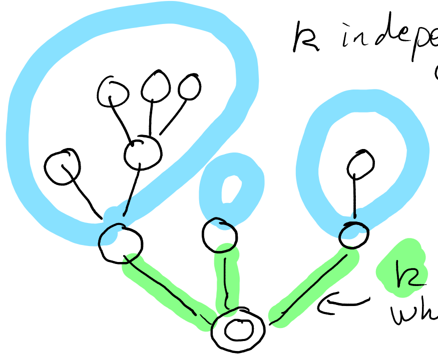
Lévy trees (Broadly)

Duquesne & le Gall

Motivation: a continuous counterpart to the trees encoded by GW branching processes.

There is a scaling limit connection between the two, we will allude to it but invariance principles are beyond the scope of this group

Let ν be a distribution on $\mathbb{Z}_{\geq 0}$. A GW_ν tree T is



k independent copies of T.

k children where $k \sim \nu$ $(k=3)$

$Z_k = \#$ of nodes at height k

then Z_k is a branching process

A GW_ν forest F is an iid sequence of copies of T.

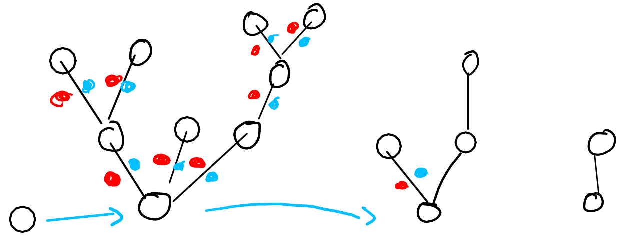
Question: what does a tree T conditioned to be "large" look like?

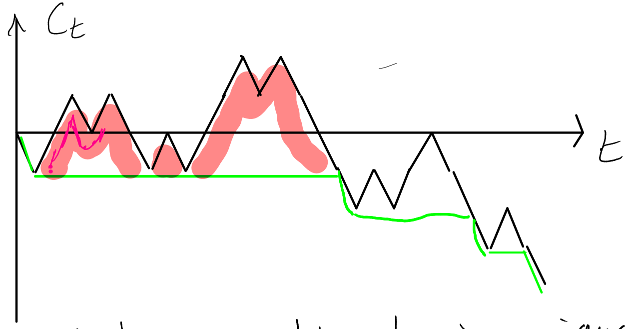
Related question: what does F look like, viewed from afar?

A simple case: $\nu = \text{Geom}(p)$

$p \geq 1/2$

Assume (sub)criticality $\sum k\nu(k) \leq 1$





The contour function is a random walk

$+1$	-1
$1-p$	p

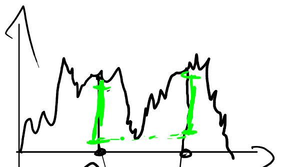
The excursions above the minimum are the trees of the forest

When $p = 1/2$, by invariance principles,

$\frac{C_{2nt}}{\sqrt{n}} \xrightarrow{d} cB$. Immediately, if $\tilde{C}_t$ is the contour of T

$\mathbb{P}\left(\frac{1}{\sqrt{n}}\tilde{C}_{2nt} \in \cdot \;\middle|\; \text{Size}(T) > n\varepsilon\right)$

$\rightarrow$ Brownian excursion of length $\geq \varepsilon$



Thm (Aldous '93)

If $\sum k\nu(k) = 1$, $\sum k^2\nu(k) < \infty$

Then the same holds and $\mathbb{P}\left(\frac{1}{\sqrt{n}}\tilde{C}_{2nt} \in \cdot \;\middle|\; \text{Size}(T) = n\right)$

$\rightarrow$ Brownian exc.

Study more general branching processes.

