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présentée par

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Rigidity and malleability aspects of groups and their representations

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Introduction

Ce mémoire, rédigé en anglais, présente une partie des travaux que j'ai effectués après ma thèse, c'est-à-dire sur la période octobre 2009–mars 2016. Il est constituté de cinq parties, qui portent toutes sur différents aspects de rigidité, ou à l'inverse de malléabilité pour des groupes. La question générale commune à tout le mémoire est de comprendre dans quelle mesure certains objets mathématiques associés à un groupe peuvent être déformés ou approchés non trivialement. On parlera de malléabilité lorsque c'est possible et de rigidité sinon. En fonctions des chapitres, ces objets seront des représentations unitaires (chapitre 1), des représentations sur des espaces de Banach (chapitre 3), des algèbres d'opérateurs ou des espaces L_p non commutatifs (chapitre 2), des graphes de Cayley (chapitre 5). Le chapitre 4 porte sur la moyennabilité, qui du point de vue de la théorie des représentations ou des algèbres d'opérateurs, et aussi une forme forte de malléabilité. Ce mémoire ne contient pas de résultats nouveaux, à l'exception d'une nouvelle preuve du Théorème 2.3, de la partie 4.4, de la proposition 5.7 et de l'annonce de l'inégalité (1.7) et de la remarque 5.16.

Les trois premiers chapitres sont très liés, et traitent de rigidité/malléabilité dans un cadre linéaire, et plus précisément dans un cadre d'algèbres d'opérateurs, d'analyse harmonique et de représentations de groupes sur des espaces de Banach.

Le premier chapitre, de nature introductive, présente une preuve détaillée de la propriété (T) pour $\mathrm{SL}(3, \mathbf{F})$ pour un corps local $\mathbf{F}$. Il n'y a pas vraiment de contribution personnelle dans cette partie, qui est essentiellement une réécriture de la preuve élaborée par Vincent Lafforgue dans son travail sur la propriété (T) renforcée. La raison pour laquelle je présente cette preuve détaillée est qu'une partie importante des mes travaux [8, 10, 6, 7, 5] (présentés dans les deuxième et troisième chapitres) repose sur des idées semblables dans un cadre différent que les représentations unitaires, ou bien pour d'autres groupes que $\mathrm{SL}(3, \mathbf{F})$.

Le deuxième chapitre traite de multiplicateurs et de propriétés d'approximation pour les groupes et les algèbres d'opérateurs associées. Dans l'aspect rigidité, je présente mes travaux [8, 7] avec Lafforgue et de Laat où je démontre que $\mathrm{SL}(n, \mathbf{F})$ pour $n \geq 3$ et ses réseaux n'ont pas la propriété AP de Haagerup et Kraus, ni des versions L_p de ces propriétés qui deviennent de plus en plus fortes lorsque n devient grand. J'ai tenu à donner quelques idées sur les preuves, et en particulier à discuter les liens et les différences avec le contenu du premier chapitre. Dans l'aspect malléabilité, je présente mon travail avec Mei sur les multiplicateurs de Herz-Schur radiaux sur les groupes hyperboliques et sur les semigroupes de la chaleur sur leurs algèbres de von Neumann [9]. Je présente aussi mon travail [1] avec Caspers sur les liens entre les versions L_p des multiplicateurs de Fourier et de Schur, et les motivations de type "malléabilité en rang supérieur" pour ce travail.

Le troisième chapitre traite d'aspects de rigidité pour les représentations de groupes sur les espaces de Banach. Il porte sur les différentes versions banachiques de la propriété (T). J'y présente quelques aspects de mon travail récent sur les projections de Kazhdan et les liens entre ces différentes propriétés [11], et surtout mes travaux [10, 6, 7, 5] seul ou avec de Laat et Mimura sur la propriété (T) renforcée banachique pour les groupes de Lie de rang supérieur. Là encore, j'explique les difficultés dans les preuves en comparant avec le contenu du premier chapitre.

