# The Lévy-Ito Decomposition (AppleBaum, §2.4)

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# (-3) Basics

Lévy Process

(L1) $X(0)=0$ a.s.

(L2) $X$ has indep stationary increments.

(L3) $X$ is stoch continuous, i.e. $X_t \xrightarrow[t \to 0]{P} 0$.

they are Feller, have càdlàg modifications,
their characteristic function is

$$\mathbb{E}[e^{i X_t}] = e^{it\eta(x)}$$

Hence the distribution of $X_1$ characterizes the Lévy Process.
and $X_1$ is ID.
( Conversely, for any ID distribution, there is a lévy process corresponding )

$\eta$ is called the Lévy symbol of $X$.

Thm (Lévy Khinchine) If $X$ is ID,
There exists $b \in \mathbb{R}$, $\sigma^2 \geq 0$, $\mu$ a measure on $\mathbb{R}$ such that $\int |x|^2 \wedge 1 \, \mu(dx) < \infty$
such that $\mathbb{E}[e^{iuX}] = e^{\eta(u)}$ with $\boxed{\eta(u) = ibu + \frac{u^2\sigma^2}{2} + \int (e^{iux} - 1 - iux)\mu(dx)}$

Rk: we have not proved LK yet! The proof comes today.

# -2 A refresher of Mathew's talk: Poisson Random Measures

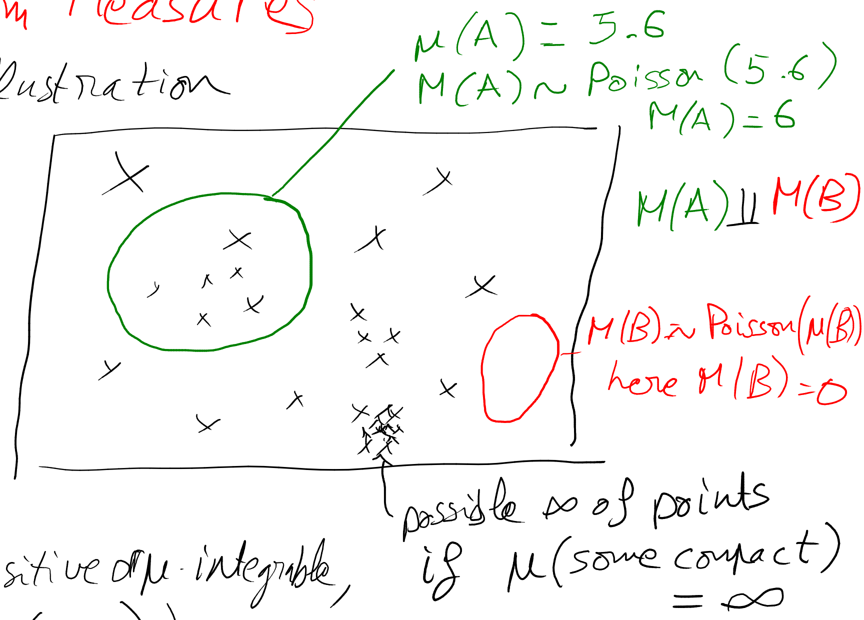
**Definition** Let $\mu$ be a $\sigma$-finite measure on some space $X$ with $\sigma$-algebra $\mathcal{X}$. A Poisson random measure on $X$ with intensity $\mu$ is a random measure on $X$ such that

(i) $M$ is a counting measure almost surely

(ii) if $\mu(A)<\infty$ then $M(A) \sim \text{Poisson}(\mu(A))$

(iii) $A \cap B = \emptyset \Rightarrow M(A) \perp\!\!\!\perp M(B)$

Illustration




**Notation** If $\alpha$ is a measure on $X$, and $f: X \to \mathbb{R}^d$ positive or $\mu$-integrable,

$\alpha(f)$ denotes $\int f(x)\,\alpha(dx)$ (so that $\alpha(A) = \alpha(\mathbb{1}_A)$)

Applebaum does not use it but it's a useful shorthand

**Thm** (Campbell's formula for sums over points of a Poisson Random measure)

Assume we have a Poisson random measure $M$ with intensity measure $\mu$ on space $X$ then for $f: X \to \mathbb{R}^d$ measurable if $\mu(|f| \wedge 1) < \infty$ then $M(|f|) < \infty$ a.s (ie $f$ is $M$-integrable a.s.)