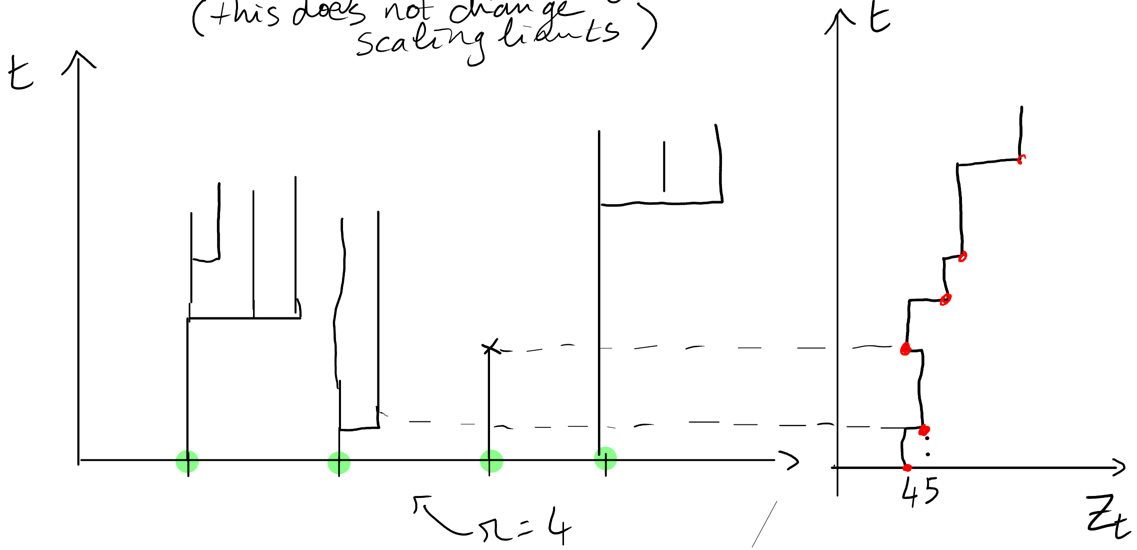
Random walks are nice tools but C is not a random walk any more.

We can look at the branching process itself:

Let $Z_k^{(n)}$ = # of nodes at height k in the first n trees.

This is no RW but a Markov process. However RWs are hidden!

To make things simpler go to the continuous time (this does not change scaling limits)



Z_t is a continuous time markov process on $\mathbb{Z}_{\geq 0}$ but its jump chain $X_k = Z_{T_k}$ is a random walk of distribution $k \mapsto \nu(k+1)$, absorbed at 0.

There is a random time-change between the two. Conditionally on X_k,

$$T_k = \sum \mathrm{Exp}(X_n) \approx \sum \frac{1}{X_k}.$$

On the other hand, the # of jumps before time t in Z_t is $\approx \int_0^t Z_s \, ds$.

A CSBP is a cadlag Markov process Z_t such that $E_x[e^{-\lambda Z_t}] = e^{-x u_t(\lambda)}$. In particular, Markov $\Rightarrow u_{t+s}(\lambda) = u_t(u_s(\lambda))$

Then a theorem of Lamperti states

Thm Set $\varphi_t = \inf\{s \geq 0,\ \int_0^s Z_u \, du > t\}$

Then $X_t = Z_{\varphi_t}$ is a killed spectrally ≥ 0 Lévy process, started at x and killed upon reaching 0.

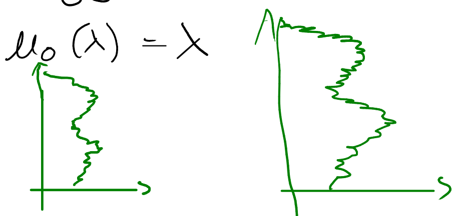
Its Laplace exponent is

$$\psi(\lambda) = -q + a\lambda + \beta\lambda^2 + \int \pi(dr)\, e^{-\lambda r} - 1 + \lambda r \mathbb{1}_{r \leq 1}$$

where $q \geq 0$, $a \in \mathbb{R}$, $\beta \geq 0$, π on $(0,\infty)$

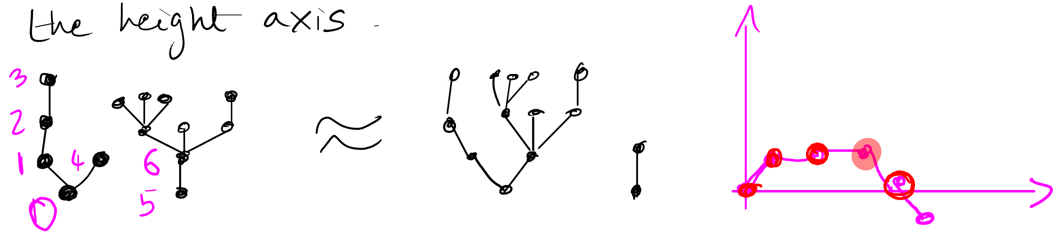
And $u_t(\lambda)$ verifies

$$\frac{\partial u_t(\lambda)}{\partial t} + \psi(u_t(\lambda)) = 0, \quad u_0(\lambda) = \lambda$$



Chap 12, Kyprianou.