Le quatrième chapitre ne porte que sur l'aspect malléabilité. C'est une présentation des mes travaux sur la moyennabilité des groupes discrets et sur ce qu'on appelle les actions *extensivement moyennables* [4, 3, 2] avec Juschenko, Matte Bon, Monod et Nekrashevych. Il contient quelques résultats nouveaux, en particulier un résultat qui unifie les approches de [3] et [2].

Le cinquième et dernier chapitre porte sur mes travaux avec Tessera [12, 13] sur les aspects de malléabilité et de rigidité pour les graphes de Cayley de groupes de présentation finie, et plus généralement pour les graphes transitifs simplement connexes à grande échelle. J'y étudie l'espace topologique de tous les tels graphes, en essayant de comprendre les points isolés (les graphes “rigides”) et les points non isolés (les graphes “malléables”).

English

This report presents a large part of my research after my Phd, that is for the period october 2009-march 2016. This report has five chapters, which all deal with different aspects of rigidity and malleability for groups. The common question is to understand to what extend certain mathematical objects associated to a group can be non-trivially deformed or approximated. We will say that we are in a malleable situation when this is possible, and in a rigid situation otherwise. Depending on the chapters, the objects will be unitary representations, (chapter 1), Banach space representations (chapter 3), operator algebras and noncommutative L_p spaces (chapter 2), Cayley graphs (chapter 5). Chapter 4 deals with amenability, which from the representation theoretical or operator algebraic point of view is also a form of malleability. This text does not contain new results, with the exception of a new proof of theorem 2.3, of the section 4.4, of proposition 5.7 and of the statement of inequality (1.7) and of remark 5.16.

The first three chapters are very related, and deal with rigidity/malleability in a linear setting, and more precisely in a setting of operator algebras, harmonic analysis and Banach space representations.

The first expository chapter presents a detailed proof of property (T) for $\mathrm{SL}(3, \mathbf{F})$ for a local field $\mathbf{F}$. There is no real personal contribution in this chapter, which is essentially a rewriting of the proof developed by Vincent Lafforgue in his work on strong property (T). The reason why I present this detailed proof is because an important part of my work [8, 10, 6, 7, 5] (presented in the second and third chapters) relies on similar ideas in a different setting than unitary representations, and for other groups than $\mathrm{SL}(3, \mathbf{F})$.

The second chapter deals with multipliers and approximation properties for groups and the associated operator algebras. In the rigidity aspect, I present my work [8, 7] with Lafforgue and de Laat where I show that $\mathrm{SL}(n, \mathbf{F})$ for $n \geq 3$ and its lattices do not have the Approximation Property of Haagerup and Kraus, neither an L_p version of these properties that becomes increasingly stronger when n is large. I give some ideas about the proofs, and in particular I discuss the relationships and differences with the content of the first chapter. In the malleability aspect, I present my work with Mei on radial Herz-Schur multipliers on hyperbolic groups and heat semigroups of their von Neumann algebras [9]. I also present my work [1] with Caspers on the relationship between L_p versions of Fourier and Schur multipliers, and “malleability in higher rank”-type motivations for this work.

The third chapter deals with rigidity aspects for group representations on Banach spaces. It covers the different Banach space versions of property (T). I present some aspects of my recent work on Kazhdan projections and the links between these different properties [11], and my work [10, 6, 7, 5] alone or with de Laat and Mimura on strong Banach property (T) for higher rank Lie groups. Again, I explain the difficulties in the proof by comparing to the content of the first chapter.

The fourth chapter deals only with the malleability aspect. This is a presentation of my work on the amenability of discrete groups and what we call *extensively amenable actions* [4, 3, 2] with Juschenko, Matte Bon, Monod and Nekrashevych. It contains some new results, in particular a result that unifies the approaches of [3] et [2].

The fifth and final chapter presents my work with Tessera [12, 13] on malleability and rigidity aspects for Cayley graphs of finitely presented groups, and more generally for large scale simply connected transitive graphs. I study the topological space of all such graphs, trying to understand the isolated points (the "rigid graphs") and non-isolated points (the "malleable graphs").

