Solutions for $\mathbb{E}$xercise sheet 2 : Properties and construction of the Brownian Motion.

Solution 1 — *Transformations.* (1) We first consider the finite-dimensional marginals of the new process $(X_t)_t$ in this case. Remark at first that they still form centered Gaussian vectors, since they are each obtained by a very simple linear transform of some f.d.m. of B. Now we only need to compute covariances: If $0 < s, t$, $\mathrm{Cov}(X_s, X_t) = \mathrm{Cov}(sB_{1/s}, tB_{1/t}) = st(s^{-1} \wedge t^{-1}) = t \wedge s$. If either $t = 0$ or $s = 0$, then we get $0 = s \wedge t$ for the covariance too.

(2) Now consider the set $U = \left\{ A \in \mathbb{R}^{\mathbb{Q}_+} : A_t \xrightarrow[t \to 0^+, t \in \mathbb{Q}_+]{} 0 \right\}$. It can be written

$$\bigcap_{n \geq 1} \bigcup_{m \in \mathbb{N}} \bigcap_{q \in \mathbb{Q}_+ : q \leq 1/m} \{A : |A_q| < 1/n\},$$

hence it belongs to the σ-algebra generated by finite-dimensional sets.

(3) By the $\pi - \lambda$ (monotone class) theorem, two measures that coincide on a π-system Π (a family of sets stable by finite intersection), coincide on the generated σ-algebra $\sigma(\Pi)$. As a result, since B and X have the same finite-dimensional marginals, then $\mathbb{P}(X_{|\mathbb{Q}_+} \in U) = \mathbb{P}(B_{|\mathbb{Q}_+} \in U) = 1$. Hence we have with probability one that:

(a) $t \mapsto X_t$ is continuous on $(0, \infty)$,

(b) $X_t \xrightarrow[t \to 0^+, t \in \mathbb{Q}]{} X_0$

wich together implies continuity on the whole of $[0, \infty)$. Now if we change the X to the constant zero function whenever X is not continuous, this makes X continuous for all ω without changing the f.d.m's. So X is a Brownian motion.

Solution 2 — *A nowhere continuous version of the Brownian motion.*
Let $(X_t)_t$ be a Brownian motion and $(U_i)_i$ be an independent sequence of independent exponential random variables with parameter 1.
Let us show the following property: with probability one, $(U_i)_i$ is dense in $[0, \infty)$. Let $a < b \in \mathbb{Q}$. $\mathbb{P}(U_1 \notin [a, b], \ldots, U_n \notin [a, b]) = \mathbb{P}(U_1 \notin [a, b])^n \to 0$ as $n \to \infty$. So $\mathbb{P}(U_i \notin [a, b] \forall i) = 0$. We have shown $\forall a < b \in \mathbb{Q}^2$, almost surely, $[a, b]$ intersects $(U_i)_i$. Because $\mathbb{Q}^2$ is countable, we can invert $\forall$ and almost surely, and we get that almost surely, $(U_i)_i$ is dense.
Now we define $B_t = X_t + \mathbb{1}_{t \notin \{U_i, i \in \mathbb{N}\}}$. By the previous property, this process is almost surely nowhere continuous, and we can check that the f.d.m's of B and X are equal almost surely (so have the same distribution) because for fixed $t_1, \ldots, t_k$, the probability that $\{t_1, \ldots, t_k\}$ intersects $\{U_i, i \in \mathbb{N}\}$ is 0 (once again by countable union).

Now we modify B on the negligible set where it is still continuous despite all our efforts, by setting $B = \mathbb{1}_{\mathbb{Q}}$, finishing the exercise.

Solution 3 — *Brownian motion is nowhere monotonous.*
Let us fix $a < b \in \mathbb{Q}$ and $a < t_0 < \ldots < t_k < b$. If the Brownian motion is increasing, it implies that $B_{t_i} - B_{t_{i_1}} \geq 0$ for every $1 \leq i \leq k$. So $\{B$ increasing on $[a,b]\} \subset \bigcap_{1 \leq i \leq k}\{B_{t_i} - B_{t_{i_1}} \geq 0\}$. This last event has probability 2^{-k} by independence. So $\{B$ increasing on $[a,b]\}$ can be included in an event of probability 0. So by countable union

$$\{B \text{ increasing on some interval }\} \subset \cup_{a<b\in\mathbb{Q}}\{B \text{ increasing on } [a,b]\}$$

can be included in an event of probability 0. So the complement property "B is increasing on no nontrivial interval" is almost sure. Same for "decreasing" by symmetry.
Remark: We did not need to show that the property "B is monotonous on no nontrivial interval" is indeed an event (i.e. is a measurable set), because the property of being almost sure or negligible can be defined for non-measurable subsets of Ω. But we can check that it is an event because of the assumption that paths are always continuous.

Solution 4 — *L^2 theory and construction of the Brownian motion.* (1) Immediate.

(2) Then setting $B_t = \langle \xi, I_t \rangle$ would yield a Gaussian process with the right covariance kernel. It can be checked by computing the characteristic function $(B_{t_1}, \ldots, B_{t_k})$.

(3) Same computation: $\mathbb{E}[\exp(it_1 Z_{i_1} + \cdots + it_p Z_{i_p})] = \mathbb{E}[\exp(i\langle t_1 e_1 + \cdots + t_p e_p, \xi\rangle] = \prod_{i=1}^p e^{-it_p^2/2}$. Hence the distribution is that of i.i.d. standard Gaussians.

$$B_t = \langle \xi, I_t \rangle = \sum_{i=0}^{\infty} \langle \xi, e_i \rangle \langle I_t, e_i \rangle = \sum_{n=0}^{\infty} Z_n \int_0^t e_i(s)ds \tag{†}$$

(4) $\|\xi\|^2 = \sum_{i=0}^{\infty} Z_i^2$ which is a.s. not convergent because it does not go to 0 (Borel-Cantelli says that there exists a subsequence of i such that $Z_i > 0$ with probability 1).

(5) Indeed the primitives of the Haar wavelets are exactly the Schauder triangular functions that appear in Lévy's construction.

(6) We get $B_t = Z_0 t + \frac{\sqrt{2}}{\pi} \sum_{i=1}^{\infty} Z_m \frac{\sin(\pi m t)}{m}$.

(7) (*)