Solutions for $\mathbb{E}$xercise sheet 3: stopping times and Markov property

Solution 1 — *Brownian motion on the circle.*

Solution 2 — *The set of zeros of B is perfect.*
Almost surely 0 is an accumulation point of Z (lecture). By countable union, and strong Markov, every first 0 after a rational is an accumulation point of Z. If Z had an isolated point, it would be a first 0 after a rational. Hence it couldn't be isolated in Z.

Solution 3 — . (1) Denote by $I = \min_{0\leq s\leq 1-t}(B_{t+s} - B_t)$ and $S = \max_{0\leq s\leq 1-t}(B_{t+s} - B_t)$. Both I and S are independent of $\mathcal{F}_t$.

$$\begin{aligned}
\mathbb{P}(L < t) &= \mathbb{P}(B_t > 0 \cap I < -B_t \cup B_t < 0 \cap S > -B_t)\\
&= 2\,\mathbb{P}(B_t < 0 \cap S > |B_t|)\\
&= \mathbb{P}(B_t \neq 0, S > |B_t|) = \mathbb{P}(S > |B_t|) = \mathbb{P}(|\widetilde{B}_{1-t}| > |B_t|)\\
&= \mathbb{P}(|\frac{Z}{\widetilde{Z}}| < \sqrt{(1-t)/t})
\end{aligned}$$

where $Z, \widetilde{Z}$ are two independent standard Gaussians. Then

$$... = \mathbb{P}(|\arg(\widetilde{Z} + iZ)| < \arcsin(\sqrt{t})) = 2\arcsin(\sqrt{t}).$$

(2) For now let $\widetilde{A} = \inf\{t \in [0,1], B_t = \min_{[0,1]} B\}$. Then $\mathbb{P}(\widetilde{A} > t) = \mathbb{P}(S > \max_{[0,t]} B - B_t)$. Once again S is independent of $\max_{[0,t]} B - B_t \in \mathcal{F}_t$, whose distribution is known to be equal to that of $|B_t|$. We end up with $\mathbb{P}(\widetilde{A} > t) = \mathbb{P}(L < t)$. By symmetry of the arcsine distribution, we have shown that $\widetilde{A} \overset{d}{=} L$.

To show that A is well-defined, consider $\widetilde{\widetilde{A}} = \sup\{t \in [0,1], B_t = \min_{[0,1]} B\}$. By time reversal and symmetry of the arcsine distribution, $\widetilde{\widetilde{A}} \overset{d}{=} \widetilde{A}$. At the same time, $\widetilde{A} \leq \widetilde{\widetilde{A}}$ almost surely. This implies that $\widetilde{A} = \widetilde{\widetilde{A}}$ almost surely.

Solution 4 — *Markov processes derived from Brownian motion.* (1) Let $B = (B^{(1)}, B^{(2)})$. We have that $(C_{a+\cdot} - C_a)$ is constructed from $B_{T^{(1)}_{a+}+\cdot} - B_{T^{(1)}_{a+}}$ the same way C is constructed from B. Hence by the strong Markov property of B, $(C_{a+\cdot} - C_a) \overset{d}{=} C$, and $(C_{a+\cdot} - C_a) \perp\!\!\!\perp \mathcal{F}_{T^{(1)}_{a+}} \supset \sigma(C_u, u \leq a)$.

(2) C is càdlàg because $T_{\cdot+}$ is. By independence of $B^{(1)}$ and $B^{(2)}$ it jumps almost surely when $T_{\cdot+}$ jumps.
(3) Firstly, $\mathrm{Cov}(X_t, X_s) = e^{-|t-s|}$. So at each time t, X_t is a standard Gaussian.

Solution 5 — *All hypotheses matter.*
Take $S = 3$ and T to be the first zero after 3. Of course the problem is that $\mathbb{E}[T] = \infty$.

Solution 6 — *Brownian gambler's ruin.*
Let $a < 0 < b$ and T be the hitting time of $\{a, b\}$.

