# $\mathbb{E}$xercise sheet 11: Miscellanea

**Exercise 1** — *Capacity and Hausdorff dimension.*
Let $f$ be a positive function on $\mathbb{R}^d$ called *potential.* The energy of a measure $\mu$ is $I_f(\mu) = \iint f(x-y)\mu(dx)\mu(dy)$. The capacity of some set $A$ is

$$\mathrm{Cap}_f(A) = [\inf\{I_f(\mu) : \mu \text{ probability measure on } A\}]^{-1}$$

At some point you will see that a closed set is polar in dimension $d \geq 2$ if and only if it has zero capacity for the radial potential $f(\epsilon) = |\log(\epsilon)|$ if $d = 2$ and $f(\epsilon) = \epsilon^{2-d}$ if $d \geq 3$. We wish to show a connexion between the notion of capacity and Hausdorff dimension.

(1) Show that if $\mu$ is a measure on $A \subset \mathbb{R}^d$,

$$\inf_{\substack{(U_i)_i \in \mathcal{P}(A)^{\mathbb{N}} \\ \forall i, \mathrm{diam}(U_i) \leq \delta \\ \bigcup_i U_i = A}} \left( \sum_{i\in\mathbb{N}} \mathrm{diam}(U_i)^\alpha \right) \geq \frac{\mu(A)^2}{\iint_{|x-y|<\delta} \mu(dx)\mu(dy)|x-y|^{-\alpha}}$$

and deduce that a set of nonzero capacity for $f(\epsilon) = \epsilon^{-\alpha}$ has Hausdorff dimension $\geq \alpha$.

(2) Show also that the image of a segment by a $\alpha$-Hölder function is of Hausdorff dimension bounded by $\frac{1}{\alpha}$.

(3) What is the Hausdorff dimension of $B([0,1])$ in $\mathbb{R}^d$ ?

**Exercise 2** — *Some more boundary value problems.*
In this exercise we will admit that for $x, y \in \mathbb{R}^d$, $t > 0$, we have $\partial_t p_t(x,t) = \frac{1}{2}\Delta_y p_t(x,y)$. (Fokker-Planck equation)

(1) Show that if $f$ is $\mathcal{C}^2$ with compact support, then under $\mathbb{P}_x$, $(f(B_t) - \frac{1}{2}\int_0^t \Delta f(B_s)ds)_t$ is a martingale. (Dynkin's formula)

(2) Let $D$ be a bounded domain and $f : \overline{D} \to \mathbb{R}$ continuous and $\mathcal{C}^2$ on the interior with bounded second derivatives. Let $T$ be the hitting time of the complement of $D$. Show that $(f(B_{t\wedge T}) - \frac{1}{2}\int_0^{t\wedge T} \Delta f(B_s)ds)_t$ is a martingale (*Hint*: use a regularization procedure to apply question 1).

(3) Show that in the sense of distributions, we have $\Delta G(x,\cdot) = -2\delta_x$, where $G$ is the Green function of the Brownian motion in the whole of $\mathbb{R}^3$ or in a bounded domain of $\mathbb{R}^2$.

(4) Show that in a bounded domain $D \subset \mathbb{R}^d$ with $f$ continuous, a solution of the *Poisson problem*

$$\Delta u = f \text{ on } D$$
$$u = 0 \text{ on } \partial D$$

must verify $u(x) = -\frac{1}{2}\,\mathbb{E}_x[\int_0^T f(B_s)ds]$.

(5) Conversely, if $f$ is Hölder and $D$ is bounded and verifies the Poincaré cone condition, show that this formula (which can be rewritten $u(x) = -\frac{1}{2}\int f(y)G(x,y)dy$) gives a solution of the Poisson problem in the sense of distributions.

## Appendix A. Hausdorff dimension

Let $(E,d)$ be a metric space. For $\alpha \geq 0$ and $A \subset E$, we define the *$\alpha$-dimensional Hausdorff measure* of $A$ follows:

$$\mathcal{H}_\alpha(A) := \lim_{\delta \to 0}\left(\inf_{\substack{(U_i)_i \in \mathcal{P}(E)^{\mathbb{N}} \\ \forall i, \mathrm{diam}(U_i) \leq \delta \\ \bigcup_i U_i \supset A}} \left(\sum_{i \in \mathbb{N}} \mathrm{diam}(U_i)^\alpha\right)\right).$$

It is well defined because the lim is actually a sup, and verifies the following property:
**Lemma** Let $\alpha \in [0,\infty)$.If $\mathcal{H}_\alpha(A) < \infty$ then for $\beta > \alpha$ $\mathcal{H}_\beta(A) = 0$. If $\mathcal{H}_\alpha(A) > 0$ then for $\beta < \alpha$ $\mathcal{H}_\beta(A) = \infty$.
This tells us that there is a transition point $\alpha \in [0,\infty]$ where the Hausdorff measure jumps from $\infty$ to 0, and we want to call that point the Hausdorff dimension of $A$.

$$\dim_{\mathcal{H}}(A) := \sup\{\alpha, \mathcal{H}_\alpha(A) = \infty\} = \inf\{\alpha, \mathcal{H}_\alpha(A) = 0\}.$$

This $\alpha$ is the only dimension for which $A$ admits a possibly non-trivial Hausdorff measure (but it may still be 0 or $\infty$ in some cases).
For instance, in $\mathbb{R}^d$, the $d$-dimensional Hausdorff measure is equal to the Lebesgue measure (you probably constructed the Lebesgue measure this way), and open sets have necessarily Hausdorff dimension $d$. Of course sets with 0 Lebesgue measure might have a strictly smaller Hausdorff dimension.