Problem: this does not completely give the shape of the "Poisson forest", but rather its projection on the height axis.



We'd prefer to look at something akin to the contour function. Back to the discrete.

Let $v_0, v_1, \ldots$ be the vertices listed in the lexicographical order. (the first appearance in the contour)

Lukasiewicz walk | Exploration process

Then set $\begin{cases} X_0 = 0 \\ X_{i+1} = X_i + \#\text{of children of } v_i - 1 \end{cases}$

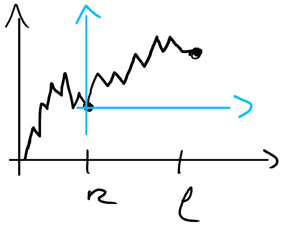
Obviously this is a random walk of distribution $k \longmapsto \nu(k+1)$.

Moreover, letting H_k = height of v_k

then $H_k = \#\left\{ i<k \,,\ X_i = \inf_{i\le j\le k} X_j \right\}$

the height function H is very similar to the contour function C and plays the same role in the scaling limit.

proof Show that $\overset{v_\ell}{\underset{v_k}{\uparrow}}$ if and only if $X_j \geq X_k$ for $k \le j \le \ell$



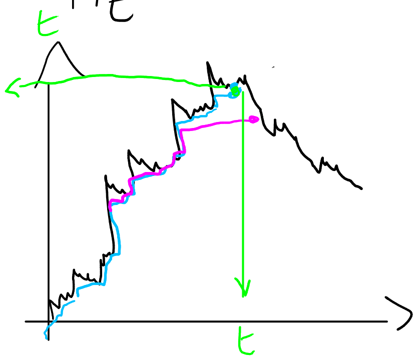
But $(X_{k+s} - X_k)_{s\geq 0}$ is the exploration process of the blue forest.

remains to show that $X_j \geq 0$ for $0 \le j \le k$ iff X_ℓ is in the first tree of the forest.

Indeed X_j = # of unexplored vertices children of explored vertices at time j -1 as long as v_j is in the first tree. ∎

As the scaling limits of random walks are lévy processes, we want a continuous analogue and define, for X a spectrally positive levy,

$$H_t = \text{local time of } \left(X_s - \inf_{[s,t]} X\right)_{0\le s\le t} \text{ at zero.}$$



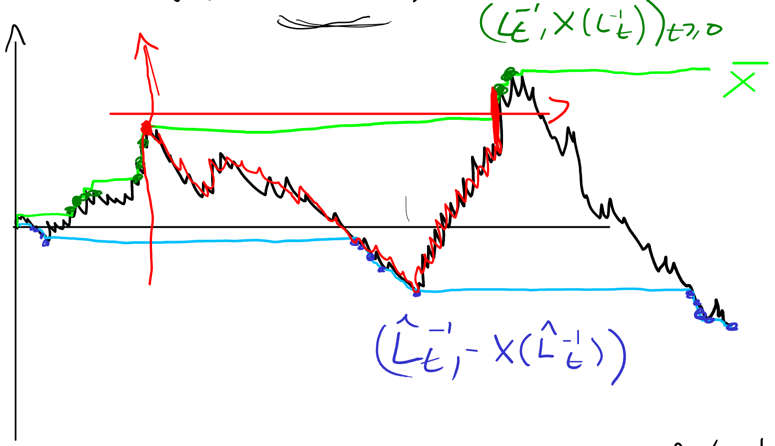
This is where I want to go next, following Duquesne & Le Gall

Let X be a spec ≥ 0 Lévy process.

Let Ψ be such that $\mathbb{E}[e^{i\theta X_t}] = e^{-t\Psi(\theta)}$

and ψ such that $\mathbb{E}[e^{-\lambda X}] = e^{t\psi(\lambda)}$

Then $\psi(\lambda) = -\Psi(i\lambda)$



Let κ be the Laplace exponent of $(L^{-1}, X \circ L^{-1})$ and $\hat{\kappa}$ similarly.