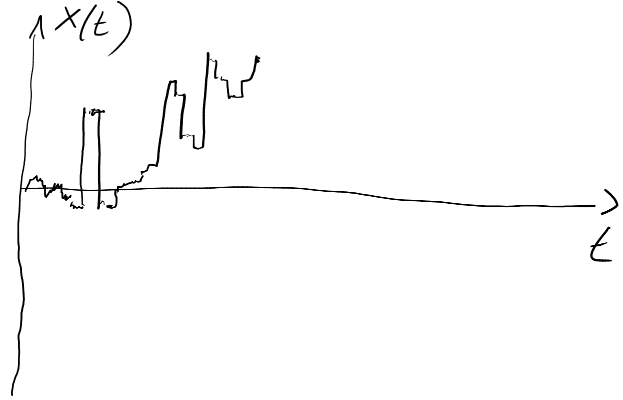
and $\mathbb{E}\left[e^{i\lambda M(f)}\right] = \int \left(e^{i\lambda f(x)} - 1\right) \mu(dx)$, in particular

$$\mathbb{E}[M(f)] = \int f(x)\,\mu(dx) \quad \text{if } \mu(|f|) < \infty$$

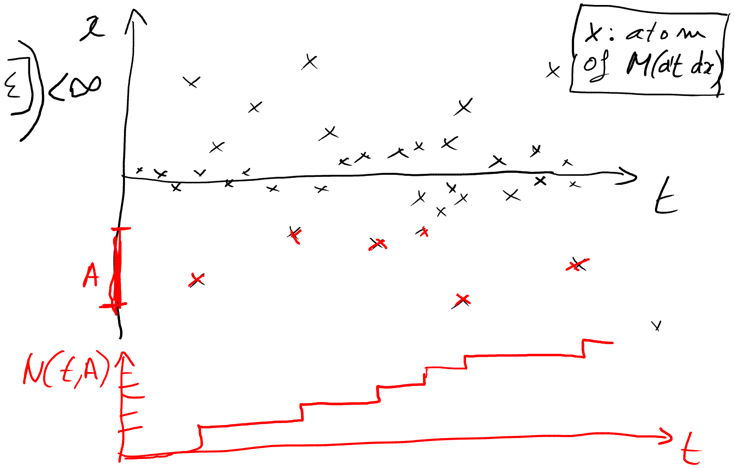
$$\mathrm{Var}(M(f)) = \int f(x)^2\,\mu(dx) \quad \text{if } \mu(|f|^2) < \infty.$$

**Definition** Let $\mu$ be a finite measure on $X$.
The Poisson Point Process on $X$ with intensity $\mu$ is the Poisson Random Measure on $\mathbb{R}_+ \times X$ with intensity Lebesgue $\otimes\, \mu$.

We then use the notation $N(t,\cdot) = M([0,t]\times \cdot)$




**Thm (2.3.5)** $\exists$ $\mu$ measure on $\mathbb{R}\setminus\{0\}$ s.t $\forall \varepsilon$, $\mu(\mathbb{R}\setminus[-\varepsilon,\varepsilon])<\infty$ such that the jump-process $M = \sum_{t:\,\Delta X(t)>0} \delta_{(t,\Delta X(t))}$ is a PPP of intensity $\mu$ on $\mathbb{R}\setminus\{0\}$ (or equivalently a Poisson Random measure on $\mathbb{R}_+ \times \mathbb{R}\setminus\{0\}$ with intensity Leb $\otimes\, \mu$




Rk The way Applebaum states it is through the processes $(N(t,A))_t$
This is equivalent.

**Remark** If $f: X \to \mathbb{R}^d$, $\mu(|f|\wedge 1)<\infty$, Campbell's formula tells us we can define $N(t,f) = \int f(x)\,N(t,dx) = \int_{[0,t]\times X} f(x)\,M(dt\,dx)$. One can easily show this is a Lévy process in the variable $t$, and Campbell's formula tells us it's Lévy symbol is $u \longmapsto \int (e^{iux}-1)\,\mu(dx)$

Subremark 1 If $\mu$ is finite and $f = \mathrm{Id}$, it's the compound Poisson process with rate $\mu(X)$ and distribution $\frac{\mu(\cdot)}{\mu(X)}$. This is a generalization with $\infty$ of summable jumps

Subremark 2 The application of the remark above to the PPP coming from the jumps of $X$ and $f=0$ around zero is called thm 2.3.7 by Applebaum. I prefer to emphasize this is a general fact about PPPs.

