$\mathbb{E}$xercise sheet 6: Harmonic functions and Brownian motion v2

version 2: typos corrected.

Exercise 1 — *Recurrence and transience.*
Show that when $d = 2$, $\mathbb{P}_x(T_{\{0\}} < \infty) = 0$ for every $x \neq 0$ while $\mathbb{P}_x(T_{B(0,\varepsilon)} < \infty) = 1$ for every x. Deduce that B visits every open set at arbitrarily large times.
Show that when $d \geq 3$, $\mathbb{P}_x(T_{B(0,\varepsilon)} < \infty) = (\varepsilon/|x|)^{d-2}$ for every $x : |x| > \varepsilon$. Deduce that $|B_t| \to \infty$ almost surely.

Exercise 2 — *Singularity removal.*
Let $d \geq 2$ and $x \in U \subset \mathbb{R}^d$ open. Suppose $h : U \setminus \{x\} \to \mathbb{R}$ is harmonic and bounded around x. Show that u can be extended to a harmonic function on the whole of U.
Better yet, show the same outcome with the relaxed condition that $u(x + \cdot)$ is negligible near x compared to the fundamental solution ($\log(\|\cdot\|)$ if $d = 2$ and $\|\cdot\|^{2-d}$ otherwise).

Exercise 3 — *Liouville's theorem.*
Let $h : \mathbb{R}^d \to \mathbb{R}$ bounded and harmonic, and $x, y \in \mathbb{R}^d$. Show that for any hyperplane H with hitting time T, $h(x) = \mathbb{E}_x[h(B_T)]$. Deduce Liouville's theorem.

Exercise 4 — *Harmonic functions and martingales.*
Let $D \subset \mathbb{R}^d$ be a domain and T its exit time. **Assume** $T < \infty$ **a.s.** Let $h : \overline{D} \to \mathbb{R}$ be continuous, bounded and harmonic inside D. Fix $x \in D$.

(1) Assume that D is bounded and verifies the Poincaré cone condition. Show that under $\mathbb{P}_x$, the process $(h(B_{t\wedge T}))_t$ is a closed martingale.
(2) In the general case, show that under $\mathbb{P}_x$, the process $(h(B_{t\wedge T}))_t$ is a martingale. (Hint: approximate D by domains that verify the cone condition.)
(3) Deduce that for an arbitrary domain, *when a bounded solution to the Dirichlet problem exists*, the Brownian expectation finds it.
(4) Look at $\log(\|B_t\|)$ if $d = 2$ and $\|B_t\|^{2-d}$ if $d \geq 3$. Are those martingales ?

Exercise 5 — *Counterexample.*
Let $D = B(0, 1) \setminus \{0\} \subset \mathbb{R}^2$ and consider the Laplace equation $\Delta u = 0$ with Dirichlet boundary conditions $u(0) = 0$ and $u(x) = 1$ for $x \in \partial B(0, 1)$. Show that the Brownian expectation does not define a solution. Show that there can't exist a solution (you may use exercise 1 or the fact that if $u(x) = g(|x|)$, then $\Delta u(x) = g''(|x|) + \frac{1}{x} g'(|x|)$.)