(1) We may show that T is integrable to apply Wald's second lemma. Here's a way to do it by comparison with a geometric variable. Let $x \leq |a| \wedge |b|$.

$$\begin{aligned}\mathbb{P}(T \geq n) &\leq \mathbb{P}(\forall k \leq n-1, \quad |B_{k+1} - B_k| < 2x) \\ &= \prod_{k=0}^{n-1} \mathbb{P}(|B_1^{(k)}| < 2x) = \rho^n\end{aligned}$$

Where $\rho < 1$. Hence T is integrable, and we can apply Wald's second lemma. We get

$$E[T] = \mathbb{E}[B_T^2] = \frac{-a}{b-a} b^2 + a^2 \frac{b}{b-a} = -ab$$

(2) Let $M = \sup_{0 \leq t \leq T_{a,b}} B_t$. Let $c \in [0, b]$. We denote $\widetilde{B}_t = B_{t+T_c} - B_{T_c}$.

$$\begin{aligned}\mathbb{P}(M \geq c \mid T_a \leq T_b) &= \frac{\mathbb{P}(T_c \leq T_a \leq T_b)}{\mathbb{P}(T_a \leq T_b)} \\ &= \frac{\mathbb{P}(T_c \leq T_a, \widetilde{T}_{a-c} \leq \widetilde{T}_{b-c})}{\mathbb{P}(T_a \leq T_b)} \\ &= \frac{\mathbb{P}(T_c \leq T_a)\, \mathbb{P}(\widetilde{T}_{a-c} \leq \widetilde{T}_{b-c})}{\mathbb{P}(T_a \leq T_b)} \\ &= \frac{\frac{-a}{c-a} \frac{b-c}{b-a}}{\frac{b}{b-a}} = \frac{-a(b-c)}{b(c-a)}.\end{aligned}$$

Solution 7 — *Girsanov theorem and hitting times with drift.*
Let B be a brownian motion, and for $\lambda \in \mathbb{R}$, denote $M_t^\theta = e^{\theta B_t - \theta^2 t/2}$. You have shown that M^λ is a $(\mathcal{F}_t)_t$-martingale, and used it to show that $\mathbb{E}[e^{-\lambda T_b}] = \mathbb{E}[e^{-|b|\sqrt{2\lambda}}]$.

(1) We check that

$$\mathbb{P}_{\theta,T}(\Omega) := \mathbb{E}[M_T^\theta] = \mathbb{E}[e^{\theta B_T}] e^{-\theta^2 T/2} = 1.$$

We use characteristic functions of fdms to characterize distribution of a process. Let $0 \le t_1 \le \ldots \le t_k \le T$ and $u_1, \ldots u_k \in R$

$$\begin{aligned}
\mathbb{E}_{\theta,T}[e^{i(u_1 B_{t_1}+\ldots+u_k B_{t_k})}] &= \mathbb{E}[e^{i(u_1 B_{t_1}+\ldots+u_k B_{t_k})} e^{\theta B_T - \theta^2 T/2}] \\
&= \mathbb{E}[e^{i(u_1 B_{t_1}+\ldots+u_k B_{t_k} - i\theta B_T)}] e^{\theta^2 T/2} \\
&= \mathbb{E}[e^{-\operatorname{Var}(u_1 B_{t_1}+\ldots+u_k B_{t_k} - i\theta B_T)/2}] e^{\theta^2 T/2} \\
&= \mathbb{E}[e^{-\operatorname{Var}(u_1 B_{t_1}+\ldots+u_k B_{t_k})/2 + \theta^2 T/2 + i\theta(u_1 t_1+\ldots+u_k t_k)}] e^{\theta^2 T/2} \\
&= \mathbb{E}[e^{i(u_1 B_{t_1}+\ldots+u_k B_{t_k})}] e^{i\theta(u_1 t_1+\ldots+u_k t_k)} \\
&= \mathbb{E}[e^{i(u_1 (B_{t_1}+\theta t_1)+\ldots+u_k (B_{t_k}+\theta t_k))}]
\end{aligned}$$

(2) No, Brownian motion with and without drift are not absolutely continuous to each other over $\mathbb{R}_+$.

