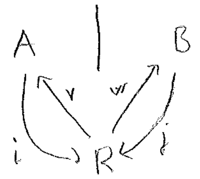
Non-closure of the set of quantum correlations

answers ↷ ↶ questions

$(P(i,j \mid v,w)) \in$

$C_r(n,k) \subseteq [0,1]^{n^2k^2}$

#questions #answers



- $C_q(n,k)$ "quantum"

  PVM's $(P_{v,i})_{i=1}^{k}$ and $(Q_{w,j})_{j=1}^{k}$ on fin. dim. Hilbert spaces H_A and H_B

  A unit vector $\xi \in H_A \otimes H_B$

  $$p(i,j \mid v,w) = \langle (P_{v,i} \otimes Q_{w,j})\xi, \xi \rangle$$

- $C_{qa}(n,k) = \overline{C_q(n,k)}$ "quantum approximate"

- $C_{qc}(n,k)$ "quantum commuting"

  PVM's $(P_{v,i})_{i=1}^{k}$ and $(Q_{w,j})_{j=1}^{k}$ on the same (possibly infinite dimensional) Hilbert space H

  satisfying $P_{v,i}Q_{w,j} = Q_{w,j}P_{v,i} \quad \forall\, v,w,i,j$

  A unit vector $\xi \in H$

  $$p(i,j \mid v,w) = \langle P_{v,i} Q_{w,j}\xi, \xi \rangle$$

<u>Terminology</u>: A tuble $(((P_{v,i})_i)_v, ((Q_{w,j})_j)_w, H, \xi)$ is called a <u>realization in C_{qc}</u> of $(p(i,j \mid v,w))$ if the above is satisfied.

<u>Exercise</u>: $p \in C_q(n,k)$ iff it has a realization in $C_{qc}(n,k)$ with H finite dimensional.

<u>Rk</u>: $C_{qa}(n,k) = C_{qc}(n,k) \ \forall\, n,k$ iff CEP is true

<u>Rk</u>: C_{qc} is closed

Goal of talk: C_q is not always closed.

- $C_{ns}(n,k)$ "non-signaling"

  Any $(p(i,j \mid v,w)) \in [0,1]^{n^2k^2}$ satisfying
  - $\sum_{i,j=1}^{k} p(i,j \mid v,w) = 1, \quad \forall\, v,w$
  - $\sum_{j=1}^{k} p(i,j \mid v,w) = \sum_{j=1}^{k} p(i,j \mid v,w') \quad \forall\, i,v,w,w'$
  - $\sum_{i=1}^{k} p(i,j \mid v,w) = \sum_{i=1}^{k} p(i,j \mid v',w) \quad \forall\, j,v,v',w$

<u>Exercise</u>: $C_{qc}(n,k) \subset C_{ns}(n,k)$

<u>Notation</u>: Marginal densities

- $p_A(i \mid v) = \sum_{j=1}^{k} p(i,j \mid v,w)$
- $p_B(j \mid w) = \sum_{i=1}^{k} p(i,j \mid v,w)$

Correlation sets and C^*-algebras

Thm: $p \in C_{qc}(n,k)$ (resp. $C_q(n,k)$) iff $\exists$ a unital C^*-algebra $\mathcal{A}$ (resp. a fin. dim. such) generated by families of projections $(e_{v,i}), (f_{w,j}) \subset \mathcal{A}$ satisfying $\sum_{i=1}^{k} e_{v,i} = 1_{\mathcal{A}} = \sum_{j=1}^{k} f_{w,j}$ and $e_{v,i} f_{w,j} = f_{w,j} e_{v,i}$, and a state ρ on $\mathcal{A}$ such that

$$p(i,j|v,w) = \rho(e_{v,i} f_{w,j})$$

GNS construction

$(\mathcal{A}, \rho)$ C^*-algebra w. a state

$\rightsquigarrow (\pi, H, \xi)$

- $\pi : \mathcal{A} \to B(H)$ *-hom.
- ξ is a cyclic unit vector ($\pi(\mathcal{A})\xi$ is dense H)
- $\rho(x) = \langle \pi(x)\xi, \xi \rangle$

Rk Marginal distributions:

- $p_A(i|v) = \rho(e_{v,i})$
- $p_B(j|w) = \rho(f_{w,j})$

Synchronous games

Def $(p(i,j|v,w))$ is called synchronous if, for all v, $p(i,j|v,v) = 0$ $i \neq j$.

