$\mathbb{E}$xercise sheet 3 : Lévy's construction, regularity (v2)

From now on, a Brownian motion is continuous.

Exercise 1 — *Simple Markov property.*
Let B be a Brownian motion and $s \geq 0$. In what measurable space does the random variable $(B_{s+t} - B_s)_{t\geq 0}$ live? Show that it is a Brownian motion, independent of B_s.

Exercise 2 — *Local regularity and long-term behavior.*
We recall from the last exercise session that $X(t) = tB(1/t)$, continued at 0 with $X(0) = 0$, is indistinguishable from a Brownian motion.

(1) Deduce that almost surely, $|B(t)| = o(t)$ as $t \to \infty$.
(2) What better bounds can be obtained from the lecture or from exercises below ?
(3) We want to show that B is almost surely not 1/2-Hölder at 0. Fix $c > 0$.
  (a) Use Fatou's lemma to show that $\mathbb{P}(\limsup_n B_{2^{-n}}/\sqrt{2^{-n}} < c)$ is < 1.
  (b) Show that $\{\limsup_n B_{2^{-n}}/\sqrt{2^{-n}} < c\}$ is a tail event for a sequence of independent random variables that appear in Lévy's construction.
  (c) Conclude. What does it say for the long-term behavior ?

Exercise 3 — *A bit more on differentiability.*
We know that almost surely, B is nowhere differentiable. Set $D^*B(t) = \limsup_{h\downarrow 0} \frac{1}{t}(B_{t+h} - B_t)$ and $D_*B(t) = \liminf_{h\downarrow 0} \frac{1}{t}(B_{t+h} - B_t)$.

(1) Show that $D^*B(0) = +\infty$ a.s. and $D_*B(0) = -\infty$ a.s.
(2) Deduce that almost surely, the Lebesgue measure of times t such that $D^*B(t) \neq +\infty$ or $D_*B(t) \neq -\infty$ is 0.
(3) Show that with probability one local maxima of B are dense. Deduce that there exists a density of exceptional random times where $D^*B(t) \leq 0$.
(4) Show that there almost surely exists an uncountable density of times t where $D^*B(t) = 0$. (Hint : consider $\tau(x) = \inf\{t \geq 0, B_t = x\}$. Show that this is almost surely a strictly increasing function whose discontinuity points are dense and deduce that $V_n = \{x \geq 0, \exists h \in (0, 1/n), \tau(x-h) < \tau(x) - nh\}$ is open and dense. What can be said about $\bigcap_{n\geq 1} V_n$?)

Exercise 4 — *The modulus of continuity.*
We consider
$$L = \limsup_{h \downarrow 0} \frac{\omega_{[0,1]}(B, h)}{\sqrt{h \log(1/h)}},$$
where $\omega_K(f, h) = \sup_{x,y \in K, |x-y|<h} |f(x) - f(y)|$ is the modulus of continuity. We will show that $L \geq \sqrt{2}$ almost surely, then that there exists $C < \infty$ fixed such that $L < C$ almost surely. This can be strengthened to show that $L = \sqrt{2}$ almost surely.

(1) Show that if X is standard Gaussian and $x > 0$, then
$$\frac{1}{\sqrt{2\pi}(x + 1/x)} e^{-x^2/2} \leq \mathbb{P}(X \geq x) \leq \frac{1}{\sqrt{2\pi}x} e^{-x^2/2}.$$
(2) Show that $L \geq \sqrt{2}$ almost surely.
(3) Recall that the Lévy construction gives B on $[0, 1]$ as the sum of a random series $B = \sum_{n=0}^{\infty} F_n$. Strengthening the estimate shown in class, show that almost surely, for n large enough, $\|F_n\|_{[0,1]} < 100\sqrt{n}2^{-n/2}$. Deduce that $\|F_n'\|_{[0,1]} < 500\sqrt{n}2^{n/2}$.
(4) Deduce that there is a fixed constant c such that almost surely, for h small enough and every $t, t + h \in [0, 1]$, $|B(t + h) - B(t)| < c\sqrt{h \log(1/h)}$ (Hint: treat summands differently according to whether $n > \log_2(1/h)$ or not). Conclude.

Exercise 5 — *Brownian bridges.*
For $x, y \in \mathbb{R}$, we define the Brownian bridge of length a as follows: let B be a standard Brownian motion and set $\beta_t^a = B_t - \frac{t}{a} B_a$ for $t \in [0, a]$.

(1) Show that β^a is a continuous centered Gaussian process, compute its covariance kernel. Show that β^a is independent of B_a.
(2) For $a < 1$ and $t \in [0, a]$, show that $\beta_t^a = \beta_t^1 - \frac{t}{a} \beta_a^1$.
(3) Let $a < 1$. Compute the density of the distribution of β_a^1 with regards to the one of B_a.
(4) Deduce that the distribution of $\beta_{|[0,a]}^1$ is absolutely continuous with regard to that of $B_{|[0,a]}$ where $a < 1$.