Chapter 1

Lafforgue's proof of Property (T) for SL_3

A topological group has Kazhdan's property (T) whenever the trivial representation is isolated in its unitary dual for the Fell topology. This means that whenever a unitary representation π of G on $\mathcal{H}$ has almost invariant vectors (*i.e.* there is a net ξ_i of unit vectors in H such that $\lim_i \|\pi(g)\xi_i - \xi_i\| = 0$ uniformly on compact subsets of G), it has a nonzero invariant vector.

The purpose of this introductory section is to give a detailed proof of Kazhdan's theorem [Kaz67] that $\mathrm{SL}_3(\mathbf{F})$ has property (T) for every local field $\mathbf{F}$. We do not give one of the classical proofs (which all rely in a way or another on the pair $\mathrm{SL}_2 \subset \mathrm{SL}_3$) but Lafforgue's recent proof [Laf08] (which relies on the pair $K \subset \mathrm{SL}_3$ of the maximal compact subgroup, for example $K = \mathrm{SO}_3$ for $\mathbf{F} = \mathbf{R}$).

There is not much personal contribution in this section : all ideas are taken from [Laf08] (where a stronger property than (T) is proved, see chapter 3), only the details have been rewritten in order to give a unified proof of the archimedean and non-archimedean case, and in order to insist (as in [HdL13]) on the role of the maximal compact subgroup and the associated Gelfand pairs. The reason why I make this presentation is because an important part of my research activity after my thesis [8, 10, 6, 7, 5] uses similar ideas, but for other groups or other settings that unitary representations. Hopefully, giving a detailed proof of the simplest case of unitary representations of SL_3 will help the reader understand the proofs and the difficulty that had to be overcome in my later work.

1.1 The setting and the main statement

Let $\mathbf{F}$ be a local field [1] (*i.e.* a field which is non-discrete and locally compact for an absolute value $|\cdot|$). From the very classical theory (see for example the first pages of [Wei74]), there are two cases:

1. $\mathbf{F}$ is archimedean, *i.e.* isomorphic to $\mathbf{R}$ or $\mathbf{C}$.
2. $\mathbf{F}$ is non-archimedean, *i.e.* isomorphic to a finite extension of $\mathbf{F}_p((t))$ or of $\mathbf{Q}_p$ for a prime number p, depending on whether $\mathbf{F}$ has characteristic > 0 or 0.

In the case $\mathbf{F}$ is non-archimedean, we will use the following notation. We will denote by

1. In this section fields are assumed to be commutative. This is only for convenience as every result is valid for non-commutative fields.

$|x|$ the usual absolute value [2] of x, by $\mathcal{O} = \{x \in \mathbf{F}, |x| \leq 1\}$ its ring of units and by $1/e$ a uniformizer [3], *i.e.* a generator of the unique maximal ideal $m = \{x \in \mathbf{F}, |x| < 1\}$ of $\mathcal{O}$. For example if $\mathbf{F} = \mathbf{Q}_p$ we have $\mathcal{O} = \mathbf{Z}_p$ and the natural choice of uniformizer is $1/e = p$. If $\mathbf{F} = \mathbf{F}_p((t))$ we have $\mathcal{O} = \mathbf{F}_p[[t]]$ and the natural choice of uniformizer is $1/e = t$.

In the case $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$, e will be the usual $e = \sum_{n\geq 0} \frac{1}{n!}$.

In this section we present Lafforgue's proof that $\mathrm{SL}_3(\mathbf{F})$ has property (T). We will treat both cases (1) and (2) simultaneously, at the cost of using the following somewhat unconventional notation. In case (2), for real numbers a, b the notation $[a, b]$ (with obvious adaptations for (a, b), $(a, b]$ or $[a, b)$) will denote $\{x \in \mathbf{Z}, a \leq x \leq b\}$ the *integer points in the segment between a and b.*