Notation: $C_r^s(n,k)$

Rk: Synchronous subsets are convex and we have the same line of inclusions.

Prop: If $(p(i,j|v,w)) \in C_{qc}^s(n,k)$ is realized $\overset{\text{in } C_{qc}}{\vee}$ by $(((P_{v,i})_i)_v, ((Q_{w,j})_j)_w, H, \xi)$ then

(i) $P_{v,i}\xi = Q_{v,i}\xi \quad \forall\, v,i$ (OBS! does not force $P_{v,i}$ and $Q_{v,i}$ to be equal)

(ii) $p(i,j|v,w) = \langle P_{v,i} P_{w,j} \xi, \xi \rangle = \langle Q_{w,j} Q_{v,i} \xi, \xi \rangle = p(j,i|w,v)$

Pf: Exercise (Hint for (i): C.S.)

Thm: $p \in C_{qc}^s(n,k)$ (resp. $C_q^s(n,k)$) iff $\exists$ unital C^*-algebra $\mathcal{A}$ (resp. a fin. dim. such) generated by a family of projections $(e_{v,i}) \subset \mathcal{A}$ satisfying $\sum_{i=1}^{k} e_{v,i} = 1_{\mathcal{A}}$, and with a tracial state τ such that

$$p(i,j|v,w) = \tau(e_{v,i} e_{w,j})$$

Pf: - If $p \in C_{qc}^s(n,k)$ has realisation $(((P_{v,i})_i)_v, ((Q_{w,j})_j)_w, H, \xi)$,

Let $\mathcal{A}$ be the C^*-algebra generated by the $P_{v,i}$'s ($Q_{w,j}$ commute with $\mathcal{A}$)

Define $\tau = \langle \cdot\, \xi, \xi \rangle$

Then $p(i,j|v,w) = \tau(P_{v,i} P_{w,j})$ by prop.

τ is tracial: For $x \in \mathcal{A}$

$\tau(P_{v,i}\, x) = \langle \overset{\curvearrowright}{P_{v,i}\, x}\, \xi, \xi \rangle = \langle \overset{\curvearrowright}{Q_{v,i}\, x}\, \xi, \xi \rangle = \langle x P_{v,i} \xi, \xi \rangle = \underline{\tau(x P_{v,i})}$ + induction + continuity

- Conversely, suppose $(\mathcal{A}, \tau)$ and $(e_{v,i})$ are as in the theorem.

For $i \neq j$, $e_{v,i} e_{v,j} = 0$, so the correlation is synchronous

Let (π, H, ξ) be the GNS construction and set $P_{v,i} = \pi(e_{v,i})$

Then $(P_{v,i})_i$ is a PVM, $\forall v$, and $p(i,j|v,w) = \tau(e_{v,i} e_{w,j}) = \langle P_{v,i} P_{w,j} \xi, \xi \rangle$

For x finite linear combination of words on $(P_{v,i})$, set

$$Q_{w,j}\, x\xi = x P_{w,j} \xi$$

- $Q_{w,j}$ is a well-defined contraction $\rightsquigarrow$ define $Q_{w,j}$ on all of H
- $Q_{w,j}^2 = Q_{w,j}^* = Q_{w,j}$ and $\sum_{j=1}^{k} Q_{w,j} = 1_H$
- $Q_{w,j} P_{v,i}(x\xi) = P_{v,i}(x Q_{w,j}\xi) = P_{v,i} Q_{w,j} x\xi \rightsquigarrow P_{v,i}$ and $Q_{w,j}$ commute

Synchronous binary answer games

Rk $(p(i,j|v,w)) \in C_{ns}(n,2)$

- $p(1,2|v,w) = p_A(1|v) - p(1,1|v,w)$
- $p(2,1|v,w) = p_B(1|w) - p(1,1|v,w)$
- $p(2,2|v,w) = 1 - p_A(1|v) - p_B(1|w) + p(1,1|v,w)$

(knowing the marginals and each probability of both answering 1 gives everything)

Rk $(p(i,j|v,w)) \in C^s_{ns}(n,2)$

That is, $p(1,2|v,v) = p(2,1|v,v) = 0$

$\rightsquigarrow p(1,1|v,v) = p_A(1|v) = p_B(1|v)$

Def (A slice of $C^s_r(n,2)$) For each $t \in [0,1]$

$$\Gamma_r(t) = \{ p \in C^s_r(n,2) \mid p_A(1|v) = p_B(1|w) = t \ \forall v,w \}$$

Rk $\Gamma_r(t)$ is non-empty and convex.