A natural choice of local time at its infimum is $\hat{L}_t = -\underline{X}_t$, yielding

$\hat{H}_t = t$ and $\hat{L}^{-1}_t = T_{-t}$

Hence $\hat{\kappa}(\alpha,\beta) = \beta + \psi^{-1}(\alpha)$.

(why $\mathbb{E}[e^{-\lambda T_{-t}}] = e^{-t\psi^{-1}(\lambda)}$? $\underbrace{e^{\lambda X_t - \psi(\lambda)t}}_{\text{martingale}}$)

In our case, one can write Wiener-Hopf as such

$$\boxed{\frac{1}{\alpha - \psi(\beta)} = \frac{k}{\kappa(\alpha,\beta)} \, \frac{\hat{k}}{\hat{\kappa}(\alpha,-\beta)}}$$

hence, choosing the normalisation of L such that $k\hat{k} = 1$,

$$\kappa(\alpha,\beta) = \frac{\alpha - \psi(\beta)}{\psi^{-1}(\alpha) - \beta}$$

In particular the Laplace exponent of $X \circ L^{-1}$ is $\frac{\psi(\lambda)}{\lambda}$ and that of L^{-1} is $\frac{\lambda}{\psi^{-1}(\lambda)}$

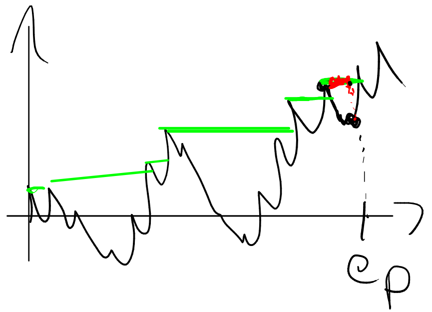
$$\varepsilon_t = \begin{cases} \text{----} & \text{if} \ \ldots \\ \delta & \text{ow} \end{cases}$$

Recall the upper excursion process $(\varepsilon_t)_{t\geq 0}$ and its measure N, define also the lower excursion process $(\hat{\varepsilon}_t)_{t\geq 0}$ ——— $\hat{N}$

Lemma 1 Assume the aforementioned normalization of local times. then $\forall g$ measurable, ≥ 0

$$\int N(de) \int_0^{\zeta(e)} g(e(s))\,ds = \int_0^{\infty} g(x)\,dx$$

Proof Consider the law of $(\bar{X}_{e_p} - X_{e_p})$ when $p \to 0$



① $$\mathbb{E}\left[e^{-\lambda(\bar{X}_{e_p} - X_{e_p})}\right] = \frac{\hat{K}(p,0)}{\hat{K}(p,\lambda)} \quad \hat{K}(p,0) \to 0, \quad \hat{K}(p,\lambda) \to \frac{1}{\lambda}$$

Hence $\frac{1}{\hat{K}(p,0)} \mathbb{P}(\bar{X}_{e_p} - X_{e_p} \in dx) \xrightarrow{\text{weak}} \text{Leb}$

Theorem (Compensation formula)

If $\mathcal{E}$ is an adapted PPP (N) on the space $\mathcal{E}$, and for $t \geq 0$ F_t is a random mes function on $\mathcal{E}$ such that

* $F_t(\delta) = 0$
* $F_t(e)$ is an adapted left-continuous process $\forall e \in \mathcal{E}$.

Then

$$\mathbb{E}\left[\sum_{s \geq 0} F_s(\mathcal{E}_s)\right] = \mathbb{E}\left[\int_0^{\infty} ds \int_{\mathcal{E}} N(de) F_s(e)\right]$$

② On the other hand, we write

$$\mathbb{E}[g(\bar{X}_{e_p} - X_{e_p})] = \mathbb{E}\left[\sum_{s \geq 0} \mathbb{1}_{L^{-1}_{s-} < e_p} F(\mathcal{E}_s, e_p - L^{-1}_{s-})\right]$$

where $F(e,t) = \mathbb{1}_{t \leq \zeta(e)}\, g(e(t))$

$$\underset{\substack{\text{compensation} \\ \text{formula}}}{=} \mathbb{E}\left[\int_0^{\infty} ds\, \mathbb{1}_{L^{-1}_{s-} < e_p} \int N(de) F(e, e_p - L^{-1}_{s-})\right] \underset{\substack{\uparrow \\ e_p \text{ is memoryless}}}{=} \left(\int_0^{\infty} ds\, \mathbb{P}(L^{-1}_{s-} < e_p)\right) \left(\int N(de)\, \mathbb{E}[F(e, e_p)]\right)$$

