$\mathbb{E}$xercise sheet 1 : Conditional probability distributions, first properties of the Brownian Motion

Version 2 : corrected typos and added an exercise 0 on Gaussian vectors.

Exercise 0 — *Gaussian vectors.*
Let X be a random vector in $\mathbb{R}^n$. We say that it is a Gaussian vector (i.e. has a multidimensional Gaussian distribution) if for every $t \in \mathbb{R}^n$, the r.v. $\langle t, X\rangle \in \mathbb{R}$ has a (possibly degenerate) Gaussian distribution.

(1) Recall the parameters, the characteristic function, and (when it exists) the p.d.f. of a Gaussian distribution on $\mathbb{R}$.
(2) Show that $t \mapsto \mathbb{E}[\langle t, X\rangle]$ is a linear form, and $(s,t) \mapsto \mathrm{Cov}[\langle s, X\rangle, \langle t, X\rangle]$ is a *positive semi-definite* bilinear form. Let them be represented by $\langle \cdot, m\rangle$ and $\langle \cdot, \Sigma \cdot\rangle$. What would be the coordinates of respectively this vector and this matrix? How would you call them?
(3) Deduce the (multidimensional) characteristic function of X, and that the distribution of X is characterized by the parameters m and Σ. Show that conversely any vector with a characteristic function of this form is Gaussian.
(4) Show that a linear transform AX of a Gaussian vector X is Gaussian, and compute its parameters.
(5) Let V_1 and V_2 be two subspaces of $\mathbb{R}^n$. Give a necessary and sufficient condition for the independence of the σ-algebras $\sigma(\langle t, X\rangle, t \in V_1)$ and $\sigma(\langle t, X\rangle, t \in V_2)$.
(6) Build two standard Gaussian variables X and Y that are uncorrelated yet not independent (they obviously do not form a Gaussian vector !)
(7) Show that the vector $(X_1, \dots X_n)$ with $X_1, \dots, X_n$ independent standard Gaussian variables, is Gaussian. Use it to build a Gaussian vector with arbitrary parameters. Deduce its p.d.f. when it has one.

Exercise 1 — *Conditioning and independence.*
Let $\mathcal{G}$ be a σ-algebra, $X \in \mathcal{G}$ and $Y \perp\!\!\!\perp \mathcal{G}$ be two random variables, and $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(X,Y) \in L^1$. Compute $\mathbb{E}[f(X,Y) \mid \mathcal{G}]$. Deduce the conditional distribution of $f(X,Y)$ given $\mathcal{G}$.

Exercise 2 — *Gaussian conditional distribution and Bayesian statistics 101.*
Let (X,Y) be a non-degenerate centered Gaussian vector in $\mathbb{R}^2$ with covariance matrix $\begin{pmatrix} \sigma_x^2 & \rho \\ \rho & \sigma_y^2 \end{pmatrix}$.

(1) Compute the conditional distribution of X given Y.
(2) If you want, look up on Wikipedia the generalized version of this, where $X \in \mathbb{R}^n$, $Y \in \mathbb{R}^m$.

(3) Let $\theta \sim \mathcal{N}(0, \tau^2)$ and $X_1, \dots X_n$ i.i.d. of distribution $\mathcal{N}(\theta, \sigma^2)$ given θ. In other terms, $X_i = \theta + Y_i$ where Y_i are i.i.d, independent of θ, and $Y_i \sim \mathcal{N}(0, \sigma^2)$. What is the conditional distribution of θ given $\overline{X} = \frac{1}{n} \sum_{i=1}^n X_i$?
(4) Take the large σ, small σ, large τ, small τ limit of this and interpret it.
(5) Find a "real-life" situation modelled by this.
(6) ($\star$) What about the conditional distribution of θ given $(X_1, \dots X_n)$?

Exercise 3 — *Borel-Kolmogorov paradox.*
Let P denote a uniform point in the sphere $\mathbb{S}^2$, i.e. for every bounded measurable f,

$$\int f(p)\, \mathbb{P}_P(dp) = \frac{1}{\mathrm{Leb}_3(B_{\mathbb{R}^3}(0,1))} \int_{B_{\mathbb{R}^3}(0,1)} f\left(\frac{p}{|p|}\right) \mathrm{Leb}_3(dp)$$

. Denote $\phi_P \in (-\pi/2, \pi/2]$ its latitude and $\theta_P \in (-\pi, \pi]$ its (almost surely defined) longitude.

(1) Compute the conditional distribution of P given ($\theta_P \mod \pi$).
(2) Compute the conditional distribution of P given ϕ_P.
(3) Justify that there is only one "right way" of specializing those answers at $\theta \mod \pi = 0$ (resp. $\phi = 0$).
(4) What is the paradox ?

Exercise 4 — *Transformations.*
Let $(B_t)_{t \geq 0}$ be a Brownian motion.

(1) Show that for any $\lambda \in \mathbb{R}_+^\star$, the process $(\lambda^{-1/2} B_{\lambda t})_{t \geq 0}$ is a Brownian motion.
(2) Show that $B_1 - B_{1-t}$ is a Brownian motion on $[0, 1]$.

Exercise 5 — *A nowhere continuous version of the Brownian motion.*
Given a Brownian motion, build a probability space with a random variable B_t for all $t \in \mathbb{R}_+$ such that:

- for every ω, the function $t \mapsto B_t(\omega)$ is nowhere continuous;
- for every $t_1 < \ldots < t_n$, the vector $(B_{t_1}, \dots, B_{t_n})$ has the same distribution as if B were a Brownian motion.

Hint: change the value of B on a countable dense random subset of $\mathbb{R}$, so that the value at a fixed deterministic time is almost surely not changed.

Exercise 6 — *Brownian motion is nowhere monotonous.*
Let B be a Brownian motion. Show that almost surely, the function $t \mapsto B_t$ is not monotonous on any nonempty open interval.

Exercise 7 — *The stationary Ornstein-Uhlenbeck process.*
For $t \in \mathbb{R}$, set $X_t = e^{-t} B_{2t}$, where B is a Brownian motion. Show that X is a continuous Gaussian process, compute its covariance function. For any given t, what is the distribution of X_t ? Does it have independent increments ?