Let us fix some notation. We denote $G = \mathrm{SL}_3(\mathbf{F})$ and $K \subset G$ the maximal subgroup given by $\mathrm{SO}(3), \mathrm{SU}(3)$ or $\mathrm{SL}_3(\mathcal{O})$ depending on whether $\mathbf{F}$ is $\mathbf{R}, \mathbf{C}$ or non-archimedean. It is useful to regard K as the isometries of determinant 1 of $\mathbf{F}^3$ equipped with its natural norm ($\|x\| = (|x_1|^2 + |x_2|^2 + |x_3|^2)^{1/2}$ in the archimedean case and $\|x\| = \max(|x_1|, |x_2|, |x_3|)$ in the non-archimedean case). Then the KAK decomposition (or polar decomposition or singular value decomposition) says that every element of G can be written as a product $g = kak'$ for a diagonal matrix a with elements powers of e in non-increasing order. In other word, it identifies the double classes $K\backslash G/K$ with the *Weyl chamber* $\Lambda = \{(r, s, t) \in (-\infty, \infty)^3, r \geq s \geq t, r + s + t = 0\}$ via the identification of (r, s, t) with the class $KD(r, s, t)K$ of

$$D(r, s, t) = \begin{pmatrix} e^r & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^t \end{pmatrix}.$$

We insist for the last time that with our convention, Λ is a subset of $\mathbf{R}^3$ in the archimedean case, and of $\mathbf{Z}^3$ is the non-archimedean case. Since K is the group of isometries, if $g \in KD(r, s, t)K$ then g and $D(r, s, t)$ have the same norm $\|g\| = |e^r|$, and so have their inverses $\|g^{-1}\| = |e^{-t}|$. This characterizes the value of r and t (and hence of $s = -r - t$) directly from g. This gives a simple way of computing r, s, t, especially in the non-archimedean case where

$$\|g\| = \max_{i,j} |g_{i,j}|. \tag{1.1}$$

We also introduce the subgroup $U \subset K$ of block-diagonal matrices

$$U = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\} \cap K.$$

Theorem 1.1. *Let π be a unitary representation of G on a Hilbert space $\mathcal{H}$, and ξ, η be $\pi(K)$-invariant unit vectors. Then for every $g_1, g_2 \in G$*

$$|\langle \pi(g_1)\xi, \eta\rangle - \langle \pi(g_2)\xi, \eta\rangle| \leq 22|e|^{1/2} \min(\max(\|g_1\|, \|g_1^{-1}\|), \max(\|g_2\|, \|g_2^{-1}\|))^{-\frac{1}{2}}.$$

This theorem implies that G has property (T). Indeed let π be a unitary representation of G with almost invariant vectors. For $g \in G$ let A_g be the operator $\iint_{K\times K} \pi(kgk')dkdk'$. The theorem implies that

$$\|A_{g_1} - A_{g_2}\| \leq 22|e|^{1/2} \min(\max(\|g_1\|, \|g_1^{-1}\|), \max(\|g_2\|, \|g_2^{-1}\|))^{-\frac{1}{2}},$$

2. $|x|$ is the factor by which the Haar measure on the additive group of $\mathbf{F}$ is scaled under the multiplication by x. For example $|e|$ is the cardinal of the residue field of $\mathbf{F}$.

3. Please forgive the very unusual notation. Usually a uniformizer is denoted by π, but we avoid this because for us π will be a representation.

which implies (Cauchy criterion) that there exists $P \in B(H)$ such that $\lim_g \|A_g - P\|_{B(\mathcal{H})} = 0$. We claim that P is the orthogonal projection on the invariant vectors. This implies that G has property (T), because if π has almost invariant vectors then A_g has norm 1 for every g, and hence P also. In particular $P \neq 0$, *i.e.* π has nonzero invariant vectors. To prove the claim, first observe that it is clear that P has norm at most 1 and that P is the identity on invariant vectors. So all we have to show is that the image of P is made of invariant vectors. For every $\xi \in \mathcal{H}$ and $g \in G$ we have

$$\int_K \pi(kg)P\xi dk = \lim_{g' \to \infty} \int_K \pi(kg)A_{g'}\xi = \lim_{g' \to \infty} \int_K A_{gkg'}\xi = P\xi.$$

This equality expresses the vector $P\xi$ as an average of vectors of same norm $\pi(kg)P\xi$. By strict convexity of Hilbert spaces, we have that $\pi(kg)P\xi = P\xi$ for every k, in particular $P\xi$ is $\pi(g)$-invariant, *quod est demonstrandum.*