(red) $= \int_0^{\infty} \mathbb{E}[e^{-pL^{-1}_s}]\,ds = \int_0^{\infty} e^{-sK(p,0)}\,ds = \frac{1}{K(p,0)}$

(yellow) $\times \frac{1}{\hat{K}(p,0)} \to \int_0^{\infty} g(x)\,dx$ by part ①

(green) $\times \frac{1}{p} \longrightarrow \int N(de) \int_0^{\infty} F(e,s)\,ds = \int N(de) \int_0^{\zeta(e)} g(e(s))\,ds$

because $\frac{1}{p}\mathbb{P}(e_p \in \cdot) \longrightarrow \text{Leb}.$

We conclude with formula (see W-H)

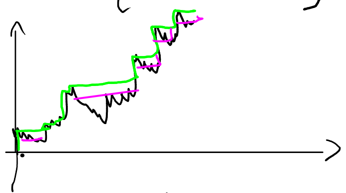
$$K(p,0)\,\hat{K}(p,0) = p$$

Lemma 2 For every $t \geq 0$,

$$\frac{1}{\varepsilon}\int_0^t \mathbb{1}\{\bar{X}_s - X_s < \varepsilon\} ds \xrightarrow[\varepsilon \to 0]{L^2} L_t$$

Proof We only need to show that for $x > 0$

$$\frac{1}{\varepsilon}\int_0^{L^{-1}(x)} \mathbb{1}\{\bar{X}_s - X_s < \varepsilon\} ds \xrightarrow[\varepsilon \to 0]{L^2} x \wedge L_\infty$$



Compute it over the excursion decomposition.

Let $(e_t)_t$ be a PPP (N). Consider η = first time of appearance of an infinite excursion.

$$\frac{1}{\varepsilon}\sum_{\substack{0 \leq t \leq x\wedge\eta \\ e_t \neq \delta}} \int_0^{\zeta(e_t)} \mathbb{1}\{e_t(s) > -\varepsilon\} ds = J_\varepsilon(x \wedge \eta)$$

By Campbell's formula $\mathbb{E}[J_\varepsilon(y)] = \frac{1}{\varepsilon} y \int N(de) \int_0^{\zeta(e)} \mathbb{1}\{e(s) > -\varepsilon\} ds$

$= y$ by lemma 1.

While $\mathrm{Var}(J_\varepsilon(y)) = \frac{x}{\varepsilon^2}\int N(de) \left[\int_0^{\zeta(e)} \mathbb{1}\{e(s) > -\varepsilon\} ds\right]^2$

Now

$$2\int N(de) \int_{0 \leq s \leq \zeta(e)} \mathbb{1}\{e(s) > -\varepsilon\} \int_{s \leq t \leq \zeta(e)} \mathbb{1}\{e(t) > -\varepsilon\} dt\, ds$$

$\underset{\text{Markov property for excursion measure}}{=} 2\int N(de) \int_{0 \leq s \leq \zeta(e)} ds\, \mathbb{1}\{e(s) > -\varepsilon\}\, \mathbb{E}_{e(s)} \int_0^{T_0} dt\, \mathbb{1}\{X(t) > -\varepsilon\}$

$$\leq 2\varepsilon \sup_{0 \geq y \geq -\varepsilon} \mathbb{E}_y \int_0^{T_{\geq 0}} dt\, \mathbb{1}\{X(t) > -\varepsilon\}$$

On the other hand for $0 \geq y \geq -\varepsilon$

$$\varepsilon = \int N(de) \int_0^{\zeta(e)} \mathbb{1}_{e(s) > -\varepsilon} ds \geq \int N(de) \int_{T_y}^{\zeta(e)} \mathbb{1}_{e(s) > -\varepsilon} ds$$

$\parallel$ strong Markov

$$\underbrace{\left(\int N(de)\, \mathbb{1}_{T_y < \infty}\right)}_{\to \infty \text{ as } \varepsilon \to 0} \mathbb{E}_y \int_0^{T_{\geq 0}} dt\, \mathbb{1}_{X(t) > -\varepsilon}$$

(because N is infinite)

Hence $J_\varepsilon(x) \to x$ in L^2.