## (-1) A refresher about last time's talk (Terence)

Recall the total variation of a function $f$ over $[0,t]$ is denoted $V_f(t)$

**Theorem 2.4.1**

Assume $M_1, M_2$ càdlàg centred martingales.
Assume $M_1(t), M_2(t) \in L^2 \ \forall t$ and furthermore $V_{M_2}(t) \in L^2 \ \forall t$.
Then $\mathbb{E}[M_1(t) M_2(t)] = \mathbb{E}\left[\sum_{s \le t} \Delta M_1(s) \Delta M_2(s)\right]$

(2) Take an arbitrary partition $\Pi = \{0 \le t_0 < t_1 < \dots < t_{n_\Pi} = t\}$ of $[0,t]$

$$\mathbb{E}[M_1(t) M_2(t)] = \mathbb{E}\Big[\sum_{\substack{0 \le j \le n-1 \\ 0 \le i \le n-1}} (M_1(t_{i+1}) - M_1(t_i))(M_2(t_{j+1}) - M_2(t_j))\Big]$$

$$\underset{\text{Martingale pty}}{=} \mathbb{E}\Big[\sum_{0 \le i \le n-1} (M_1(t_{i+1}) - M_1(t_i))(M_2(t_{i+1}) - M_2(t_i))\Big] = \mathbb{E}[*_\Pi]$$

(3) Fix $\varepsilon > 0$

For a cadlag function, one can enumerate jumps in the decreasing order of magnitude. assume $j_1, \dots, j_{n_\varepsilon}$ are the jumps $\ge \varepsilon$ of $M_1$.
There is a partition $\Pi_\varepsilon$ fine enough such that

* if $[t_i, t_{i+1}]$ contains a jump $j_\ell$, $\ell \le n_\varepsilon$ then it's the only such $\ell$ and
$$|M_k(t_{i+1}) - M_k(t_i) - \Delta M_k(j_\ell)| < \frac{\varepsilon}{2^\ell}\Big(\sup_{[0,t]} |M_1| + \sup_{[0,t]} |M_2|\Big)^{-1} \quad k = 1,2$$

* if $[t_i, t_{i+1}]$ does not, then $|M_1(t_{i+1}) - M_1(t_i)| < 2\varepsilon$

Proof (1) First of all, is $\sum_{s \le t} \Delta M_1(s) \Delta M_2(s)$ summable?
Answer: yes, this is nonzero only for countably many $s \le t$, and

$$\sum_{s \le t} |\Delta M_1(s)||\Delta M_2(s)| \le 2 \sup_{[0,t]} M_1 \times V_{M_2}(t)$$

Moreover, $\mathbb{E}\big[\sup_{[0,t]} M_1 \times V_{M_2}(t)\big]$

$$\underset{(CS)}{\le} \mathbb{E}\big[(\sup_{[0,t]} M_1)^2\big] + \mathbb{E}\big[V_{M_2}(t)^2\big]$$

$$\underset{(Doob)}{\le} 4 \sup_{[0,t]} \mathbb{E}[M_1(t)^2] + \mathbb{E}[V_{M_2}(t)^2]$$

So partial sums are dominated by a $L^1$ variable and the convergence of the series is in $L^1$ by dominated convergence

(4) Then $*_{\Pi_\varepsilon} - \sum_{\substack{s \le t \\ |\Delta M_1(s)| < \varepsilon}} \Delta M_1(s) \Delta M_2(s)$

$$\le \sum_{\ell=1}^{n_\varepsilon} \frac{2\varepsilon}{2^\ell} + \sum_{i=0}^{n_{\Pi_\varepsilon}} 2\varepsilon\, |M_2(t_{i+1}) - M_2(t_i)|$$

$$\le (2 + 2 V_{M_2}(t))\varepsilon \longrightarrow 0 \text{ in } L^1 \text{ as } \varepsilon \to 0$$

Hence theorem as $\mathbb{E}[*_{\Pi_\varepsilon}] = \mathbb{E}[M_1(t) M_2(t)]$ and $\sum_{s \le t} \Delta M_1(s) \Delta M_2(s)$ converges in $L^1$

Application:

<u>Thm</u> (2.4.6) If $A_1, A_2$ are disjoint and bounded below. $\int_{A_1} x\, N(t, dx)$ and $\int_{A_2} x\, N(t, dx)$ are independent stochastic processes.