Def (The correlation function) $f_r : [0,1] \to \mathbb{R}$

$$f_r(t) = \inf \{ F(p) \mid p \in \Gamma_r(t) \}$$

where $F : C_{ns}(n,2) \longrightarrow \mathbb{R}$ is the function

$$F(p) = \sum_{v \neq w} p(1,1|v,w)$$

Rk If $C^s_r(n,2)$ is closed, the infimum is attained, for all $t \in [0,1]$

Rk $f_q(t) = f_{qa}(t) \geq f_{qc}(t) \geq f_{ns}(t) \geq 0$

Prop f_r is a convex function

Pf: By convexity of $C^s_r(n,2)$,

$$\lambda \Gamma_r(t_1) + (1-\lambda)\Gamma_r(t_2) \subset \Gamma_r(\lambda t_1 + (1-\lambda)t_2), \quad t_1, t_2, \lambda \in [0,1]$$

we have

$$\begin{aligned}\lambda f_r(t_1) + (1-\lambda) f_r(t_2) &= \lambda \inf\{F(p) \mid p \in \Gamma_r(t_1)\} + (1-\lambda)\inf\{F(p) \mid p \in \Gamma_r(t_2)\} \\ &= \inf\{F(p) \mid p \in \lambda\Gamma_r(t_1) + (1-\lambda)\Gamma_r(t_2)\} \\ &\geq \inf\{F(p) \mid p \in \Gamma_r(\lambda t_1 + (1-\lambda)t_2)\} \\ &= f_r(\lambda t_1 + (1-\lambda)t_2)\end{aligned}$$

$\square$

The quantum correlation function

Lemma: Let $t \in [0,1]$ be irrational and suppose $f_q(t)$ is attained in the infimum defining it.
Then there exist rational numbers $r, s \in [0,1]$ s.t. $t \in [r,s]$ and the restriction of f_q to $[r,s]$ is linear.

Pf: Take $p \in \Gamma_q(t)$ s.t. $f_q(t) = F(p)$.
There is a unital fin. dim. C^*-algebra $\mathcal{A}$ generated by projections $\{P_v\}_{v=1}^{n}$
and w. a tracial state $\tau : \mathcal{A} \to \mathbb{C}$ w. $\tau(P_v) = t \ \ \forall v$ and such that

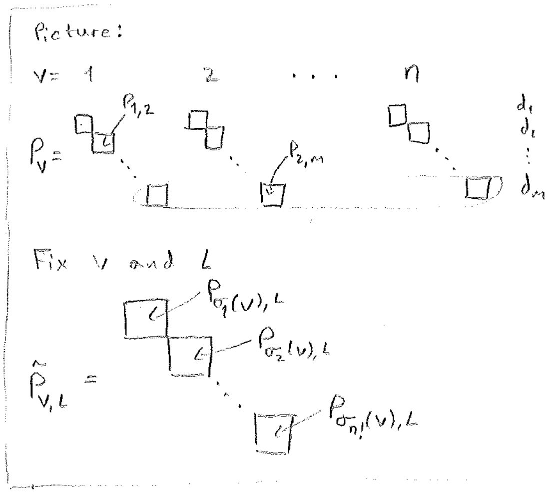
$$f_q(t) = \sum_{v \neq w} \tau(P_v P_w)$$

Write $\mathcal{A} = \bigoplus_{\ell=1}^{m} M_{d_\ell}(\mathbb{C})$ and $\tau = \oplus_{\ell=1}^{m} \lambda_\ell \underset{\text{normalized trace}}{tr_{d_\ell}}$ where $\lambda_\ell > 0$ with $\sum_{\ell=1}^{m} \lambda_\ell = 1$

Moreover, write $P_v = \bigoplus_{\ell=1}^{m} P_{v,\ell}$ with $P_{v_\ell} \in M_{d_\ell}(\mathbb{C})$ a projection

For each v and each $1 \leq \ell \leq m$, set

$$\tilde{P}_{v,\ell} = \bigoplus_{\sigma \in S_n} P_{\sigma(v),\ell} \in \bigoplus_{\sigma \in S_n} M_{d_\ell}(\mathbb{C}) =: \mathcal{A}_\ell$$



Define tracial state $\tau_\ell : \mathcal{A}_\ell \to \mathbb{C}$ by $\tau_\ell = \frac{1}{n!} \bigoplus_{\sigma \in S_n} tr_{d_\ell}$

