Solutions for $\mathbb{E}$xercise sheet 9: Harmonic functions and Brownian motion

Solution 1 — *Harmonic functions and martingales.* (1) Let us show that for every x, under $\mathbb{P}_x$, $t \mapsto h(B_{t\wedge T})$ is a martingale that is closed by $h(B_T)$. Thus we shall show that for every x, $\mathbb{E}_x[h(B_T) \mid \mathcal{F}_t] = h(B_{t\wedge T})$. Indeed we can compute

$$\begin{aligned}\mathbb{E}_x[h(B_T) \mid \mathcal{F}_t] &= \mathbb{E}_x[h(B_{t\wedge T})\,\mathbb{1}_{T<t} + h(B_T)\,\mathbb{1}_{t\leq T} \mid \mathcal{F}_t] \\ &= h(B_{t\wedge T})\,\mathbb{1}_{T<t} + \mathbb{E}_x[h(B_T)\,\mathbb{1}_{t\leq T} \mid \mathcal{F}_t] \\ &= h(B_{t\wedge T})\,\mathbb{1}_{T<t} + \mathbb{E}_x[h(B^t_{T^t})\,\mathbb{1}_{t\leq T} \mid \mathcal{F}_t] \\ &= h(B_{t\wedge T})\,\mathbb{1}_{T<t} + \mathbb{E}_x[h(B^t_{T^t})\,\mathbb{1}_{t\leq T} \mid \mathcal{F}_t] \\ &= h(B_{t\wedge T})\,\mathbb{1}_{T<t} + \mathbb{E}_{B_t}[h(B_T)]\,\mathbb{1}_{t\leq T} \\ &= h(B_{T\wedge t})\,\mathbb{1}_{T<t} + h(B_t)\,\mathbb{1}_{T\geq t} = h(B_{t\wedge T})\end{aligned}$$

Where B^t denoted the Brownian motion restricted from time t onward, and T^t the hitting time of ∂D for this process.

(2) Consider $D_\varepsilon = B(0, \varepsilon^{-1}) \cap \{x, d(x, D^\complement) > \varepsilon\}$, which is open and bounded. Then $D_\varepsilon^\complement = B(0,\varepsilon^{-1})^\complement \cup \bigcup_{x\in D^\complement} \overline{B}(0,\varepsilon)$. We can show that D_ε verifies the Poincaré cone condition.

Let T_ε be the hitting time of $D^\complement$. We can now apply question 1 and get that

$$\mathbb{E}[h(B_{t\wedge T_\varepsilon}) \mid \mathcal{F}_s] = h(B_{s\wedge T_\varepsilon})$$

We can now use continuity of paths, continuity of h, and the (conditional) dominated convergence theorem to conclude.

Solution 2 — *A lemma for the Poincaré cone condition.*
Let C be an open cone based in 0. We wish to show that the function $\varphi(x) = \mathbb{P}_x(T_{\partial B(0,1)} < T_{\partial C})$ is bounded away from 1 on $\overline{B}(0, 1/2) \setminus C$.

(1) We have that φ is harmonic on the interior of $B(0,1)\setminus C$. That is because it can be rewritten as $\mathbb{E}[u(B_{T_{\partial(B(0,1)\setminus C)}})]$ for $u = \mathbb{1}_{B(0,1)^\complement \setminus C}$.
We can't use the maximum principle for φ on $\overline{B}(0,1/2)\setminus C$ because we don't know if φ is continuous on the boundary of this set.

(2) Let $\widetilde{C}$ be another open cone such that $\overline{\widetilde{C}} \subset C$ (for instance take $\widetilde{C}$ to be C translated away from 0). Define $\psi(x) = \mathbb{P}_x(T_{\partial B(0,1)} < T_{\partial \widetilde{C}})$. It is clear from an inclusion of events, that $\psi(x) \geq \varphi(x)$. It is also clear that ψ is harmonic on $U = B(0,1)\setminus\overline{\widetilde{C}}$, for the same reasons as φ. Now $P = \overline{(B(0,1/2)\setminus C)}$ is completely included in U, so ψ is continuous on the compact P. If $\sup_P \psi = 1$, then by compactness we found

a point of $P \subset U$ where ψ reaches one. As $\psi \leq 1$, the maximum principle tells us that $\psi \equiv 1$, which seems absurd.
To see why this is absurd, take a finite union F (for *flower*) of rotations of $\widetilde{C}$ that disconnects 0 from infinity. If $\varphi(0) = 1$, then almost surely B does not touch $\widetilde{C}$ before exiting $B(0,1)$ and by rotation invariance and countable union, it also almost surely does not touch F before exiting $B(0,1)$. Hence it does not touch F, hence it stays bounded almost surely. This is clearly absurd.

Solution 3 — *Counterexample.*
Set $T = T_{\partial D}$ and $h(x) = \mathbb{E}_x[u(B_T)]$. This does not define a solution to the Laplace equation, because since the Brownian motion started outside of 0 almost surely does not hit 0, we have $h(0) = 0$ and $h(x) = 1$ for all $x \in \overline{D} \setminus \{0\}$. Hence h is not continuous.
Suppose a solution h exists. Then a rotation of h is still a solution, and hence equals h thanks to the maximum principle. Thus h is rotation invariant hence radial ($h(x) = g(|x|), x \in \overline{D}$, for some $g : \mathbb{R}_+ \to \mathbb{R}$ that must be twice differentiable.) We deduce that $0 = g''(x) + \frac{1}{x}g'(x)$ for all $0 < x < 1$, an ODE whose solutions are of the form $x \mapsto A + B\log(x)$, none of which fits our purpose. Hence a solution cannot exist.

Solution 4 — *Singularity removal.*
See TD10