Solutions for $\mathbb{E}$xercise sheet 5: Martingales

Solution 1 — *All hypotheses matter.*
Take $S = 3$ and T to be the first zero after 3. Of course the problem is that $\mathbb{E}[T] = \infty$.

Solution 2 — *Brownian gambler's ruin.*
Let $a < 0 < b$ and T be the hitting time of $\{a, b\}$.

(1) Let $p = \mathbb{P}\{T = T_a\}$. We can apply the optional stopping theorem to B at T since $|B_{t\wedge T}| < |a| \vee b < \infty$. Hence $0 = \mathbb{E}[B_T] = ap + (1-p)b$. Solving for p yields $p = b/(b-a)$.

(2) Look at the martingale $B^2_{t\wedge T} - (t \wedge T)$ at t. By the martingale property we have

$$\mathbb{E}[B^2_{t\wedge T}] = \mathbb{E}[t \wedge T].$$

We apply the dominated convergence theorem at the left and the monotone convergence theorem at the right. We get $\mathbb{E}[B_T^2] = \mathbb{E}[T]$ hence $\mathbb{E}[T] = |a|b$.

Alternatively we may show that T is integrable to apply Wald's second lemma. Here's a way to do it by comparison with a geometric variable.

$$\begin{aligned}
\mathbb{P}(T \geq n) &= \prod_{k=0}^{n-1} \mathbb{P}(T \geq k+1 \mid T \geq k) \\
&= \prod_{k=0}^{n-1} \mathbb{E}\left(\frac{\mathbb{1}[T \geq k]}{\mathbb{P}(T \geq k)} \mathbb{E}\left(\mathbb{1}[T \geq k+1] \mid \mathcal{F}_k\right)\right) \\
&= \prod_{k=0}^{n-1} \mathbb{E}\left(\frac{\mathbb{1}[T \geq k]}{\mathbb{P}(T \geq k)} \mathbb{E}\left(\mathbb{1}\left[\max_{[0,1]} B^{(k)} < 1 - B_k \text{ and } \min_{[0,1]} B^{(k)} > -1 + B_k\right] \mid \mathcal{F}_k\right)\right) \\
&= \prod_{k=0}^{n-1} \mathbb{E}\left(\frac{\mathbb{1}[T \geq k]}{\mathbb{P}(T \geq k)} \mathbb{P}\left(\max_{[0,1]} B < 1 - x \text{ and } \min_{[0,1]} B > -1 + x\right)_{x = B_k}\right)
\end{aligned}$$

The probability inside depends on x and we need to bound it unifomly for $x \in [-1, 1]$. But it is clearly bounded by $\alpha = \mathbb{P}(\max_{[0,1]} |B| < 2)$, which is < 1. Hence $\mathbb{P}(T \geq n) \leq \alpha^n$ and T is integrable.

Solution 3 — *Exponential martingale and computations.*
We recall that for every $\lambda \in \mathbb{R}$, the process $e^{\lambda B_t - t\lambda^2/2}$ is a martingale, called the exponential martingale.

(1) Let $a, \lambda > 0$ and $X_t = e^{\lambda B_t - t\lambda^2/2}$. Then $|X_{t\wedge T}| \leq e^{\lambda a}$. So the optional stopping theorem applies and $1 = \mathbb{E}[X_T] = e^{\lambda a - T\lambda^2/2}$. Setting $\lambda^2/2 = \mu$, we get $\mathbb{E}[e^{-\mu T}] = e^{-\sqrt{2\mu}}$.

(2) (a) Let $B = (B^{(1)}, B^{(2)})$. We have that $(C_{a+\cdot} - C_a)$ is constructed from $B_{T^{(1)}_{a+}+\cdot} - B_{T^{(1)}_{a+}}$ the same way C is constructed from B. Hence by the strong Markov property of B, $(C_{a+\cdot} - C_a) \overset{d}{=} C$, and $(C_{a+\cdot} - C_a) \perp\!\!\!\perp \mathcal{F}_{T^{(1)}_{a+}} \supset \sigma(C_u, u \leq a)$. As a result the Markov transition kernel is

$$\nu_t(x, dy) = \mathbb{P}(x + C_t \in dy).$$

(b) C is càdlàg because $T_{\cdot+}$ is. By independence of $B^{(1)}$ and $B^{(2)}$ it jumps almost surely when $T_{\cdot+}$ jumps.

(c) Set $X_t = e^{\lambda(B^{(1)}_t + iB^{(2)}_t)}$. We compute by Fubini $\mathbb{E}[X_t] = e^{\lambda^2 t/2} e^{-\lambda^2 t/2} = 1$. This plus the independent increments give that X is a complex martingale. Moreover $|X_{t\wedge T^{(1)}_{a+}}| = \exp(\lambda B^{(1)}_{T^{(1)}_{a+}}) \leq e^{\lambda a}$. Hence the optional stopping theorem applies and

$$1 = \mathbb{E}[X_{T^{(1)}_{a+}}] = \mathbb{E}[e^{i\lambda(-ia + C_a)}] = e^{\lambda a}\,\mathbb{E}[e^{i\lambda C_a}].$$

Hence $\mathbb{E}[e^{i\lambda C_a}] = e^{-\lambda a}$ for $\lambda \geq 0$. For negative λ we use the fact that $C_a \overset{d}{=} -C_a$, and finally get $\mathbb{E}[e^{i\lambda C_a}] = e^{-|\lambda| a}$

(3) These hypotheses imply that $\mathbb{E}[e^{\lambda(X_{t+s} - X_t)} \mid \mathcal{F}_t] = e^{-s\lambda^2/2}$. So almost surely for every $\lambda \in \mathbb{Q}$,

$$\int e^{\lambda x}\,\mathbb{P}((X_{t+s} - X_t) \in dx \mid \mathcal{F}_t) = \int e^{\lambda x}\,\mathbb{P}(B_s \in dx)$$

Hence by injectivity of the moment-generating function, almost surely $\mathbb{P}((X_{t+s} - X_t) \in dx \mid \mathcal{F}_t) = \mathbb{P}(B_s \in dx)$, which is exactly to say that $(X_{t+s} - X_t)$ is a Gaussian of variance s, independent of $\mathcal{F}_t$. This characterizes the Brownian motion.

Solution 4 — *Hitting time of a line.*
Let $X_t = e^{2aB_t - 2a^2 t}$ be the exponential martingale with $\lambda = 2a$. For $n \geq 0$, $X_{t\wedge T}$ is trivially bounded by $e^{2ab} < \infty$, and the optional stopping theorem applied to $n \wedge T$ gives

$$1 = \mathbb{E}[X_{T\wedge n}] = \mathbb{E}[X_T\,\mathbb{1}_{T<n}] + \mathbb{E}[X_n\,\mathbb{1}_{n\leq T}]$$

As $n \to \infty$, the first term goes to $\mathbb{E}[X_T\,\mathbb{1}_{T<\infty}]$ by monotone convergence. The integrand in the second term is bounded by $e^{2ab} < \infty$, and goes to 0 almost surely, since $B_t = o(t)$ almost surely. So by dominated convergence the expectation goes to 0. We get $1 = \mathbb{E}[X_T\,\mathbb{1}_{T<\infty}] = \mathbb{E}[e^{2ab}\,\mathbb{1}_{t<\infty}]$ so $\mathbb{P}(T < \infty) = e^{-2ab}$.

Solution 5 — *Martingales derived from B.*
Those martingales are the derivative w.r.t λ of the exponential martingale.