Master 1 — Spring 2020
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$\mathbb{E}$xercise sheet 4-5: Markov and martingale property v2

1. Markov processes

Exercise 1 — *Brownian motion on the circle.*
Define a Brownian motion on the circle $\mathbb{S}^1$ by setting $X_t = e^{iB_t}$ for $t \geq 0$. Show that the last point visited by X in $\mathbb{S}^1$ is uniformly distributed.
More precisely, we define $T = \min\{t \geq 0 : \sup_{s \leq t} B_s - \inf_{s \leq t} B_s = 2\pi\}$ and want to compute the distribution of X_T.

Exercise 2 — *The set of zeros of B is perfect.*
Let B be a Brownian motion, and $Z = \{t \geq 0 : \ B_t = 0\}$. Show that almost surely, Z is a closed set without isolated points.

Exercise 3 — *Arcsine law.*
Let $L = \max\{t \in [0,1] : B_t = 0\}$ and $A = \arg\max_{[0,1]} B$.

(1) Compute the distribution of L.
(2) Show that A is well-defined and that $A \overset{d}{=} L$.
(3) Those are not stopping times. Why ?

Exercise 4 — *Markov processes derived from Brownian motion.*
Let $(B^{(1)}, B^{(2)})$ be a two-dimensional Brownian motion started at zero. Denote T_a the hitting time of a by $B^{(1)}$. For every $a \geq 0$, we let

$$C_a := B^{(2)}_{T_a^{(1)}}.$$

In the lecture, using martingale theory, you have computed the Laplace transform and the distribution of T_1 (Lévy distribution) and C_1 (Cauchy distribution). We will now consider the process $(T_a)_a$ and $(C_a)_a$. They are respectively called the 1/2-*stable subordinator*, and the *Cauchy process*.

(1) Show that they have stationary independent increments.
(2) Show that they are not continuous.
(3) *(Ornstein-Uhlenbeck process)* For $t \in \mathbb{R}$, set $X_t = e^{-t} B_{e^{2t}}$, where B is a Brownian motion. Show that X is a continuous Gaussian process, compute its covariance function. For any given t, what is the distribution of X_t ?

2. Martingales

Exercise 5 — *Don't skip an hypothesis.*
Find two stopping times S and T with $S \leq T < \infty$ a.s. and $\mathbb{E}[S] < \infty$, such that $\mathbb{E}[B_S^2] > \mathbb{E}[B_T^2]$.

Exercise 6 — *Brownian gambler's ruin.*
For any $c \in \mathbb{R}$, we let $T_c := \inf\{t \geq 0 \ : \ B_t = c\}$ be the hitting time of c by $(B_t)_{t\geq 0}$. Let $a, b \in \mathbb{R}$ such that $a < 0 < b$, we let $T_{a,b} := T_a \wedge T_b$ be the hitting time of $\{a, b\}$ by $(B_t)_{t\geq 0}$. You know that $\mathbb{P}(T_{a,b} = T_a) = \frac{b}{b-a}$.

(1) Compute $\mathbb{E}[T_{a,b}]$.
(2) Compute the distribution of $\sup_{0\leq t\leq T_{a,b}} B_t$ conditioned on $T_{a,b} = T_a$.
(3) (*) Compute the Laplace transform of $T_{a,b}$.

Exercise 7 — *Girsanov theorem and hitting times with drift.*
Let B be a brownian motion, and for $\lambda \in \mathbb{R}$, denote $M_t^\theta = e^{\theta B_t - \theta^2 t/2}$. You have shown that M^λ is a $(\mathcal{F}_t)_t$-martingale, and used it to show that $\mathbb{E}[e^{-\lambda T_b}] = \mathbb{E}[e^{-|b|\sqrt{2\lambda}}]$.

(1) Consider the following measure defined by absolute continuity.
$$\mathbb{P}_{\theta,T}(A) := \mathbb{E}[\mathbb{1}_A \, M_T^\theta].$$
Show that it is a probability measure, and that the distribution of $B|_{[0,T]}$ under $\mathbb{P}_{\theta,T}$ is the same as the one of the drifted Brownian motion $(B_t + \theta t)_{0\leq t\leq T}$ under $\mathbb{P}$.
(2) Could you do this with $T = \infty$?
(3) Use this to show that if T_b^θ is the hitting time of b by $(B_t + \theta t)_t$, then
$$\mathbb{E}[e^{-\lambda T_b^\theta} \mathbb{1}_{T_b^\theta < \infty}] = e^{-|b|\sqrt{\theta^2 + 2\lambda} + \theta b}.$$
(4) What is $\mathbb{P}(T_b^\lambda < \infty)$? Consider the law of T_b^λ conditioned on $T_b^\lambda < \infty$. Comment on the duality phenomenon that appears.

Exercise 8 — *The binary splitting martingale.*
Let X be centered with finite variance and $(X_n)_n$. We want to show that there is a filtration $\mathcal{G}$ and a $\mathcal{G}$-martingale $(X_n)_n$ such that

(1) $X_n \to X$ almost surely and in L^2.
(2) Conditional on $\mathcal{G}_n$, X_{n+1} is supported on a finite set of size at most two.

Let $\mathcal{G}_0$ the trivial σ-field, and for $n \geq 0$, set $X_n = \mathbb{E}[X \mid \mathcal{G}_n]$, $\xi_n = \text{sgn}(X - X_n)$ and $\mathcal{G}_{n+1} = \sigma(\xi_0, \ldots, \xi_n)$. By definition, $(X_n)_n$ is a $(\mathcal{G}_n)_n$ martingale closed by $X_\infty = \mathbb{E}[X \mid \mathcal{G}_\infty]$. So $X_n \to X_\infty$ almost surely and in L^1.

(1) Draw a picture to understand what's going on.
(2) Express $X_{n+1} - X_n$ so that its positive and negative part are explicit. Use this to compute $|X_{n+1} - X_n|$.
(3) Deduce that $|X_n - X|$ goes to 0 in L^1 and that $X_\infty = X$.
(4) It is clear that $X_n \in L^2$. Show that $\mathbb{E}[X_n(X - X_n)] = 0$ and deduce that $(X_n)_n$ is bounded in L^2. Conclude.

Exercise 9 — *Martingales derived from B.*
Show that $(B_t^2 - t)_{t\geq 0}$ and $(B_t^3 - 3tB_t)_{t\geq 0}$ are martingales. Guess the other ones.