$\mathbb{E}$xercise sheet 4: Markov processes and regularity properties

version 2: the order of reasoning in exercise 3 has been changed + some minor tweaks

1. Markov processes

Exercise 1 — *The Ornstein-Uhlenbeck process.*
For $t \in \mathbb{R}$, set $X_t = e^{-t} B_{e^{2t}}$, where B is a Brownian motion.

(1) Show that X is a continuous Gaussian process, compute its covariance function.For any given t, what is the distribution of X_t ?
(2) Show that it is X_t Markov process and compute its transition kernel.
(3) Given some probability measure μ, construct (using B) a process with this kernel started from the distribution μ at time 0. What choice of μ gives a stationary process ? Show that independently of μ there is convergence to the stationary distribution.

Exercise 2 — *Cauchy process.*
Let $(B^{(1)}, B^{(2)})$ be a two-dimensional Brownian motion, and $T_a = \inf\{t \geq 0, B_t^{(1)} \geq a\}$ for $a \geq 0$. Set $C_a = B^{(2)}_{T_a}$.

(1) Compute the distribution of C_a.
(2) Show that $(C_{a+\cdot} - C_a)$ is independent of $\mathcal{F}_{T_a}$ and distributed like C. Deduce that C is a Markov process and give its transition kernel.
(3) Is C continuous ?

2. Regularity of B

Exercise 3 — *One-sided and two-sided local minima.*
A (strict) local minimum of B is a time t for which there exists $\varepsilon > 0$ such that $B_s \leq B_t, s \in [t-\varepsilon, t+\varepsilon]$ ($B_s < B_t, s \in [t-\varepsilon, t+\varepsilon], s \neq t$). A right decrease point is a time t for which there exists $\varepsilon > 0$ such that $B_s \leq B_t, s \in [t, t+\varepsilon]$.

(1) What is the Lebesgue measure of those points ?
(2) Show that almost surely there is a density of local minima.
(3) Show the minimum values of the Brownian motion on two distinct rational intervals are almost surely different.
(4) Deduce that almost surely, all local minima are strict, with distinct values. Deduce that they are at most countable.
(5) Show that there is almost surely an uncountable number of right decrease points.

Exercise 4 — *Quadratic and absolute variation.*
For fixed t, a partition $\underline{t}$ is a finite sequence $0 = t_0 \leq t_1 \leq \ldots \leq t_{\#\underline{t}} = t$ and its mesh-size

is $|\underline{t}| = \max_{1\le i\le \#\underline{t}} |t_i - t_{i-1}|$. The absolute (resp. quadratic) variation of B between 0 and t is

$$\lim_{\epsilon\to 0}\sup_{|\underline{t}|\le\epsilon}\sum_{i=1}^{\#\underline{t}}|B_{t_i}-B_{t_{i-1}}| \qquad \left(\text{resp.}\quad \lim_{\epsilon\to 0}\sup_{|\underline{t}|\le\epsilon}\sum_{i=1}^{\#\underline{t}}(B_{t_i}-B_{t_{i-1}})^2.\right)$$

(1) If $\underline{t}^{(k)}$ is a sequence of partitions with $|\underline{t}^{(k)}| \to 0$, then show that $\lim_{k\to\infty}\sum_{i=1}^{\#\underline{t}^{(k)}}(B_{t_i^{(k)}} - B_{t_{i-1}^{(k)}})^2$ exists in the L^2 sense. What is it ?

(2) If $(\underline{t}^{(k)})_k$ is such that $\sum_{k=1}^{\infty}\sum_{j=1}^{\#\underline{t}^{(k)}}(t_i^{(k)} - t_{i-1}^{(k)})^2 < \infty$, then the convergence is almost sure.

(3) Deduce that the Brownian motion almost surely does not have finite absolute variation (i.e. is not a bounded variation function).