<u>More general theorem</u>, same proof

If $X_1$ and $X_2$ are Lévy processes and $X_2$ has square-integrable truncated variation (ie $\sup_{\substack{n \geq 1 \\ 0 = t_0 \leq \cdots \leq t_n = t}} \sum_{i=0}^{n-1} |X_2(t_{i+1}) - X_2(t_i)| \wedge 1$ is in $L^2$ for all $t \geq 0$)

If their jump-times are disjoint, they are independent.

<u>proof</u> For $k = 1, 2$, and $\theta \in \mathbb{R}$, define $M_k^\theta(t) = e^{i\theta X_k(t) - t\eta_{X_k}(\theta)}$. this is a martingale

Then one can apply (2.4.1) to $M_1^{\theta_1}$ and $M_2^{\theta_2}$ hence $\mathbb{E}[M_1(t) M_2(t)] = 0 \; \forall t$

Then $\mathbb{E}[M_1^{\theta_1}(t) M_2^{\theta_2}(s)] \underset{\substack{\text{wlog} \\ t \geq s}}{=} \underbrace{\mathbb{E}[M_1^{\theta_1}(s) M_2^{\theta_2}(s)]}_{0} + \underbrace{\mathbb{E}[(M_1^{\theta_1}(t) - M_1^{\theta_1}(s)) M_2^{\theta_2}(s)]}_{0 \text{ (bc } M_1, M_2 \text{ martingales same filtration)}}$

$= 0$

As a result $\mathbb{E}\left[e^{i\theta_1 X_1(t) + i\theta_2 X_2(s)}\right] = \mathbb{E}[e^{i\theta_1 X_1(t)}]\, \mathbb{E}[e^{i\theta_2 X_2(s)}]$ $\forall \theta_1, \theta_2$ $\forall s, t$.

Hence $X_1(t) \perp\!\!\!\perp X_2(s)$ $\forall s, t$. Independence of $X_1, X_2$ as processes ($\Leftrightarrow$ independence of finite-dim marginals) follows by Lévyness (pf by induction)

$\square$

◎ Statement.

Thm (Lévy-Ito) (24.16) + (24.12) + (2.4.13)

Let $X$ be Lévy, recall the jump poisson process $N(t, dx)$

- There exists
  - $b \in \mathbb{R}$
  - a centered BM $B$ w/ covariance $\sigma^2 \geq 0$ adapted to the filtration of $X$.

such that

$$X(t) = bt + B(t) + Y_d(t) + \int_{|x| \geq 1} x \, N(t, dx)$$

$$\int_{|x| > \varepsilon} \mu(dx) < \infty$$

$Y_d(t)$ is a centred martingale and a Lévy process, whose jumps are the points of $N$ of magnitude $< 1$. It is adapted to $N$ and its Lévy symbol is $u \mapsto \int_{|x|<1} (e^{iux} - 1 - iux) \, \mu(dx)$

- All 4 terms of the sum are $\perp\!\!\!\perp$ processes

- $\int_{|x|<1} x^2 \mu(dx) < \infty$.

Corollary LK

pf take characteristic functions $\square$

# (1) Three intermediate results

**Theorem 2.4.7**

If a Lévy process has bounded jumps, (ie $\exists C>0$ s.t almost surely $|\Delta X_s| \leq C$ $\forall s \geq 0$)
Then it has bounded moments of all orders (ie $\mathbb{E}[|X_s|^m] < \infty$ $\forall m \geq 0$)

Proof: Let $C$ be the bound on jumps.
Let $T_0 = 0$ and inductively $T_{i+1} = \inf\{t \geq T_i, |X_t - X_{T_i}| > C\}$
Then for $t \leq T_1$, $|X_t| \leq 2C$. And for $t \leq T_n$ $|X_t| \leq 2nC$.
By Strong Markov, $T_n$ is a sum of iid copies of $T_1$. $T_1 > 0$ a.s by stochasticity

$$\mathbb{P}(|X_t| > 2nC) \leq \mathbb{P}(T_n > t) \underset{\text{Chernoff bound}}{\leq} e^t \underbrace{\mathbb{E}[e^{-T_1}]}_{<1}{}^n$$