1.2 Proof of Theorem 1.1

For $\delta \in \mathbf{F}$ and $|\delta| \leq 1$ we introduce the following matrix $k_\delta \in K$ with entry $(1,1)$ equal to δ :

$$k_\delta = \begin{pmatrix} \delta & -\sqrt{1-|\delta|^2} & 0 \\ \sqrt{1-|\delta|^2} & \overline{\delta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ if } \mathbf{F} = \mathbf{R} \text{ or } \mathbf{C},$$

$$k_\delta = \begin{pmatrix} \delta & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ if } \mathbf{F} \text{ is non-archimedean.}$$

Proposition 1.2. *For every unitary representation π of K on a Hilbert space $\mathcal{H}$ and every $\pi(U)$-invariant unit vectors $\xi, \eta \in \mathcal{H}$ we have*

$$|\langle \pi(k_\delta)\xi, \eta\rangle - \langle \pi(k_0)\xi, \eta\rangle| \leq 2\sqrt{|\delta|}.$$

Proof. The proof is very different depending on whether $\mathbf{F}$ is archimedean or not, and is postponed to §1.3 and §1.4. ☐

Let us deduce Theorem 1.1. Fix π, $\mathcal{H}$, ξ, η as in Theorem 1.1. Then the matrix coefficient $\langle \pi(g)\xi, \eta\rangle$ is K-biinvariant :

$$\langle \pi(kgk')\xi, \eta\rangle = \langle \pi(g)\xi, \eta\rangle \text{ for all } k, k' \text{ in K.}$$

It therefore defines a function on Λ, that we denote by c :

$$c(r,s,t) = \langle \pi(g)\xi, \eta\rangle \text{ for all } g \in KD(r,s,t)K.$$

For every $\alpha \in [0, \infty)$ consider the following matrix :

$$D_\alpha = \begin{pmatrix} e^{2\alpha} & 0 & 0 \\ 0 & e^{-\alpha} & 0 \\ 0 & 0 & e^{-\alpha} \end{pmatrix}.$$

Since D_α commutes with every element of U, the coefficient $k \in K \mapsto \langle \pi(D_\alpha k D_\alpha)\xi, \eta\rangle$ is a coefficient of a unitary representation of K with respect to U-invariant unit vectors $\pi(D_\alpha)\xi$ and $\pi(D_\alpha^{-1})\eta$. We can apply Proposition 1.2, but before that we compute the A-part in the KAK decomposition of $D_\alpha k_\delta D_\alpha$.

Lemma 1.3. *For every $r \in [\alpha, 4\alpha]$ there is $\delta \in \mathbf{F}$ such that $|\delta| \leq |e^{r-4\alpha}| \leq 1$ and*

$$D_\alpha k_\delta D_\alpha \in KD(r, 2\alpha - r, -2\alpha)K.$$

Proof. For $\delta \neq 0$, $g = D_\alpha k_\delta D_\alpha$ is block diagonal with one eigenvalue $e^{-2\alpha}$ and another block of the form DkD for $D = \mathrm{diag}(e^{2\alpha}, e^{-\alpha})$ and k an isometry. In particular $\|g^{-1}\| = |e^{2\alpha}|$. If we define $r_\alpha(\delta) \in [0, \infty)$ by $\|g\| = |e^{r_\alpha(\delta)}|$ we therefore have that $g \in KD(r_\alpha(\delta), 2\alpha - r_\alpha(\delta), -2\alpha)K$. By saying that the norm of g is larger that the absolute value of its $(1,1)$ entry we get the desired inequality $|\delta e^{4\alpha}| \leq |e^{r_\alpha(\delta)}|$. It remains to show that r_α is surjective. In the archimedean case, r_α is continuous on the interval $[0,1]$ so its image contains the interval $[r_\alpha(0), r_\alpha(1)] = [\alpha, 4\alpha]$. The non-archimedan case is obvious because by (1.1) we have the exact formula $|e^{r_\alpha(\delta)}| = \max(|e^\alpha|, |\delta e^{4\alpha}|)$, which implies that $r_\alpha(e^{-n}) = 4\alpha - n$ for all $n \in [0, 3\alpha]$ and in particular that r_α is surjective. $\square$

