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Solutions for $\mathbb{E}$xercise sheet 3 : Lévy's construction, regularity

Solution 1 — *Simple Markov property.* (1) We know that B belongs to the measurable space $(\mathbb{R}^{\mathbb{R}_+}, \mathcal{B}(R)^{\otimes \mathbb{R}_+})$. But because of continuity of paths, B also belongs to $\mathcal{C}(\mathbb{R}_+)$. This is a topological space (for uniform convergence over every compact), which provides the Borel σ-algebra $\mathcal{B}(\mathcal{C}(\mathbb{R}_+))$. The question is now: is B measurable with regard to this apparently stronger σ-algebra? This is the case, since actually

$$(\mathcal{B}(\mathbb{R})^{\otimes \mathbb{R}_+})_{|\mathcal{C}(\mathbb{R}_+)} = \mathcal{B}(\mathcal{C}(\mathbb{R}_+)).$$

We proceed to the proof of this statement. First of all, $(\mathcal{B}(\mathbb{R})^{\otimes \mathbb{R}_+})_{|\mathcal{C}(\mathbb{R}_+)} \supset \mathcal{B}(\mathcal{C}(\mathbb{R}_+))$ because evaluations are continuous w.r.t. the topology of $\mathcal{C}([0,1])$.

Conversely, to show $(\mathcal{B}(\mathbb{R})^{\otimes \mathbb{R}_+})_{|\mathcal{C}(\mathbb{R}_+)} \subset \mathcal{B}(\mathcal{C}(\mathbb{R}_+)$ it suffices to show that every semi-norm $\|\cdot\|_K$ is measurable w.r.t. $(\mathcal{B}(\mathbb{R})^{\otimes \mathbb{R}_+})_{|\mathcal{C}(\mathbb{R}_+)}$ because then an open ball can be rewritten as $\|f - \cdot\|^{-1}([0, l))$, hence is measurable. But we have

$$\|f\|_K = \sup_{t \in K, t \in \mathbb{Q}} |f(t) - f(t)|.$$

which immediately gives measurability of $\|\cdot\|_K$.

This shows that B is measurable with regard to $B(\mathcal{C}(\mathbb{R}_+))$, and that $B(\mathcal{C}(\mathbb{R}_+))$ is generated by cylinder sets, just like $B(R)^{\otimes \mathbb{R}_+}$. In particular, the distribution of an element of $\mathcal{C}(\mathbb{R}_+)$ is characterized by its finite-dimensional marginals.

This will be helpful in the future, because it will provide for free measurability of lots of functional of B: maximum over an interval, hitting times, ...

(2) The fact that $\widetilde{B} = (B_{t+s} - B_t)_{s \geq 0}$ is a Brownian motion is immediate by the definition. Let's show the independence property. We will show the even stronger statement:

$$\widetilde{B} \perp\!\!\!\perp (B_s)_{0 \leq s \leq t}.$$

By the lecture, we only have to show that finite-dimensional marginals are independent. Let us consider $0 \leq s_1 \leq \ldots \leq s_k \leq t$ and $0 \leq s'_1, \ldots \leq s'_l$.

$$(B_{s_1}, \ldots B_{s_k} \widetilde{B}_{s'_1}, \ldots, \widetilde{B}_{s'_1}) = (B_{s_1}, \ldots B_{s_k}, B_{t+s'_1} - B_t, \ldots, B_{t+s'_1} - B_t)$$

is a Gaussian vector because

- B is a Gaussian process
- affine transforms preserve Gaussianity.

