T_a < T_{-a}, |B_1 - 2a| < \varepsilon) + \mathbb{P}(T_a < T_{-a}, |B_1 + 2a| < \varepsilon)\\ &= \mathbb{P}(T_a < T_{-a}, |B_1 - 2a| < \varepsilon) + \mathbb{P}(\widetilde{T}_a < \widetilde{T}_{-a}, |\widetilde{B}_1 - 4a| < \varepsilon)\end{aligned}$$

Hence the result.

(c) Keep working and deduce an explicit series which equals $\mathbb{P}(S > a, |B_1| < \varepsilon)$.

We may show exactly as above the equalities, for $k \geq 1$

$$\mathbb{P}(T_a < T_{-a}, |B_1 - 2ka| < \varepsilon) = \mathbb{P}(|B_1 - 2ka| < \varepsilon) - \mathbb{P}(T_a < T_{-a}, |B_1 - (2k+1)a| < \varepsilon)$$

Hence, combining all this gives

$$\begin{aligned}\mathbb{P}(T_a < T_{-a}, |B_1 - 2a| < \varepsilon) = &\sum_{k=1}^{n} (-1)^{k+1}\,\mathbb{P}(|B_1 - 2ka| < \varepsilon)\\ &+ (-1)^n\,\mathbb{P}(T_a < T_{-a}, |B_1 - (2n+1)a| < \varepsilon).\end{aligned}$$

As the remainder goes to 0 (at the speed $e^{-n^2/2}$...), we have an equality:

$$\mathbb{P}(S > a, |B_1| < \varepsilon) = 2\sum_{k=1}^{\infty} (-1)^{k+1}\,\mathbb{P}(|B_1 - 2ka| < \varepsilon)$$

(2) Deduce the cumulative distribution function of K.

We deduce the expression of the conditional probability

$$\mathbb{P}(S > a||B_1| < \varepsilon) = 2\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\mathbb{P}(|B_1 - 2ka| < \varepsilon)}{\mathbb{P}(|B_1| < \varepsilon)}$$

The summands converge each to $(-1)^{k+1}e^{-2k^2a^2}$, and assuming $\varepsilon < 1$, $k \mapsto e^{-2(k-1)^2a^2}e^{1/2}$ is a summable dominating function. With the dominated convergence theorem we get

$$\begin{aligned}\mathbb{P}(S > a||B_1| < \varepsilon) &\underset{\varepsilon\to 0}{\longrightarrow} 2\sum_{k=1}^{\infty}(-1)^{k+1}e^{-2k^2a^2} \\ &\underset{\varepsilon\to 0}{\longrightarrow} \mathbb{P}(K > a) \text{ if } a \text{ is a point of continuity for } K\end{aligned}$$

As we have an equality between a decreasing continuous function and a decreasing function outside of a countable number of points, we have equality everywhere and we are done.