Actually by Doob $\sup_{0 \leq z \leq x} |J_\varepsilon(z) - z|^2 \to 0$

Hence $J_\varepsilon(x \wedge \eta) \to x \wedge \eta$ in L^2 $\square$

The previous lemma allows us to define H.

For every $t \geq 0$, there is a subsequence ε_k such that $\frac{1}{\varepsilon_k}\int_0^t \mathbb{1}_{\overline{X}_s - X_s < \varepsilon_k}\, ds \longrightarrow L_t$ a.s

By diagonal extraction and countable union, there is a subsequence ε_k such that this holds a.s for every $t \in \mathbb{Q}$

Then by monotonicity and continuity of $(L_t)_t$, this holds for all $t \in \mathbb{R}$ simultaneously.

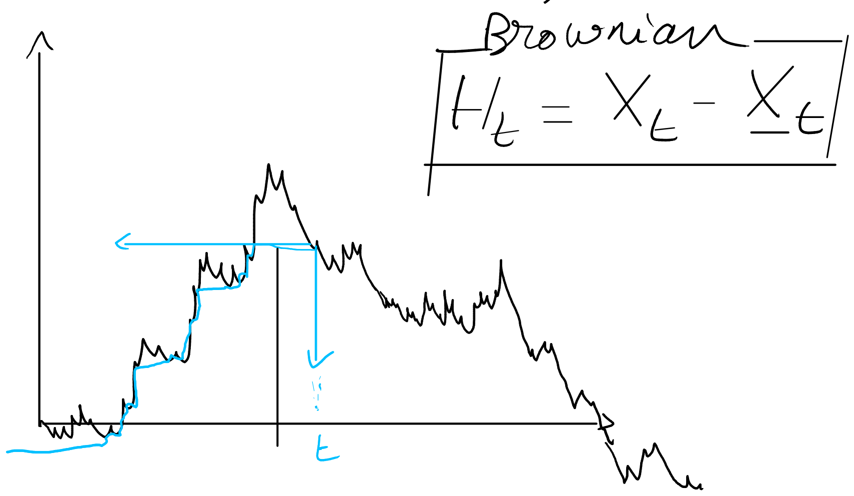
Hence the local time process is a measurable function of the trajectory of a Lévy process.

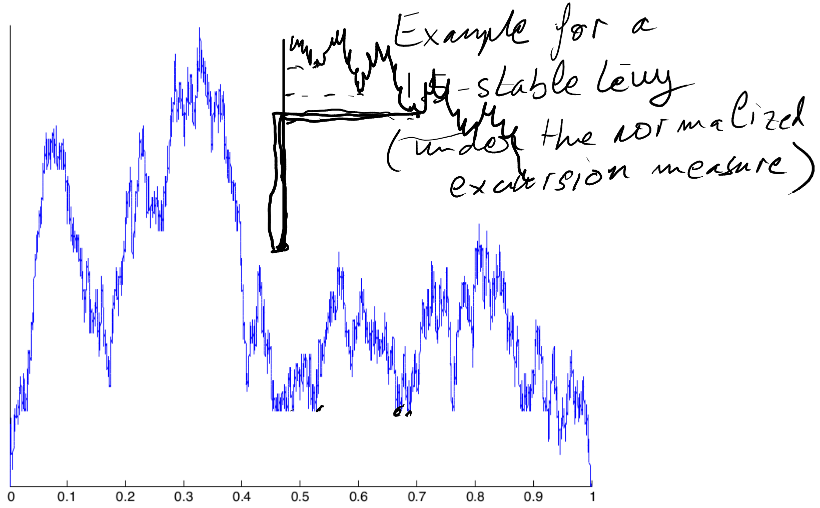
Set $\Lambda_t(u) = \lim\limits_{k \to \infty} \frac{1}{\varepsilon_k}\int_0^t \mathbb{1}_{\{\sup_{[0,s]} u - u_s < \varepsilon_k\}}\, ds$

Define the reversed Lévy process $\overleftarrow{X}^{(t)}(s) = X(t) - X((t-s)-)$ for $s \leq t$.

We know $(\overleftarrow{X}^{(t)}(s))_{0 \leq s \leq t} \overset{d}{=} (X(s))_{0 \leq s \leq t}$

<u>Def</u> The height process is $H_t = \Lambda_t(\overleftarrow{X}^{(t)})$





Continuity properties.