So (first exercise session) it suffices to find that crossed covariances are zero to prove independence. We take $0 \leq s \leq t$, $s" \geq 0$ and compute

$$Cov(\widetilde{B}_{s'}, B_s) = \mathrm{Cov}(B_{s'+t} - B_t, B_s) = \mathrm{Cov}(B_{s'+t}, B_s) - \mathrm{Cov}(B_t, B_s) = s - s = 0.$$

Solution 2 — *Local regularity and long-term behavior.* (1) Immediate since almost surely $X_t = o(1)$ as $t \to 0$

(2) We know from the lecture that almost surely,
- X is not locally $1/2+\epsilon$-Hölder at 0, more precisely
$$\limsup_{h\to 0} \frac{X_h}{h^{1/2+\epsilon}} = \infty$$
,
- X is $1/2-\epsilon$-Hölder on $[0,1]$, in particular there exists C random such that
$$\limsup_{h\to 0} \left| \frac{X_h}{h^{1/2-\epsilon/2}} \right| < C,$$
which implies that
$$\lim_{h\to 0} \left| \frac{X_h}{h^{1/2-\epsilon}} \right| = 0$$

Translating on B_t, this means that
$$\limsup_{t\to\infty} \frac{B_t}{t^{1/2-\epsilon}} = \infty, \quad \liminf_{t\to\infty} \frac{B_t}{t^{1/2-\epsilon}} = -\infty \text{ (because } B \overset{d}{=} -B)$$
and
$$\limsup_{t\to\infty} \left| \frac{B_t}{t^{1/2+\epsilon}} \right| = 0.$$

Exercise 4 below will allow us to improve the upper bound to
$$\limsup_{t\to\infty} \left| \frac{B_t}{\sqrt{t\log t}} \right| \le C.$$

Remark: It turns out that this is not sharp. The law of iterated logarithm tells us that actually $\limsup_{t\to\infty} \left| \frac{B_t}{\sqrt{2t\log\log t}} \right| = 1$.

(3) We will now proceed to improve the lower bound by showing that Brownian motion is not 1/2-Hölder at 0.

(a) By Fatou's lemma,
$$\begin{aligned}\mathbb{P}((\limsup_{n\to\infty} B_{2^{-n}}/\sqrt{2^{-n}}) < c) &\le \mathbb{P}(\liminf_{n\to\infty}\{B_{2^{-n}} < c\sqrt{2^{-n}}\}) \\ &\le \liminf_{n\to\infty} \mathbb{P}(B_{2^{-n}} < c\sqrt{2^{-n}}) = \liminf_{n\to\infty} \mathbb{P}(B_1 \le c) < 1.\end{aligned}$$

(b) Lévy's construction tells us that $B_{2^{-n}} = 2^{-n}N_0 + \sum_{k=0}^{n-1} 2^{-n+k/2} N_{0,k}$. Hence
$$\frac{B_{2^{-n}}}{\sqrt{2^{-n}}} = 2^{-n/2}N_0 + \sum_{k=0}^{n-1} 2^{(k-n)/2} N_{0,k}$$
Each fixed term of the sum goes to 0 separately. So the lim inf does not change when the first few terms are removed. We deduce that $\liminf_{n\to\infty} B_{2^{-n}}/\sqrt{2^{-n}}$ is measurable w.r.t. the σ-algebra $\sigma(N_{0,k}, k \ge K)$ for all fixed K, thus to the tail σ-algebra.

(c) $\{\limsup_{n\to\infty} B_{2^{-n}}/\sqrt{2^{-n}} < c\}$ is a tail event for a sequence of independent random variables, hence by Kolmogorov's 0-1 law it has probability 0 or 1, and it is not 1 because of question 3a. Hence with probability one $\limsup_{t\to 0} B_t/\sqrt{t} \geq \limsup_{n\to\infty} B_{2^{-n}}/\sqrt{2^{-n}} = \infty$. So B is not locally Hölder at 0 and by time-reversal, $\limsup_{t\to\infty} B_t/\sqrt{t} = +\infty$. Since $B \stackrel{d}{=} -B$, it comes that $\liminf B_t/\sqrt{t} = -\infty$ too.

Solution 3 — *A bit more on differentiability.*
Set $D^*B(t) = \limsup_{h\downarrow 0} \frac{1}{t}(B_{t+h} - B_t)$ and $D_*B(t) = \liminf_{h\downarrow 0} \frac{1}{t}(B_{t+h} - B_t)$.

(1) We showed earlier that almost surely, $\limsup B_t = +\infty$ and $\liminf B_t = -\infty$ almost surely. Hence the claim by time inversion and simple Markov property.