Observe that, for any v and w

$$\tau_\ell(\tilde{P}_{v,\ell}) = \frac{1}{n!} \sum_{\sigma \in S_n} tr_{d_\ell}(P_{\sigma(v),\ell}) = \frac{1}{n!} \sum_{\sigma \in S_n} tr_{d_\ell}(P_{\sigma \circ \sigma'(v),\ell})$$

$$= \tau_\ell(\tilde{P}_{w,\ell}) =: \underset{\nwarrow \text{ this is rational}}{r_\ell}$$

Summarizing: We have a unital $\overset{\text{fin. dim}}{C^*}$-algebra $\mathcal{A}_\ell$ w. a tracial state τ_ℓ and all marginals equal to r_ℓ.
Hence, $\{\tilde{P}_{v,\ell}\}_{v=1}^{n}$ defines a synchronous quantum correlation in the slice $\Gamma_q(r_\ell)$, and thus

$$f_q(r_\ell) \leq \sum_{v \neq w} \tau_\ell(\tilde{P}_{v,\ell} \tilde{P}_{w,\ell})$$

Set $\tilde{\mathcal{A}} = \bigoplus_{\ell=1}^{m} \mathcal{A}_\ell$, define a tracial state $\tilde{\tau} : \tilde{\mathcal{A}} \to \mathbb{C}$ by $\tilde{\tau} = \bigoplus_{\ell=1}^{m} \lambda_\ell \tau_\ell$.
Define projections $\tilde{P}_v = \bigoplus_{\ell=1}^{m} \tilde{P}_{v,\ell}$. Then

$$\tilde{\tau}(\tilde{P}_v) = \sum_{\ell=1}^{m} \lambda_\ell \underbrace{\tau_\ell(\tilde{P}_{v,\ell})}_{= r_\ell} = \frac{1}{n!} \sum_{\sigma \in S_n} \sum_{\ell=1}^{m} \lambda_\ell \, tr_{d_\ell}(P_{\sigma(v),\ell}) = \frac{1}{n!} \sum_{\sigma \in S_n} \tau(P_{\sigma(v)}) = t \qquad \boxed{t = \sum_{\ell=1}^{m} \lambda_\ell r_\ell}$$

$$\tilde{\tau}(\tilde{P}_v \tilde{P}_w) = \sum_{\ell=1}^{m} \lambda_\ell \tau_\ell(\tilde{P}_{v,\ell} \tilde{P}_{w,\ell}) = \frac{1}{n!} \sum_{\sigma \in S_n} \tau(P_{\sigma(v)} P_{\sigma(w)})$$

$$\rightsquigarrow f_q(t) = \sum_{v \neq w} \tau(P_v P_w) \overset{\text{for any } \sigma \in S_n}{=} \sum_{v \neq w} \tau(P_{\sigma(v)} P_{\sigma(w)}) = \sum_{v \neq w} \tilde{\tau}(\tilde{P}_v \tilde{P}_w)$$

Thus,

$$f_q(t) = \sum_{\ell=1}^{m} \lambda_\ell \sum_{v \neq w} \tau_\ell(\tilde{P}_{v,\ell} \tilde{P}_{w,\ell}) \geq \sum_{\ell=1}^{m} \lambda_\ell f_q(r_\ell)$$

But f_q is convex, so this is an equality (by Jensen)
Since t is irrational and all the r_ℓ's are rational, it is not the case that all r_ℓ's are equal.
So the equality in Jensens inequality implies that f_q is linear on an interval containing the r_ℓ's.

Talk: Non-closure of quantum correlations

Recap from Emilie's part:

Correlation sets:

$$\begin{tikzcd}
C_q^s(n,k) & \subseteq & C_{qs}^s(n,k) & \subseteq & C_{qa}^s(n,k) & \subseteq & C_{qc}^s(n,k) & \subseteq & C_{ns}^s(n,k) \\
 & & & & \overline{C_q(n,k)} \arrow[u, equal] & & & & \\
\underline{\text{closed ?}} \arrow[uu] \arrow[uurr] & & & & & & \text{closed !} \arrow[uu] & &
\end{tikzcd}$$

Define

$$\Gamma_r(t) = \left\{ \big(p(i,j|v,w)\big) \in C_r^s(n,2) \; : \; p_A(1|v) = p_B(1|w) = t \;\; \forall v,w \right\}$$

$$F(p) = \sum_{v \neq w} p(1,1|v,w)$$

and

$$f_r(t) = \inf \left\{ F(p) \mid p \in \Gamma_r(t) \right\} \qquad (*)$$

Proposition: Let $t \in [0,1]$ be irrational.
If $f_q(t)$ is attained in $(*)$, then $\exists\, r,s \in \mathbb{Q}$ $r<t<s$ s.th.
$$f_q\big|_{[r,s]} \text{ linear.}$$

