$\mathbb{E}$xercise sheet 1: Gaussian vectors, random walks, conditioning.

Exercise 1 — *Gaussian vectors.*
Let X be a random vector in $\mathbb{R}^n$. We say that it is a Gaussian vector (i.e. has a multidimensional Gaussian distribution) if for every $t \in \mathbb{R}^n$, the r.v. $\langle t, X\rangle \in \mathbb{R}$ has a (possibly degenerate) Gaussian distribution.

(1) Recall the parameters, the characteristic function, and (when it exists) the p.d.f. of a Gaussian distribution on $\mathbb{R}$.
(2) Show that $t \mapsto \mathbb{E}[\langle t, X\rangle]$ is a linear form, and $(s,t) \mapsto \mathrm{Cov}[\langle s, X\rangle, \langle t, X\rangle]$ is a *positive semi-definite* bilinear form. Let them be represented by $\langle \cdot, m\rangle$ and $\langle \cdot, \Sigma \cdot\rangle$. What would be the coordinates of respectively this vector and this matrix? How would you call them?
(3) Deduce the (multidimensional) characteristic function of X, and that the distribution of X is characterized by the parameters m and Σ. Show that conversely any vector with a characteristic function of this form is Gaussian.
(4) Show that a linear transform AX of a Gaussian vector X is Gaussian, and compute its parameters.
(5) Let V_1 and V_2 be two subspaces of $\mathbb{R}^n$. Give a necessary and sufficient condition for the independence of the σ-algebras $\sigma(\langle t, X\rangle, t \in V_1)$ and $\sigma(\langle t, X\rangle, t \in V_2)$.
(6) Build two standard Gaussian variables X and Y that are uncorrelated yet not independent (they obviously do not form a Gaussian vector !)
(7) Show that the vector $(X_1, \dots X_n)$ with $X_1, \dots, X_n$ independent standard Gaussian variables, is Gaussian. Use it to build a Gaussian vector with arbitrary parameters. Deduce its p.d.f. when it has one.

Exercise 2 — *Limits of Gaussian variables.*
Let $(X_n)_{n\geq 0}$ be a sequence of Gaussian variables.

(1) Give a necessary and sufficient condition for convergence in distribution, show that the limit is always Gaussian, and determine its parameters.
(2) If $X_n \to X$ in probability, show that for every $p \geq 1$, $X_n \to X$ in L^p.

Exercise 3 — *Some estimates.*
Let $(X_n)_{n\geq 0}$ be the simple symmetric random walk.

(1) Compute the asymptotic for the probability of $X_{2n} = 0$.
(2) Compute the probability that the first return to zero is at time $2n$. Compute its asymptotic.
(3) Deduce the asymptotic for the probability that the first return to zero is after time n.

(4) Deduce the asymptotic of $\mathbb{E}[|X_n|]$.

Exercise 4 — *Maximum and hitting times.*
Let $(X_n)_{n\geq 0}$ be the simple symmetric random walk and $M_n = \max_{1\leq i\leq n} X_i$.

(1) Find a limit in distribution (after suitable rescaling) for the pair (X_n, M_n).
(2) For $k \geq 0$, let
$$\tau_k = \min\{n \geq 0 : X_n = k\}.$$
Show that $(\tau_k)_k$ is a random walk. What is $\mathbb{P}(\tau_k \leq n)$? Find a limit in distribution (after suitable rescaling) for τ_k.

Exercise 5 — *Conditioning and independence.*
Let $\mathcal{G}$ be a σ-algebra, $X \in \mathcal{G}$ and $Y \perp\!\!\!\perp \mathcal{G}$ be two random variables, and $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(X,Y) \in L^1$. Compute $\mathbb{E}[f(X,Y) \mid \mathcal{G}]$. Deduce the conditional distribution of $f(X,Y)$ given $\mathcal{G}$.

Exercise 6 — *Gaussian conditional distribution and Bayesian statistics 101.*
Let (X,Y) be a non-degenerate centered Gaussian vector in $\mathbb{R}^2$ with covariance matrix $\left(\begin{smallmatrix}\sigma_x^2 & \rho \\ \rho & \sigma_y^2\end{smallmatrix}\right)$.

(1) Compute the conditional distribution of X given Y.
(2) If you want, look up on Wikipedia the generalized version of this, where $X \in \mathbb{R}^n$, $Y \in \mathbb{R}^m$.
(3) Let $\theta \sim \mathcal{N}(0,\tau^2)$ and $X_1, \dots X_n$ i.i.d. of distribution $\mathcal{N}(\theta,\sigma^2)$ given θ. In other terms, $X_i = \theta + Y_i$ where Y_i are i.i.d, independent of θ, and $Y_i \sim \mathcal{N}(0,\sigma^2)$. What is the conditional distribution of θ given $\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i$?
(4) Take the large σ, small σ, large τ, small τ limit of this and interpret it.
(5) Find a "real-life" situation modelled by this.
(6) $(\star)$ What about the conditional distribution of θ given $(X_1, \dots X_n)$?