$\mathbb{E}$xercise sheet 5 : Martingales

Exercise 1 — *All hypotheses matter.*
Find two stopping times S and T with $S \leq T < \infty$ a.s. and $\mathbb{E}[S] < \infty$, such that $\mathbb{E}[B_S^2] > \mathbb{E}[B_T^2]$.

Exercise 2 — *Brownian gambler's ruin.*
For any $c \in \mathbb{R}$, we let

$$T_c := \inf\{t \geq 0 \ : \ B_t = c\}$$

be the hitting time of c by $(B_t)_{t\geq 0}$. Let $a, b \in \mathbb{R}$ such that $a < 0 < b$, we let $T_{a,b} := T_a \wedge T_b$ be the hitting time of $\{a, b\}$ by $(B_t)_{t\geq 0}$.

1. Compute $\mathbb{P}(T_{a,b} = T_a)$.
2. Compute $\mathbb{E}[T_{a,b}]$.

Exercise 3 — *Exponential martingale and computations.*
We recall that for every $\lambda \in \mathbb{R}$, the process $(e^{\lambda B_t - t\lambda^2/2})_{t\geq 0}$ is a martingale, called the exponential martingale. We let for any $a > 0$,

$$T_{a+} := \inf\{t \geq 0 \ : \ B_t > a\}$$

1. Use the exponential martingale to compute the Laplace transform of the hitting time T_{a^+} of a given level $a > 0$..
2. Let $(B^{(1)}, B^{(2)})$ be a two-dimensional Brownian motion. For every $a \geq 0$, we let

$$C_a := B^{(2)}_{T^{(1)}_{a+}}.$$

   (a) Show that for any $b > 0$, the process $(C_{b+a} - C_b)_{a\geq 0}$ is independent of $\mathcal{F}_{T_{b+}}$ and has the same law as $(C_a)_{a\geq 0}$. Deduce that $(C_a)_{a\geq 0}$ is a Markov process and give its transition kernel.

   (b) Is $(C_a)_{a\geq 0}$ continuous ?

   (c) Show that $(C_a)_{a\geq 0}$ is self-similar for an exponent that you will specify

   (d) Show that $(e^{\lambda(B_t^{(1)} + iB_t^{(2)})})_{t\geq 0}$ is a complex martingale, and deduce the characteristic function of C_a for $a > 0$ fixed.

   A random process with such stationary increments and with such marginals is called a Cauchy process (the marginals are actually Cauchy distributed)

3. Show that if $(X_t)_{t\geq 0}$ is a process such that for any $\lambda \in \mathbb{R}$, $(e^{\lambda X_t - t\lambda^2/2})_{t\geq 0}$ is a continuous martingale, then $(X_t)_{t\geq 0}$ is a Brownian motion.

Exercise 4 — *Hitting time of a line.*
Let $a, b \geq 0$ and $T = \inf\{t \geq 0, B_t = at + b\}$. Compute $\mathbb{P}(T < \infty)$.

Exercise 5 — *Martingales derived from B.*
Show that $(B_t^2 - t)_{t\geq 0}$ and $(B_t^3 - 3tB_t)_{t\geq 0}$ are martingales. Guess the other ones.