By Proposition 1.2 we therefore get

$$|c(\alpha, \alpha, -2\alpha) - c(r, 2\alpha - r, -2\alpha)| \leq 2|\delta|^{1/2} \leq 2|e|^{r/2 - 2\alpha}.$$

In particular for every $(r,s,t), (r', s', t') \in \Lambda$, if there exists α such that $t = t' = -2\alpha$ we have that,

$$|c(r,s,t) - c(r', s', t')| \leq 2(|e|^{r/2+t} + |e|^{r'/2+t'}). \tag{1.2}$$

We now prove the same inequality in the case when $t = t'$ cannot be written as $t = t' = -2\alpha$. This is the case when $\mathbf{F}$ is non-archimedean and t is odd, *i.e.* can be written $t = t' = 1 - 2\alpha$. Then consider the matrices

$$D'_\alpha = \begin{pmatrix} e^{2\alpha-1} & 0 & 0 \\ 0 & e^{-\alpha} & 0 \\ 0 & 0 & e^{-\alpha} \end{pmatrix}, D''_\alpha \begin{pmatrix} e^{2\alpha-1} & 0 & 0 \\ 0 & e^{-\alpha+1} & 0 \\ 0 & 0 & e^{-\alpha+1} \end{pmatrix}.$$

Observe that D'_α, D''_α have determinant e^{-1}, e. As in the previous lemma we can compute that $g = D'_\alpha k_\delta D''_\alpha$ satisfies $\|g^{-1}\| = |e^{-t}|$ and $\|g\| = \max(|\delta e^{4\alpha-2}|, |e^\alpha|)$, so that for every $(r,s,t) \in \Lambda$ there is $\delta = e^{r+2t}$ such that $D'_\alpha k_\delta D''_\alpha \in KD(r,s,t)K$. Although D'_α, D''_α do not have determinant 1, it is still true that $k \mapsto \langle \pi(D'_\alpha k D''_\alpha)\xi, \eta\rangle$ is a U-biinvariant coefficient of K : if u_0 is the (non-inner) automorphism of G given by

$$u_0(g) = \begin{pmatrix} e^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} g \begin{pmatrix} e & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.3}$$

and π_0 is the representation $\pi \circ u_0$ of G, then $\pi(D'_\alpha g D''_\alpha) = \pi_0(D_\alpha g D_{\alpha-1})$. Therefore by Proposition 1.2 we get (1.2) also in that case.

Notice that if $s \geq -1$ we have $r/2 + t = (r+s+t)/2 + t/2 - s/2 \leq (t+1)/2$. Therefore (1.2) implies

$$|c(r,s,t) - c(r', s', t')| \leq 4\sqrt{|e|}|e|^{t/2} \text{ if } t = t' \text{ and } s, s' \geq -1. \tag{1.4}$$

By applying this to the representation $g \mapsto \pi((g^t)^{-1})$ we get that

$$|c(r,s,t) - c(r', s', t')| \leq 4\sqrt{|e|}|e|^{-r/2} \text{ if } r = r' \text{ and } s, s' \leq 1. \tag{1.5}$$

In particular if $c_r = c(r, 0, -r)$ and $1 \leq r_1 \leq r_2 \leq r_1 + 1$,

$$\begin{aligned} |c_{r_2} - c_{r_1}| &\leq |c_{r_2} - c(r_2, r_1 - r_2, -r_1)| + |c(r_2, r_1 - r_2, -r_1) - c_{r_1}| \\ &\leq 4\sqrt{|e|}(|e|^{-r_2/2} + |e|^{-r_1/2}). \end{aligned}$$