Goal: Find $t \in [0,1]$ irrational s.th. there is no intervall $[r,t] \ni t$ where
$$f_q\big|_{[r,t]} \text{ is linear!}$$

__Def:__ (Vectorial correlations)

$\big(p(i,j\,|\,v,w)\big)_{\substack{i,j\in\{1,\dots,k\}\\ v,w\in\{1,\dots,n\}}}$ is called a __vectorial correlation__

if $\exists$ Hilbertspace H and vectors $\left.\begin{matrix}\{x_{v,i}\}\\ \{y_{w,i}\}\\ h\end{matrix}\right\}\in H$ s.th.

$$\begin{cases} (1) & \|h\|=1 \\ (2) & \langle x_{v,i}, x_{v,j}\rangle = 0 \quad \forall v\ \forall i\neq j \\ (3) & \langle y_{w,i}, y_{w,j}\rangle = 0 \quad \forall w\ \forall i\neq j \\ (4) & h=\sum_i x_{v,i} = \sum_j y_{w,j} \quad \forall v,w \\ (5) & p(i,j\,|\,v,w) = \langle x_{v,i}, y_{w,j}\rangle \quad \forall v,w,i,j \end{cases}$$

Write $C_{vec}(n,k) = \left\{\begin{matrix}\text{set of all vectorial corr.}\\ \text{for } n,k\end{matrix}\right\}$.

__Exercise:__ Show $C_{qc}(n,k) \subseteq C_{vec}(n,k) \subseteq C_{ns}(n,k)$

__Exercise:__ Show that $p\in C^{S}_{vec}(n,k)$ iff $\exists$ H. space H & $\{x_{v,i}\}\subset H$, $h\in H$ s.th.

(1) $\|h\|=1$

(2) $\langle x_{v,i}, x_{v,j}\rangle = 0 \quad \forall v\ \forall i\neq j$

(3) $h=\sum_i x_{v,i} \quad \forall v$

(4) $p(i,j\,|\,v,w) = \langle x_{v,i}, x_{w,j}\rangle \quad \forall v,w,i,j$

Computing $f_{vec}(t)$:

$$f_{vec}(t) = \inf\{\mathcal{F}(p) \;:\; p \in \Gamma_{vec}(t)\}$$

$$\mathcal{F}(p) = \sum_{v \neq w} p(1,1 \mid v,w)$$

$$\& \quad \Gamma_{vec}(t) = \left\{ p \in C^{S}_{vec}(n,2) \;:\; \underset{= t}{p_A(1 \mid v) = p_B(1 \mid w)} \right\}$$

Lemma (Symmetrization):

$$f_{vec}(t) = \inf\{\mathcal{F}(p) : p \in \tilde{\Gamma}_{vec}(t)\}$$

where

$$\tilde{\Gamma}_{vec}(t) = \left\{ p \in \Gamma_{vec}(t) \;:\; p(1,1 \mid v,w) = p(1,1 \mid x,y) \quad \forall v \neq w \;\; \forall x \neq y \right\}$$

Proof: For $\pi \in S_n$ define $\beta_\pi : C^S_{vec} \longrightarrow C^S_{vec}$

$$\beta_\pi((p(i,j \mid v,w)) = p(i,j \mid \pi^{-1}(v), \pi^{-1}(w)))$$

Note: $\mathcal{F} = \mathcal{F} \circ \beta_\pi$

Define
$$\begin{cases} Avg : C^S_{vec} \longrightarrow C^S_{vec} \\ Avg(p) = \frac{1}{n!} \sum_{\pi \in S_n} \beta_\pi(p) \end{cases}$$

Again: $\mathcal{F} = \mathcal{F} \circ Avg$

Since $Avg(\Gamma_{vec}(t)) = \tilde{\Gamma}_{vec}(t)$ we find

$$\begin{aligned} f_{vec}(t) &= \inf\{\mathcal{F}(p) \;:\; p \in \Gamma_{vec}(t)\} \\ &= \inf\{\mathcal{F} \circ Avg(p) : p \in \Gamma_{vec}(t)\} \\ &= \inf\{\mathcal{F}(p) : p \in \tilde{\Gamma}_{vec}(t)\}. \end{aligned}$$

$\square$

How does $p \in \tilde{\Gamma}_{vec}(t)$ look like?