$X_t$ has exponential tails hence all moments $\square$

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Let $a \geq 0$
Define now $Y_a = X - \int_{|x| \geq a} x N(t, dx)$ if second term well-define.
(if $a > 0$, OK, $\mu(\{x : |x| \geq a\}) < \infty$ and this is just a compound Poisson process. When $a = 0$, only if $\underbrace{\int |x| \wedge 1 \mu(dx)}_{<\infty}$ by Campbell's theorem)

**Theorem 2.4.8** $Y_a$ is a Lévy process

Pf ⚠ The substraction of two independent Lévy processes is automatically Lévy but no independence here.
But for all $t \geq 0$, $(Y_a(t+s) - Y_a(t))_{s \geq 0} = \hat{F}((X(t+s) - X(t))_{s \geq 0})$ a.s, where $a$ is the functional that takes a cadlag function and removes all jumps $\geq a$. Moreover $Y_a(t) \in \sigma(X_s, s \leq t)$.
This allows to show that $Y_a$ has stationary independent increments $\square$

**Theorem 2.4.15** A centered Lévy process (hence martingale) with no jumps (hence continuous) is a centred Brownian motion with variance $\sigma^2 \geq 0$

**Pf** By 2.4.7, It has all moments. Moments are obtained by derivation of $\theta \mapsto \mathbb{E}[e^{i\theta X_t}] = e^{t\eta(\theta)}$
$\eta(0) = 0$ and $\eta'(0) = 0$ because centered. Hence $\exists$ polynomial s.t $\mathbb{E}[X_t^m] = at + a_2 t^2 + \dots + a_{m-1} t^{m-1}$
In particular $\mathbb{E}[X_t^2] = at$ (Makes sense bc Lévy) and as $t \to 0$ $\mathbb{E}[X_t^m] = at + o(t^2)$ for $m \geq 2$.

Now consider $f(t) = \mathbb{E}[e^{i\theta X_t}]$. Take a partition $0 = t_0 \leq \dots \leq t_n = t$. Denote $\delta_j = X_{t_{j+1}} - X_{t_j}$

$$f(t) - f(0) = \mathbb{E}\left[\sum_{i=0}^{n-1} e^{i\theta X_{t_{i+1}}} - e^{i\theta X_{t_i}}\right]$$

By Taylor applied to $x \mapsto e^{i\theta x}$ at the order 1, $\exists u \in [X_{t_i}, X_{t_{i+1}}]$

$$\left|e^{i\theta X_{t_{i+1}}} - e^{i\theta X_{t_i}} - i\theta \delta_i e^{i\theta X_{t_i}} - i\theta^2 \frac{\delta_i^2}{2} e^{i\theta X_{t_i}}\right| = \left|i\theta^2 \frac{\delta_i^2}{2}\left(e^{i\theta u} - e^{i\theta X_{t_i}}\right)\right|$$
$$\leq |\theta|^2 \frac{\delta_i^2}{2}\left(|\theta \delta_i| \wedge 2\right)$$

Hence
$$\left|\sum_{i=0}^{n-1} \mathbb{E}\left[e^{i\theta X_{t_{i+1}}} - e^{i\theta X_{t_i}} - i\theta e^{i\theta X_{t_i}} \delta_i + \theta^2 e^{i\theta X_{t_i}} \frac{\delta_i^2}{2}\right]\right| \leq |\theta|^2 \mathbb{E}\left[\sum_{i=0}^{n-1} \delta_i^2 \left(|\theta \delta_i| \wedge 2\right)\right]$$
$$\leq |\theta|^2 \mathbb{E}\left[\underbrace{\max_{i=0}^{n-1} |\theta \delta_i| \wedge 2}\times \underbrace{\sum_{i=0}^{n-1} \delta_i^2}\right]$$

$\parallel$ Lévy: $\delta_i \perp\!\!\!\perp X_{t_i}$

$$\left|f(t) - f(0) - 0 + \frac{a\theta^2}{2} \sum_{i=0}^{n-1} (t_{i+1} - t_i) \mathbb{E}[e^{i\theta X_{t_i}}]\right|$$

Now make the mesh of the partition go to zero $(n \to \infty)$.

$$f(t) - f(0) + \frac{a\theta^2}{2} \int_0^t f(s)\, ds$$

goes to zero a.s + bounded hence goes to zero in $L^2$ by dominated convergence

Hence $\mathbb{E}[\smile \times \smile] \to 0$ by Cauchy Schwarz.