(2) $\mathbb{E}[\mathrm{Leb}\{t \geq 0, D^*B(t) \neq +\infty \text{ or } D_*B(t) \neq -\infty\}] = \int_{\mathbb{R}} dt\, \mathbb{P}(D^*B(t) \neq +\infty \text{ or } D_*B(t) \neq -\infty) = \int_R 0 = 0$, where we used Fubini and Markov.

(3) Let us show that for $p < q \in \mathbb{Q}_+$ there is a local minium for B in (p,q) almost surely. By simple Markov property, there exist almost surely arbitrarily small t such that $B_{p+t} - B_p$ is strictly negative. Taking $t < q - p$, it means that we cant find $a \in (p,q)$ such that $B_a > B_p$. By time reversal, we can also show that there is $b \in (p,q)$ such that $B_b > B_q$.

Hence the minimum of B on (p,q) is reached inside (p,q) and this provides a local minimum for B.

By countable union this is the case for every (p,q), proving that local minima are dense. And clearly at a local minimum, we have $D^*B \leq 0$.

(4) We consider $\tau(x) = \inf\{t \geq 0, B_t = x\}$. This is by definition strictly increasing function, and if it were continuous on some open interval, then B would be monotonous on some open interval, which it is almost surely not. Now if we consider $V_n = \{x \geq 0, \exists h \in (0, 1/n), \tau(x-h) < \tau(x) - nh\}$, it is open because τ is càglàd strictly increasing. It is dense because otherwise we found an open interval of x where $\forall h \in (0, 1/n)$, $\tau(x) - nh \leq \tau(x-h) \leq \tau(x)$, implying continuity on some open interval. Then by the Baire category theorem, $\bigcap_{n\geq 1} V_n$ is uncountable and dense. Let x be in this set, and $t = \tau(x)$. Then there exists a sequence $t_n \uparrow t$, $B^*(t_n) > t - 1/n$, $t_n < t - nB^*(t_n)$. Hence the lower left derivative of B at t is 0. The upper left derivative is 0 too by definition. We get the claim by time reversal.

Solution 4 — *The precise constant (Lévy, 1937).* (1) The upper bound comes from the inequality $\int_x^\infty e^{-t^2/2}dt \le \int_x^\infty \frac{t}{x}e^{-t^2/2}dt$. The lower bound can be obtained by differentiating the difference.

(2) Let $c < \sqrt{2}$ and compute $\mathbb{P}(E_{k,n}) := \mathbb{P}(B_{(k+1)2^{-n}} - B_{k2^{-n}} \ge c\sqrt{2^{-n}\log(2^n)}) = \mathbb{P}(B_1 \ge c\sqrt{n\log 2}) \ge \frac{1}{1000c\sqrt{n}}2^{-c^2n/2}$. Then

$$\mathbb{P}(\forall 0 \le k \le 2^{-n}, B_{(k+1)2^{-n}} - B_{k2^{-n}} < c\sqrt{2^{-n}\log(2^n)}) = \mathbb{P}(\bigcap_k E_{k,n}^\complement)$$
$$= \prod_k (1 - \mathbb{P}(E_{k,n})) \le (1 - \frac{1}{1000c\sqrt{n}}2^{-c^2n/2})^{2^n} \le \exp(-2^n\frac{1}{1000c\sqrt{n}}2^{-c^2n/2})$$
$$= \exp(-\frac{1}{1000c\sqrt{n}}2^{(1-c^2/2)n}) = \text{summable in} n.$$

So by Borel-Cantelli, we get that infinitely often in n, there is an increment of length 2^{-n} that exceeds $c\sqrt{2^{-n}\log(2^n)}$. This implies the claim.

(3) We have $\|F_n\|_{[k2^{-n},(k+1)2^{-n}]} \overset{d}{=} 2^{-(n+1)/2}|Z|$ where Z is standard Gaussian. Then $\mathbb{P}(\|F_n\|_{[k2^{-n},(k+1)2^{-n}]} \ge 100\sqrt{n}2^{-n/2}) \le \mathbb{P}(|Z| > 10\sqrt{n}) \le e^{-10n/2}$.