Theorem 3 (Le Gall, Le Jan '98, cf Duquesne-Le Gall)

H has a continuous modification

$\iff \int_1^\infty \frac{1}{\psi(x)}\,dx < \infty$

$(\iff$ The CSBP with mechanism ψ is extinct almost surely.)

Too hard. We can imagine why necessary.

Proposition 4 If $\frac{\psi(\lambda)}{\lambda^\gamma} \to \infty$, then

H has a continuous modification that is

$1-\frac{1}{\gamma}-\varepsilon$ Hölder for $\varepsilon > 0$

Example: if X is α-stable, H is

$1-\frac{1}{\alpha}$-Hölder.

Define $\Theta(\lambda) = \frac{\lambda}{\psi^{-1}(\lambda)}$ Laplace exponent of L^{-1}.

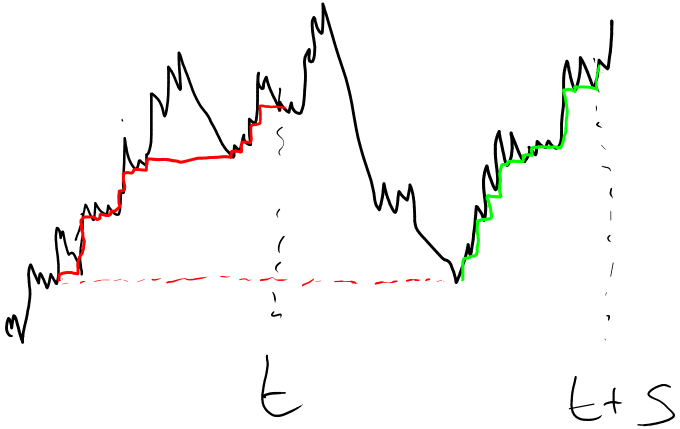
Then Lemma 5 For $t \geq 0$, $s \geq 0$, $q > 0$

$$\mathbb{E}\left[\left|H_{t+s} - \inf_{[t,t+s]} H\right|^q\right] < C_q\, \Theta(1/s)^{-q}$$

$$\mathbb{E}\left[\left|H_t - \inf_{[t,t+s]} H\right|^q\right] < \text{same}$$

Proposition 4 follows from the Lemma 5 and Kolmogorov's continuity criterion $\square$

Proof of lemma 5



$$H_{t+s} - \inf_{[t,t+s]} H \overset{d}{=} L_s \quad \text{by looking at } \overleftarrow{X}^{(t+s)}$$

$$H_t - \inf_{[t,t+s]} H \overset{d}{=} L_{(T_{-\tilde{X}_s} \wedge t)} \quad (\tilde{X} \text{ is an independent copy of } X)$$

by looking at $\overleftarrow{X}^{(t)}$ which is independent of $X(t+\cdot)$

Fact: The processes $(L_s)_{s\geq 0}$ and $(L_{T_{-\tilde{X}_s}})_{s\geq 0}$ have the same distribution!

Proof We consider right-continuous inverses of these processes. Define $U_x = \inf\{ s\geq 0, L_{T_{-\tilde{X}_s}} > x\}$. We have $U_x = \tilde{T}_-(X_{L^{-1}_x})$ is the composition of two subordinators with exponents ψ^{-1} and $\psi(\lambda)/\lambda$. (WH)

Hence U is a subordinator with exponent $\dfrac{\psi(\psi^{-1}(\lambda))}{\psi^{-1}(\lambda)} = \dfrac{\lambda}{\psi^{-1}(\lambda)} = \Theta(\lambda)$ Same as L^{-1} $\square$.

End proof lemma The fact above gives that $\bullet$ is stoch dominated by $\bullet$. For the moments of $\bullet$,

$$E\Big[\big(H_{t+s} - \inf_{[t,t+s]} H\big)^p\Big] = \int p x^{p-1} P(L_s > x)\, dx = \int p x^{p-1} \underbrace{P(L_x^{-1} < s)}_{\underset{\text{Markov}}{\leq}\, e\, E[e^{-\frac{1}{s} L_x^{-1}}]} dx \leq \int p x^{p-1} e^{-\Theta(\frac{1}{s})x}\, dx = C_p\, \Theta\left(\tfrac{1}{s}\right)^{-p}$$

$\square$