Bounded in $L^2$

Because $\mathbb{E}\left[\left(\sum \delta_i^2\right)\right]$
$= \sum \mathbb{E}[\delta_i^4] + 2\sum_{i<j} \mathbb{E}[\delta_i^2]\mathbb{E}[\delta_j^2]$
$\leq \sum (t_{i+1} - t_i)(1 + o(1)) + 2\sum_{i,j} (t_{i+1} - t_i)(t_{j+1} - t_j)$
$\leq 2t + 4t^2$ for $n$ large

As a result $f(t) = f(0) + \frac{a\theta^2}{2} \int_0^t f(s)\, ds$ Hence $f(t) = e^{-\frac{a\theta^2}{2} t}$ implying increments of $X$ are Gaussian. $X$ is BM $\blacksquare$

## ② Proof of Lévy-Ito, easy case

Recall $Y_a(t) = X(t) - \int_{|x|\geq a} x\, N(t,dx)$. Since $Y_a$ is Lévy (Thm 2.4.8) and has bounded jumps, it has bounded moments (Thm 2.4.7)

One can define $\hat{Y}_a(t) = Y_a(t) - \underbrace{\mathbb{E}[Y_a(t)]}_{= t\,\mathbb{E}[Y_a(1)]}$

Let us sketch the proof in the special case $\int (|x| \wedge 1)\, \mu(dx) < \infty$
then $Y_0$ is well-defined and

$$X(t) = t\,\mathbb{E}[Y_1(1)] + \underbrace{\hat{Y}_0(t)}_{\substack{\text{Continuous} \\ \text{centered} \\ \text{Lévy} = \text{BM.} \\ \text{by } (2.4.15)}} + \underbrace{\hat{Y}_1(t) - \hat{Y}_0(t)}_{= \int_{|x|<1} x\, N(t,dx) - \mathbb{E}\left[\int_{|x|<1} x\, N(t,dx)\right]} + \int_{|x|\geq 1} x\, N(t,dx)$$

hence has Lévy symbol $u \mapsto \int (e^{iux} - 1 - iux)\, \mu(dx)$
By Campbell's thm.

For the independence property first of all $\int_{|x|\geq 1} x\, N(t,dx) \perp\!\!\!\perp \hat{Y}_1$ By 2.4.5
secondly, $\hat{Y}_0$ and $\hat{Y}_1 - \hat{Y}_0 \in \sigma(\hat{Y}_1)$ and $\hat{Y}_1 - \hat{Y}_0 \perp\!\!\!\perp \hat{Y}_1$ By 2.4.5 again. $\square$

## 3 Proof general case.

$$X(t) = t\,\mathbb{E}[Y_1(1)] + \underbrace{\hat{Y}_a(t)}_{\text{Lévy process with jumps} \le a} + \overbrace{\underbrace{\hat{Y}_1(t) - \hat{Y}_a(t)}_{\text{Compensated compound Poisson}}}^{Z_a} + \underbrace{\int_{|x|\ge 1} x\, N(t,dx)}_{\text{Lévy (compound Poisson process)}}$$

Compensated compound Poisson: $\int_{a\le|x|<1} x\, N(t,dx) - \mathbb{E}[\,-\,]$

So Lévy symbol is $\boxed{+}$

$$u \mapsto \int_{a\le|x|<1} (e^{iux} - 1 - iux)\,\mu(dx)$$

1. The three terms are independent by a double application of Thm 2.4.5 as in the special case above

Now we want to take $a \to 0$.

Because bounded jumps, $\hat{Y}_a(t), Z_a(t) \in L^2$. Because orthogonal (2.4.1), $\mathrm{Var}(\hat{Y}_a(t)) + \mathrm{Var}(Z_a(t)) = \mathrm{Var}(\hat{Y}_1)$

Hence $\mathrm{Var}(Z_a)$ is bounded as $a \downarrow 0$.

Moreover if $a' < a$, $Z_{a'}(t) = \underbrace{Z_a(t) + Z_{a'}(t) - Z_a(t)}_{\text{orthogonal (2.4.1)}}$ hence $\mathrm{Var}(Z_{a'}(t)) - \mathrm{Var}(Z_a(t))$
$= \mathrm{Var}(Z_{a'}(t) - Z_a(t))$ (*)

As a result $\mathrm{Var}(Z_a(t))$ is increasing and bounded as $a \downarrow 0$. Hence it converges, hence it is Cauchy and by (*) $Z_a(t)$ itself is Cauchy in $L^2$.