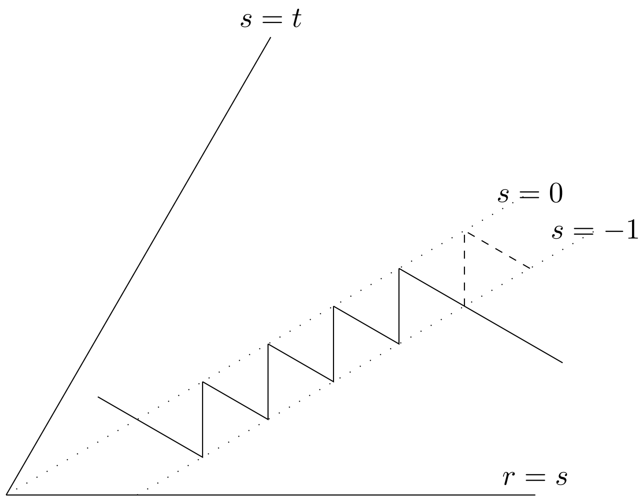

Figure 1.1 – The zig-zag path in the Weyl chamber Λ.

This implies (since $\sum_{k\geq 0} |e|^{-k/2} \leq 3.5$) that for every $r, r' \geq 1$,

$$|c_r - c_{r'}| \leq 14\sqrt{|e|} \max(e^{-r/2}, e^{-r'/2}).$$

Since $14 + 4 + 4 = 22$, it follows easily from the above estimates that

$$|c(r,s,t) - c(r',s',t')| \leq 22\sqrt{|e|} \max(\min(e^{-r/2}, e^{t/2}), \min(e^{-r'/2}, e^{t'/2})),$$

which is exactly Theorem 1.1. The previous computations are best understood on a picture (see Figure 1.1) : (1.4) expresses that c is almost constant on lines of slope $-\frac{1}{2}$ in the region $s \geq -1$, whereas (1.5) expresses that c is almost constant on vertical lines in the region $s \leq 0$. These estimates are combined by the zig-zag path in Figure 1.1.

We end this section by a remark that the reduction from Proposition 1.2 to Theorem 1.1 was very combinatorial in nature. For example, it would work with minor changes if π was not a unitary representation but merely a representation whose restriction to K is unitary and such that $\|\pi(g)\|$ does not grow too fast. Also, the argument would have worked identically if in Proposition 1.2, $|\delta|^{1/2}$ was replaced by $|\delta|^{\theta}$ for some $\theta > 0$. We record here the output of the method.

For $\theta > 0$ denote $\mathcal{C}_\theta$ the set of all U-biinvariant functions $\varphi \colon K \to \mathbf{C}$ such that $|\varphi(k_\delta) - \varphi(k_0)| \leq |\delta|^\theta$.

Proposition 1.4. *Let $\varphi \colon G \to \mathbf{C}$ be a K-biinvariant function and $C > 0$, $\gamma < \theta$ such that the map $k \mapsto \varphi \circ u(D_\alpha k D_\beta)$ belongs to $C|e|^{\gamma(\alpha+\beta)}\mathcal{C}_\theta$ for every $\alpha, \beta \in [0,\infty)$ and $u \in \{id, g \mapsto (g^t)^{-1}, u_0\}$* [4]*. Then there exists $l \in \mathbf{C}$ such that*

$$|\varphi(g) - l| \leq C' \max(\|g\|, \|g^{-1}\|)^{-\theta-\gamma}$$

for some C' depending on C, γ.

A particular case is:

Proposition 1.5. *Let $\gamma < \theta$. Let E be a Banach space of functions on G such that the restriction to K of every norm 1 U-biinvariant function $\varphi \in E$ belongs to $\mathcal{C}_\theta$. Assume also*

4. where, if $\mathbf{F}$ is non-archimedean, u_0 is given by (1.3) and otherwise u_0 is the identity.

that E is stable by automorphisms of G and by translation as follows : for every $g_1, g_2 \in G$, $\|\varphi(g_1 \cdot g_1)\|_E \le C\|g_1\|^\gamma \|g_2\|^\gamma \|\varphi\|_E$ for some C independant from g_1, g_2.

Then there exists $l \in \mathbf{C}$ such that for every K-biinvariant function $\varphi \in E$

$$|\varphi(g) - l| \le C' \max(\|g\|, \|g^{-1}\|)^{-\theta-\gamma}\|\varphi\|_E$$

for some C' depending only on E.

