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Solutions for $\mathbb{E}$xercise sheet 6: Harmonic functions and Brownian motion

Solution 1 — *Recurrence and transience.*
Let us denote $T_r = T_{\partial B(0,r)}$. We will use the fact that for $0 < r \leq x \leq R$,

$$\mathbb{P}_x \{T_r < T_R\} = \begin{cases} \frac{R-|x|}{R-r} & \text{if } d = 1 \\ \frac{\log R - \log |x|}{\log R - \log r} & \text{if } d = 2 \\ \frac{R^{2-d} - |x|^{2-d}}{R^{2-d} - r^{2-d}} & \text{if } d \geqslant 3 \end{cases}$$

(1) When $d = 2$. Assume $x \neq 0$. By continuity of Brownian trajectories and dominated convergence,

$$\mathbb{P}_x(T_0 < T_R) = \lim_{r \to 0} \mathbb{P}_x(T_r < T_R) = 0$$

We have that if $T_0 < \infty$, then $T_R > T_0$ for large enough R by continuity of trajectories and the fact that Brownian motion is unbounded. So by dominated convergence,

$$\mathbb{P}_x(T_0 < T_R) = \lim_{R \to \infty} \mathbb{P}_x(T_0 < T_R) = \lim 0 = 0$$

On the other hand, for every $x \geq 0, \varepsilon > 0$, we have

$$\mathbb{P}_x(T_\varepsilon < T_R) = \begin{cases} \frac{\log R - \log |x|}{\log R - \log r} \text{ if } \varepsilon < x \text{ by formula above} \\ 1 \text{ otherwise, obviously} \end{cases}$$

so that

$$\mathbb{P}_x(T_\varepsilon < \infty) = \lim_{R \to \infty} \mathbb{P}_x(T_\varepsilon < T_R) = 1.$$

We finally prove that Brownian motion is recurrent, that is that it visits every open set at arbitratily large times almost surely. If we show this property for every fixed small closed ball, then it will be enough by countable union. By translation invariance we can assume the small ball is centered at 0. Let $\varepsilon > 0$. We define the sequence of stopping times $T_0 = 0$, and

$$T_{k+1} = \inf\{t \geq \lceil T_k \rceil + 1, B_t \in \partial B(0, \varepsilon)\}$$

Let us show that $\mathbb{P}_x(T_k < \infty) = 1$ for every k. As $T_k + 1 - T_k \geq 1$ almost surely, this will be enough to provide an unbounded sequence of times t where $B_t \in B(0, \varepsilon)$.

Let us work by induction. Of cours $\mathbb{P}(T_0 < \infty) = 0$. Now assume $\mathbb{P}(T_k) = 1$. Then

$$\begin{aligned}
\mathbb{P}_x(T_{k+1} < \infty) &\geq \mathbb{E}_x[\mathbb{1}_{T_k<\infty}\, \mathbb{1}_{T_{k+1}<\infty}] \\
&= \mathbb{E}_x[\mathbb{1}_{T_k<\infty}\, \mathbb{1}_{T_1<\infty} \circ \theta_{\lceil T_k \rceil + 1}] \\
&= \mathbb{E}_x[\mathbb{1}_{T_k<\infty}\, \mathbb{E}_x[\mathbb{1}_{T_1<\infty} \circ \theta_{\lceil T_k \rceil + 1} \mid \mathcal{F}_{\lceil T_k \rceil + 1}]] \\
&= \mathbb{E}_x[\mathbb{1}_{T_k<\infty}\, \mathbb{E}_{B_{\lceil T_k \rceil + 1}}[\mathbb{1}_{T_1<\infty}]] \\
&= \mathbb{E}_x[\mathbb{1}_{T_k<\infty}\, 1] \\
&= 1
\end{aligned}$$

Here we have used the strong Markov property written in a similar form as the one you are used to from last semester in the discrete setting. Let us state it:

Theorem : Let T be a stopping time, and θ_T be map $\Omega \to \Omega$ that shifts of the Brownian trajectory by time T. Then for every random variable $F : \Omega \to \mathbb{R}$ such that $F \circ \theta_T \in L^1$,

$$\mathbb{E}_x[F \circ \theta_T \mid \mathcal{F}_T] = \mathbb{E}_{B_T}[F] \text{ a.s.}$$

If you don't like the θ_T notation, you can also use
Theorem : Let T be a stopping time, and $F : \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}$ measurable such that $F((B_{T+s})_{s\geq 0}) \in L^1$. Then

$$\mathbb{E}_x[F((B_{T+s})_{s\geq 0}) \mid \mathcal{F}_T] = \mathbb{E}_{B_T}[F((B_s)_{s\geq 0})] \text{ a.s.}$$

You can admit each of these theorems and use them whenever needed. To prove that it is equivalent to the version of strong Markov given in class, one can use exercise 5, TD 1.

(2) When $d \geq 3$, we similarly get $\mathbb{P}_x(T_{B(0,\varepsilon)} < \infty) = (\varepsilon/|x|)^{d-2}$ for every $x : |x| > \varepsilon$. We deduce

$$\mathbb{P}_x(T_n \circ \theta_{T_{2^n}} < \infty) = (n/2n)^{d-2} \to 0 \text{ exponentially fast}$$

By Borel-Cantelli, there exists almost surely n_0 such that for $n \geq n_0$, the Brownian motion does not return to distance n after T_{2^n}. This implies that $B_t \to \infty$.