Denote $B(t)$ the $L^2$-limit of $\hat{Y}_a(t)$ and $Y_d(t)$ the $L^2$-limit of $Z_a(t)$.

Independence and being levy pass to the limit in $\mathbb{P}$. Being centered passes to the limit in $L^2$.

② Now, because of Doob's inequality for martingales ( $\mathbb{E}\left[\sup_{s\le t}|M_s|\right] \le E\left[\sup_{s\le t} M_s^2\right]^{1/2} \le 2\,\mathbb{E}\left[M_t^2\right]^{1/2}$ )
we can obtain some regularity of convergence

Extract $a_k \downarrow 0$ such that $\mathbb{E}\left[\sup_{[0,t]}|\hat{Y}_{a_{k+1}} - \hat{Y}_{a_k}|\right] \le 2\,\mathbb{E}\left[\left(\hat{Y}_{a_{k+1}}(t) - \hat{Y}_{a_k}(t)\right)^2\right]^{1/2} \le 1/2^k$.

(possible bc $\hat{Y}_{a_k}$ is Cauchy when $a \downarrow 0$).

Almost surely $\sum \sup_{[0,t]} |\hat{Y}_{a_{k+1}} - \hat{Y}_{a_k}|$ is a convergent series. On that event of probability one the sequence $(Y_{a_k})_k$ of càdlàg functions on $[0,t]$ is Cauchy for the uniform norm hence there exists a function $\tilde{B}$ that is its uniform limit.

For fixed $t$ we have found a subsequence such that almost surely $Y_{a_k}$ cv uniformly on $[0,t]$

Now by diagonal extraction there is a subsequence such that a.s $Y_{a_k}$ cv uniformly on every compact to a function $\tilde{B}$ (and similarly for $Z_a \to \tilde{Y}_d$ )

The uniform limit of càdlàg functions is càdlàg and if $f_n \to f$ uniformly then $\Delta f_n \to \Delta f$ uniformly.

Hence * $\tilde{B}$ is continuous a.s. because $\Delta \hat{Y}_a \le a$ a.s.

* The jumps of $\tilde{Y}_d$ are exactly given by the restriction of $N$ to $\{|x|<1\}$

$Z_a$ are exactly given by the restriction of $N$ to $\{a \le |x| < 1\}$

We already knew the Lévy processes $B$ and $Y_d$ had càdlàg modifications, but here we explicated those modifications $\tilde{B}$ and $\tilde{Y}_d$, yielding further properties

As a byproduct, $\tilde{B}$ being a continuous Lévy, is a BM (2.4.15)

③

Finally we need to show that $\int x^2 \wedge 1 \, \mu(dx) < \infty$. We know $\int_{|x| \geq 1} \mu(dx) < \infty$.

But $\mathrm{Var}(Z_a(1)) = \int_{a \leq |x| < 1} x^2 \mu(dx)$ converges to $\mathrm{Var}(Y_d(1))$

Hence $\int_{|x|<1} x^2 \mu(dx) < \infty$ and hence $\int x^2 \wedge 1 \, \mu(x) dx < \infty$

This shows that $\mu$ integrates $\mathbb{1}_{|x|<1}(e^{iux} - 1 - iux)$ and one can take a limit in ⊕ showing that the lévy symbol of $Y_d$ is $\int_{|x|<1} (e^{iux} - 1 - iux) \mu(dx)$.

We're done! $\square$

# 4 A few remarks

1) In the easy case $\int |x|\wedge 1\, \mu(dx) < \infty$, the compensation of jumps of magnitude $\leq 1$ is not really necessary. Lévy-Ito becomes

$$X(t) = b' t + B(t) + \int x \; N(t,dx)$$

(the drift has changed: $b' = b + \int_{|x|\leq 1} x\,\mu(dx)$ )

This is the preferred form to write the Lévy-Ito decomposition and the Lévy-Khinchine representation $\eta(u) = ib'u + \frac{\sigma^2 u^2}{2} + \int (e^{iux} - 1)\mu(dx)$ for such cases.

2) Thm 2.4.25 Lévy processes have finite variation iff

* easy case described above
* $\sigma = 0$ (no Brownian part)

3) In the case $\int |x|\wedge 1\, \mu(dx) = \infty$, then the small jumps are not summable even though they manage to form a càdlàg function. This is really "martingale magic".

4) The cutoff at 1 for which jumps are compensated is artificial. One could replace by a cutoff at $M>0$ or even a smooth cutoff function, up to updating the drift term.