1.) $p(1,1|v,w) = p(1,1|x,y) =: \frac{s}{n(n-1)}$

$\forall v \neq w, \ \forall x \neq y$

$\hookrightarrow \ \underline{\underline{F(p) = s}}$.

2.) $t = p_A(1|v) = p(1,1|v,v) + \overset{\text{by synchr.}}{\overset{=0}{p(1,2|v,v)}}$

$$\Longrightarrow \boxed{p(1,1|v,w) = \begin{cases} t & v = w \\ \frac{s}{n(n-1)} & v \neq w. \end{cases}}$$

3.) $t = p_A(1|v) \overset{v \neq w}{=} \underset{\frac{s}{n(n-1)}}{\underset{\parallel}{p(1,1|v,w)}} + p(1,2|v,w)$

$$\Longrightarrow \boxed{p(1,2|v,w) = p(2,1|v,w) = \begin{cases} 0 & v = w \\ t - \frac{s}{n(n-1)} & v \neq w. \end{cases}}$$

4.) $p(2,2|v,w) = 1 - \sum_{ij} p(i,j|v,w)$

$$\Longrightarrow \boxed{p(2,2|v,w) = \begin{cases} 1 - t & v = w \\ 1 - 2t + \frac{s}{n(n-1)} & v \neq w \end{cases}}$$

Theorem:

$$\mathrm{frec}(t) = \min_{s \geq 0} \left. \begin{array}{l} \text{s.th.} \quad \frac{s}{n(n-1)} \leq t \\ \qquad (2t-1) \leq \frac{s}{n(n-1)} \\ \& \quad \underset{\overset{\cap}{M_{n+1}}}{\begin{pmatrix} 1 & t & \cdots & t \\ t & t & & \frac{s}{n(n-1)} \\ \vdots & & \ddots & \\ t & \frac{s}{n(n-1)} & & t \end{pmatrix}} \geq 0 \end{array} \right\} (*)$$

Proof: Consider $s \geq 0$ satisfying $(*)$.

Since every positive matrix is a Gram matrix

$$\exists \{h, x_{1,1}, x_{2,1}, \ldots, x_{n,1}\} \subseteq \mathbb{C}^{n+1}$$

s.th.

$$\begin{cases} \langle h, h \rangle = 1 \\ \langle h, x_{v,1} \rangle = t \quad \forall v \in \{1, \ldots, n\} \\ \langle x_{v,1}, x_{w,1} \rangle = \frac{s}{n(n-1)} \quad \forall v \neq w \;\&\; \langle x_{v,1}, x_{v,1} \rangle = t \;\; \forall v \end{cases}$$

Set $x_{v,2} = h - x_{v,1}$. Then:

$$\langle x_{v,1}, x_{v,2} \rangle = \langle x_{v,1}, h \rangle - \langle x_{v,1}, x_{v,1} \rangle$$
$$= t - t = 0.$$

& $h = x_{v,1} + x_{v,2}$

Set

$$p(i,j \mid v,w) = \langle x_{v,i} \mid x_{w,j} \rangle$$

$$\Longrightarrow \quad \begin{matrix} p \in \Gamma_{vec}(t) \\ F(p) = s \end{matrix} \quad \Longrightarrow \quad f_{vec}(t) \leq \min\big(s \geq 0 : (*)\big)$$

For converse assume that $p \in \Gamma_{vec}(t)$ satisfies $F(p) = s$.

$$\text{Lemma} \Longrightarrow \quad \exists\, \tilde{p} \in \tilde{\Gamma}_{vec}(t) \text{ s.th. } F(\tilde{p}) = s.$$

Previous discussion

$$\Longrightarrow \quad \frac{s}{n(n-1)} \leq t$$

$$\& \quad (2t-1) \leq \frac{s}{n(n-1)}$$

Build Gram matrix from realization of $\tilde{p}$

$$\Longrightarrow \begin{pmatrix} 1 & t & \cdots & t \\ t & t & & \frac{s}{n(n-1)} \\ \vdots & & \ddots & \\ t & \frac{s}{n(n-1)} & & t \end{pmatrix} \geq 0.$$

$$\rightsquigarrow \quad \min\big(s \geq 0 \text{ s.th. } (*)\big) \leq f_{vec}(t).$$

$\square$

Corollary:

For $n=5$ we have

$$f_{vec}(t) = \begin{cases} 0 & 0 \leq t \leq \frac{1}{5} \\ 5t(5t-1) & \frac{1}{5} \leq t \leq \frac{4}{5} \\ 20(2t-1) & \frac{4}{5} \leq t \leq 1 \end{cases}$$

quadratic.