Solution 2 — *Singularity removal.*
Assume without loss of generality that U is a ball centered at x. Let $\widetilde{h}(y) = \mathbb{E}_y[h(B_T)]$, where $T = T_{U^\complement}$. This is well defined because almost surely $B_T \in \partial U$, and of course $\widetilde{h}$ is harmonic on the whole of U. To show that $h(y) = \widetilde{h}(y)$ forall $y \neq x$, proceed as follows. Define $T_\epsilon = T_{U^\complement \cup B(x,\epsilon)}$. Then by harmonicity of h, $h(y) = \mathbb{E}_y[h(B_{T_\epsilon})]$. Furthermore, since almost surely x is not hit by B, we have $B_{T_\epsilon} \to B_T$ as $\epsilon \to 0$. Applying the dominated convergence theorem yields $h(y) = \mathbb{E}_y[h(B_{T_\epsilon})] \underset{\epsilon \downarrow 0}{\longrightarrow} \mathbb{E}_y[h(B_T)] = \widetilde{h}(y)$ and we are done.

Whith the relaxed condition that $u(x+\epsilon) = o(f(\epsilon))$ where f is a fundamental solution, we define the same $T, \widetilde{h}, T_\epsilon$. Now

$$h(y) = \mathbb{E}_y[h(B_{T\epsilon})] = \mathbb{E}_y[\mathbb{1}_{T_\epsilon < T}\, h(B_{T_\epsilon})] + \mathbb{E}[\mathbb{1}_{T_\epsilon = T}\, h(B_T)]$$

The first term is bounded by $\frac{C}{f(\epsilon)} o(f(\epsilon)) \to 0$ and the second term goes to $\mathbb{E}_y[h(B_T)] = \widetilde{h}(y)$. Hence we still have $h(y) = \widetilde{h}(y)$.

Solution 3 — *Liouville's theorem.*

Solution 4 — *Harmonic functions and martingales.* (1) Let us show that for every x, under $\mathbb{P}_x$, $t \mapsto h(B_{t\wedge T})$ is a martingale that is closed by $h(B_T)$. Thus we shall show that for every x, $\mathbb{E}_x[h(B_T) \mid \mathcal{F}_t] = h(B_{t\wedge T})$. Indeed we can compute

$$\begin{aligned}
\mathbb{E}_x[h(B_T) \mid \mathcal{F}_t] &= \mathbb{E}_x[h(B_{t\wedge T})\, \mathbb{1}_{T<t} + h(B_T)\, \mathbb{1}_{t\leq T} \mid \mathcal{F}_t] \\
&= h(B_{t\wedge T})\, \mathbb{1}_{T<t} + \mathbb{E}_x[h(B_T)\, \mathbb{1}_{t\leq T} \mid \mathcal{F}_t] \\
&= h(B_{t\wedge T})\, \mathbb{1}_{T<t} + \mathbb{E}_x[h(B^t_{T^t})\, \mathbb{1}_{t\leq T} \mid \mathcal{F}_t] \\
&= h(B_{t\wedge T})\, \mathbb{1}_{T<t} + \mathbb{E}_x[h(B^t_{T^t})\, \mathbb{1}_{t\leq T} \mid \mathcal{F}_t] \\
&= h(B_{t\wedge T})\, \mathbb{1}_{T<t} + \mathbb{E}_{B_t}[h(B_T)]\, \mathbb{1}_{t\leq T} \\
&= h(B_{T\wedge t})\, \mathbb{1}_{T<t} + h(B_t)\, \mathbb{1}_{T\geq t} = h(B_{t\wedge T})
\end{aligned}$$

Where B^t denoted the Brownian motion restricted from time t onward, and T^t the hitting time of ∂D for this process.

(2) Consider $D_\varepsilon = B(0, \varepsilon^{-1}) \cap \{x, d(x, D^\complement) > \varepsilon\}$, which is open and bounded. Then $D_\varepsilon^\complement = B(0, \varepsilon^{-1})^\complement \cup \bigcup_{x\in D^\complement} \overline{B}(0, \varepsilon)$. We can show that D_ε verifies the Poincaré cone condition.

Let T_ε be the hitting time of $D^\complement$. We can now apply question 1 and get that

$$\mathbb{E}[h(B_{t\wedge T_\varepsilon}) \mid \mathcal{F}_s] = h(B_{s\wedge T_\varepsilon})$$

We can now use continuity of paths, continuity of h, and the (conditional) dominated convergence theorem to conclude.

Solution 5 — *Counterexample.*
Set $T = T_{\partial D}$ and $h(x) = \mathbb{E}_x[u(B_T)]$. This does not define a solution to the Laplace equation, because since the Brownian motion started outside of 0 almost surely does not hit 0, we have $h(0) = 0$ and $h(x) = 1$ for all $x \in \overline{D} \setminus \{0\}$. Hence h is not continuous.
Suppose a solution h exists. Then a rotation of h is still a solution, and hence equals h thanks to the maximum principle. Thus h is rotation invariant hence radial ($h(x) = g(|x|), x \in \overline{D}$, for some $g : \mathbb{R}_+ \to \mathbb{R}$ that must be twice differentiable.) We deduce that $0 = g''(x) + \frac{1}{x} g'(x)$ for all $0 < x < 1$, an ODE whose solutions are of the form $x \mapsto A + B\log(x)$, none of which fits our purpose. Hence a solution cannot exist.