By union bound, $\mathbb{P}(\|F_n\|_{[0,1]} \ge 100\sqrt{n}2^{-n/2}) \le 2^{-n}e^{-10n/2}$ which is summable. So there is N random, such that for $n \ge N$, $\|F_n\|_{[0,1]} \le 100\sqrt{n}2^{-n/2}$. From the shape of F_n, the statement of F_n' follows deterministically.

(4) Finally, we have

$$|B_{t+h} - B_t| \le h\sum_{n=0}^{N}\|F_n'\| + h\sum_{n=N}^{\log_2(1/h)} 500\sqrt{n}2^{n/2} + \sum_{n=\log_2(1/h)}^{\infty} 100\sqrt{n}2^{-n/2}$$
$$\le \sqrt{h\log(1/h)} + 2000h\sqrt{\log(1/h)}\sqrt{1/h} + 2000\sqrt{h\log(1/h)}\sum_{n=\log(1/h)}^{\infty} 1.1^{-n}$$

as soon as h is small enough.

Solution 5 — *Brownian bridges.*
(to be completed)

(1) β^a is a Gaussian process as a linear transform of a Gaussian process. We compute the covariance.

$$\mathrm{Cov}(\beta_t^a, \beta_s^a) = t \wedge s - ts/a - st/a + st/a = t \wedge s - st/a$$

For independence, since everybody is jointly Gaussian, we compute the crossed covariance

$$\mathrm{Cov}(\beta_t^a, B_a) = t - \tfrac{t}{a}a = 0.$$

(2) $\beta_t^1 - \frac{t}{a}\beta_a^1 = B_t - tB_1 - \frac{t}{a}(B_a - aB_1) = B_t - \frac{t}{a}B_a = \beta_t^a$

(3) We divide the densities and obtain
$$\frac{\mathbb{P}(\beta^1_a \in dx)}{\mathbb{P}(B_a \in dx)} = \frac{1}{\sqrt{1-a}} e^{-x^2(1/(1-a)-1)/2a} = \frac{1}{\sqrt{1-a}} e^{-x^2/2(1-a)}$$

(4) We have $\beta^1_t = \frac{t}{a}\beta^1_a + \beta^a_t$. We also remark that these two components are independent: $\mathrm{Cov}(\frac{t}{a}\beta^1_a, \beta^a_t) = \mathrm{Cov}(\frac{t}{a}B_a - aB_1, B_t - \frac{t}{a}B_a) = t^2/a - at - t^2/a + ta = 0$. So
$$\beta_1|_{[0,a]} \stackrel{d}{=} \frac{1}{a}\beta^1_a \mathrm{Id} \stackrel{\perp\!\!\!\perp}{+} \beta^a.$$
At the same time,
$$B_1|_{[0,a]} \stackrel{d}{=} \frac{1}{a}B_a \mathrm{Id} \stackrel{\perp\!\!\!\perp}{+} \beta^a.$$
So
$$\begin{aligned}
\mathbb{E}[h(\beta_1|_{[0,a]})] &= \int_{C([0,a])} \mathbb{P}(\beta^a \in d\phi) \int_{\mathbb{R}} \mathbb{P}(\beta^1_a \in dx) h(\tfrac{x}{a}\mathrm{Id} + \phi) \\
&= \int_{C([0,a])} \mathbb{P}(\beta^a \in d\phi) \int_{\mathbb{R}} \mathbb{P}(B_a \in dx) h(\tfrac{x}{a}\mathrm{Id} + \phi) \frac{1}{\sqrt{1-a}} e^{-x^2/2(1-a)} \\
&= \int_{C([0,a])} \mathbb{P}(\beta^a \in d\phi) \int_{\mathbb{R}} \mathbb{P}(B_a \in dx) h(\tfrac{x}{a}\mathrm{Id} + \phi) \frac{1}{\sqrt{1-a}} e^{-(\frac{x}{a}\mathrm{Id}+\phi)(a)^2/2(1-a)} \\
&= \mathbb{E}[h(B_1|_{[0,a]}) \frac{1}{\sqrt{1-a}} e^{-B_a^2/2(1-a)}]
\end{aligned}$$
Hence absolute continuity.