Thm: $C_q^S(5,2)$ is not closed.

Proof: Consider

$$I_5 := \left[\frac{\sqrt{5}-1}{2\sqrt{5}}, \frac{\sqrt{5}+1}{2\sqrt{5}}\right] \subset \left[\frac{1}{5}, \frac{4}{5}\right]$$

Claim: $f_{vec}(t) = f_q(t) \quad \forall t \in \mathbb{Q} \cap I_5$

Since $f_{vec}(t) = 5t(5t-1)$ not linear on I_5

Claim $\Longrightarrow$ $f_q(t) = \inf\left(\mathcal{F}(p) : p \in \Gamma_q(t)\right)$

not attained in any irrational $t \in I_5$.

(otherwise $\exists r,s \in \mathbb{Q}$ s.th. $t \in [r,s]$ & $f_q|_{[r,s]}$ linear)

$\Longrightarrow$ $C_q^S(5,2)$ not closed.

To prove the claim, note $f_{vec}(t) \leq f_q(t)$.

$$\overset{\shortparallel}{5t(5t-1)} \quad \forall t.$$

Show: $\underset{5t(5t-1)}{\underset{\shortparallel}{f_{vec}(t)}} \geq f_q(t) \quad \forall t \in I_5 \cap \mathbb{Q} \Big\} (*)$

Magical lemma (Kruglyak, Rabanovich, Samoilenko 2003)

If $t \in I_5 \cap \mathbb{Q}$, then $\exists k \in \mathbb{N}$ & $P_1, \dots, P_5 \in M_k$ projections s.th.

$$P_1 + \dots + P_5 = 5t \mathbb{1}_k$$

Proof of (*):

For $t \in I_5 \cap \mathbb{Q}$ consider projections as in lemma.

Set
$$\begin{cases} \tilde{P}_v = P_v \oplus \dots \oplus P_{v+4} \in M_k \oplus \dots \oplus M_k \subseteq M_{5k} \\ \text{for } v \in \{0, \dots, 4\} \end{cases}$$

Note: $\sum_{v=0}^{4} \tilde{P}_v = 5t \mathbb{1}_{5k}$

Let tr_{5k} denote normalized trace (state) on M_{5k}.

1.) $$tr_{5k}(\tilde{P}_v) = \frac{1}{5k} Tr(\tilde{P}_v)$$
$$= \frac{1}{5k} \sum_{v=0}^{4} Tr(P_v) = \frac{t}{5k} Tr(\mathbb{1}_{5k}) = \underline{\underline{t}}.$$

2.) $$\sum_{v,w} \tilde{P}_v \tilde{P}_w = 25t^2 \mathbb{1}_{5k}$$

$$\|$$

$$\sum_{v} \tilde{P}_v + \sum_{v \neq w} \tilde{P}_v \tilde{P}_w$$

$$\Longrightarrow \sum_{v \neq w} \tilde{P}_v \tilde{P}_w = (25t^2 - 5t)\mathbb{1}_{5k}$$

$$\Longrightarrow \sum_{v \neq w} tr_{5k}(\tilde{P}_v \tilde{P}_w) = 5t(5t-1)$$

Finally define

$$p(i,j|v,w) = tr_{5k}(Q_{v,i} Q_{w,j})$$

with

$$Q_{v,i} = \begin{cases} \tilde{P}_v & i=0 \\ (\mathbb{1}_{5k} - \tilde{P}_v) & i=1 \end{cases}$$

$$\Longrightarrow f_q(t) \leq 5t(5t-1)$$

$\square$

Corollary:

$C_q(5,2)$ and $C_{qs}(5,2)$ are not closed.

$C_{qs}(5,2) \neq C_{qa}(5,2) = \overline{C_q(5,2)}$.

A few words about the magic:

We will not prove the magical Lemma. But we can clarify a few constructions.