(3) Let T_b^θ be the hitting time of b by $(B_t + \theta t)_t$. Then using question 1 and optional stopping,

$$\begin{aligned}
\mathbb{E}[e^{-\lambda T_b^\theta} \mathbb{1}_{T_b^\theta < U}] &= \mathbb{E}[e^{-\lambda T_b} \mathbb{1}_{T_b < U} M_U^\theta] \\
&= \mathbb{E}[e^{-\lambda T_b} \mathbb{1}_{T_b < U} M_{T_b}^\theta] \\
&= \mathbb{E}[e^{-\lambda T_b} \mathbb{1}_{T_b < U} e^{\theta b - \theta^2 T_b/2}] \\
&= e^{\theta b} \mathbb{E}[e^{-(\lambda - \theta^2/2) T_b} \mathbb{1}_{T_b < U}]
\end{aligned}$$

Using dominated convergence, we get

$$\mathbb{E}[e^{-\lambda T_b^\theta} \mathbb{1}_{T_b^\theta < \infty}] = e^{\theta b} \mathbb{E}[e^{-(\lambda - \theta^2/2) T_b}] = e^{-|b|\sqrt{\theta^2 + 2\lambda} + b\theta}.$$

(4) Then taking $\lambda = 0$, $\mathbb{P}(T_b^\lambda < \infty) = e^{-|b\theta| + b\theta} = e^{2b\theta \wedge 0}$. We observe, that

$$\mathbb{E}[e^{-\lambda T_b^\theta} \mid T_b^\theta < \infty] = e^{-|b|\sqrt{\theta^2 + 2\lambda} - |b\theta|}.$$

It is independent on the sign of b. So a Brownian motion with negative drift, conditioned on hitting a positive level, will behave as a Brownian motion with the reverse (positive) drift.

Solution 8 — *The binary splitting martingale.* (1) We write

$$\begin{aligned}
X_{n+1} - X_n &= \mathbb{E}[X - X_n \mid \mathcal{G}_n] \\
&= \mathbb{E}[(X - X_n) \mathbb{1}_{X > X_n} \mid \mathcal{G}_n] \mathbb{1}_{X > X_n} + \mathbb{E}[(X - X_n) \mathbb{1}_{X < X_n} \mid \mathcal{G}_n] \mathbb{1}_{X < X_n} .
\end{aligned}$$

where we used the fact that the sign of $(X - X_n)$ is $\mathcal{G}_n$-measurable. The first term is almost surely positive, the second one is almost surely negative, and almost surely only one of them is nonzero. Hence they almost surely they form a decomposition of $X_{n+1} - X_n$ into a positive and negative part. Then

$$\begin{aligned}
|X_{n+1} - X_n| &= \mathbb{E}[(X - X_n) \mathbb{1}_{X > X_n} \mid \mathcal{G}_n] \mathbb{1}_{X > X_n} - \mathbb{E}[(X - X_n) \mathbb{1}_{X < X_n} \mid \mathcal{G}_n] \mathbb{1}_{X < X_n} \\
&= \mathbb{E}[|X - X_n| \mid \mathcal{G}_n].
\end{aligned}$$

(2) We deduce $\mathbb{E}[|X_n - X|] = \mathbb{E}[|X_{n+1} - X_n|]$, and this last expression goes to 0 as $(X_n)_n$ is L^1-convergent. Thus $|X_n - X|$ goes to 0 in L^1 and by uniqueness (up to a.s. equality) of the L^1 limit we get that $X_\infty = X$ a.s. Hence X_n converges a.s. and L^1 to X.

Solution 9 — *Martingales derived from B.*
Those martingales are the derivative w.r.t λ of the exponential martingale.