Suppose: $\alpha \mathbb{1}_k = P_1 + \dots + P_n$ (*) $(\& \; P_i \neq 0)$

$$\Longrightarrow \quad \alpha k = \sum_i \operatorname{rk}(P_i) \in \mathbb{N} \,. \quad \& \quad \alpha \geq 1 .$$

Two tricks lead to new decompositions:

(1) (*) $\Longrightarrow$ $\exists\, Q_1, \dots, Q_n$ projections s.th.

$$(n-\alpha)\mathbb{1}_k = Q_1 + \dots + Q_n$$

$$\left(\hookrightarrow \text{Choose } \; Q_i = \mathbb{1} - P_i \, . \right).$$

(2) $\underset{\& \, \alpha > 1}{(*)}$ $\Longrightarrow$ $\exists\, Q_1, \dots, Q_n$ projections s.th.

$$\frac{\alpha}{\alpha - 1} \mathbb{1}_{k(\alpha-1)} = Q_1 + \dots + Q_n$$

Construction for (2):

Suppose (*).

in particular $P_i \neq 0$.

Write $P_i = V_i V_i^*$ for $V_i : \mathbb{C}^{\operatorname{rk}(P_i)} \longrightarrow \mathbb{C}^k$ isometry.

and define

$$V = (V_1, \dots, V_n) \; : \mathbb{C}^{\alpha k} \longrightarrow \mathbb{C}^k .$$

Note that

$$VV^* = \sum_i V_i V_i^* = \alpha \mathbb{1}_k$$

$\Longrightarrow$ $\frac{1}{\sqrt{\alpha}} V^*$ is an isometry.

$\hookrightarrow$ Extend to unitary

$$U = \begin{array}{r} k \\ (\alpha-1)k \end{array} \overbrace{\begin{pmatrix} \frac{1}{\sqrt{\alpha}} V_1 & \cdots & \frac{1}{\sqrt{\alpha}} V_n \\ \sqrt{\frac{\alpha-1}{\alpha}}\, W_1 & \cdots & \sqrt{\frac{\alpha-1}{\alpha}}\, W_n \end{pmatrix}}^{k\alpha} \Big| \; k\alpha$$

$$U^*U = \begin{pmatrix} \frac{1}{\alpha} V_1^* V_1 + \frac{\alpha-1}{\alpha} W_1^* W_1 & & * \\ & \ddots & \\ * & & \end{pmatrix} = \begin{pmatrix} \mathbb{1}_{rk(P_1)} & & \\ & \ddots & \\ & & \ddots \end{pmatrix}$$

$$\Longrightarrow \underset{\substack{\| \\ \frac{1}{\alpha} \mathbb{1}_{rk(P_1)}}}{\frac{1}{\alpha} V_i^* V_i} + \frac{\alpha-1}{\alpha} W_i^* W_i = \mathbb{1}_{rk(P_1)}$$

$$\Longrightarrow W_i^* W_i = \mathbb{1}_{rk(P_1)}$$

$\hookrightarrow$ W_i isometry.

$$U^*U = \begin{pmatrix} \frac{\alpha-1}{\alpha} \sum_i W_i W_i^* & & * \\ * & & * \end{pmatrix} = \begin{pmatrix} \mathbb{1}_{k(\alpha-1)} & & \\ & \ddots & \\ & & \ddots \end{pmatrix}$$

$\Longrightarrow$ $\underset{\text{projections}}{Q_i = W_i W_i^*}$ satisfy $\sum_i Q_i = \frac{\alpha}{\alpha-1} \mathbb{1}_{k(\alpha-1)}$

Lemma: TFAE

(1) $\alpha \mathbb{1}_k = P_1 + \dots + P_n$ for proj. P_i

(2) $\frac{\alpha}{\alpha-1} \mathbb{1}_{k(\alpha-1)} = \tilde{P}_1 + \dots + \tilde{P}_n$ for proj. $\tilde{P}_i$

(3) $(n-\alpha) \mathbb{1}_k = P_1' + \dots + P_n'$ for proj. P_i'

(4) $\frac{(n-1)\alpha - n}{\alpha-1} \mathbb{1}_{k(\alpha-1)} = Q_1 + \dots + Q_n$ for proj. Q_i

Interval in theorem

$$I_5 = \left[\frac{\sqrt{5}-1}{2\sqrt{5}}, \frac{\sqrt{5}+1}{2\sqrt{5}} \right]$$

$$\Longrightarrow \quad 5I_5 = \left[\frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2} \right]$$

$$\frac{5+\sqrt{5}}{2} = 2 + \phi$$

ϕ ← golden section 1.618...

Recall: Fibonacci numbers

$$\begin{cases} f_0 = f_1 = 1, \quad f_{n+1} = f_n + f_{n-1} \\ \phi = \lim_{n \to \infty} \frac{f_{2n+1}}{f_{2n}} \end{cases}$$