1.3 Proof of Proposition 1.2, real case

The orginal proof (for a different constant than 2) for the case of $\mathbf{R}$ is from [Laf08], and a proof with 2 replaced by 4 can be found in [HdL13]. Both proofs go as follows : by the Peter-Weyl theorem it is enough to prove the lemma for the irreducible representations of SO(3). For the n-th irreducible representation of SO(3) the quantity $\langle \pi(k_\delta)\xi, \eta\rangle$ is equal to $\pm P_n(\delta)$, the value at δ of the n-th Legendre polynomial. So we have to prove that $\sup_n |P_n(\delta) - P_n(0)| \le 2\sqrt{|\delta|}$. By bounding

$$|P_n(\delta) - P_n(0)| \le \min(|P_n(0)| + |P_n(\delta)|, |\delta| \max_{t\in[0,1]} |P_n'(t\delta)|)$$

and using the Bernstein inequality $|P_n(x)| \le \min(1, \sqrt{\frac{2}{\pi n}}(1-x^2)^{-\frac{1}{4}})$ [Sze75, Theorem 7.3.3] and the formula

$$(1-x^2)P_n'(x) = -nxP_n(x) + nP_{n-1}(x)$$

expressing P_n' in terms of P_n and P_{n-1} one deduces the proposition with constant 2 (and actually with the constant $\sqrt{\frac{2}{\pi}} + o_{x\to 0}(1)$). The case of $\mathbf{C}$ when $\delta \in [-1,1]$ follows from the case of $\mathbf{R}$ by seeing SO_3 as a subgroup of SU_3. The general case of $\delta \in \mathbf{C} \setminus [-1,1]$ follows from the case of $\delta \in [-1,1]$ by writing, if θ is an argument of δ,

$$k_\delta = \begin{pmatrix} e^{i\theta/2} & 0 & 0 \\ 0 & e^{-i\theta/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} k_{|\delta|} \begin{pmatrix} e^{i\theta/2} & 0 & 0 \\ 0 & e^{-i\theta/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and noticing that the matrix $\begin{pmatrix} e^{i\theta/2} & 0 & 0 \\ 0 & e^{-i\theta/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ normalizes U.

For further use, we can rephrase the preceding in terms of the operators $T_\delta \colon L_2(\mathrm{SO}(3)) \to L_2(\mathrm{SO}(3))$ defined by

$$T_\delta(k) = \iint_{U\times U} f(uk_\delta^{-1}u'k)dudu'.$$

Alternatively, the operators can be regarded as operators on $L_2(\mathrm{SO}(2)\backslash\mathrm{SO}(3)) = L_2(S^2)$, and then for a function $f \in L_2(S^2)$ they satisfy that $T_\delta f(x)$ is the average of f on the circle $\{y \in S^2 | \langle x, y\rangle = \delta\}$.

The preceding computations show that $\|T_\delta - T_0\| \le 2\sqrt{|\delta|}$. In fact, using that the multiplicity in $L_2(\mathrm{SO}(3))$ of the n-th irreducible representation of SO(3) is equal to $(2n+1)$, we get that there is a constant C such that for every $p > 4$ and $|\delta|, |\delta'| \le \frac{1}{2}$,

$$\begin{aligned} \|T_\delta - T_0\|_{S_p}^p &= \sum_{n\ge 1}(2n+1)|P_n(\delta) - P_n(0)|^p \\ &\le \frac{C^p}{(p-4)}|\delta|^{\frac{p}{2}-2}. \end{aligned}$$

We therefore get that

$$\|T_\delta - T_0\|_{S^p} \leq \frac{C}{(p-4)^{1/p}} |\delta|^{\frac{1}{2}-\frac{2}{p}}. \tag{1.6}$$

A finer analysis of the asymptotic behaviour of Legendre polynomials could show that

$$\|T_\delta - T_{\delta'}\|_{S^p} \geq \frac{C^{-1}}{(p-4)^{1/p}} |\delta - \delta'|^{\frac{1}{2}-\frac{2}{p}}. \tag{1.7}$$

for every $\delta, \delta' \in [-1/2, 1/2]$.

1.4 Proof of Proposition 1.2, non-archimedean case

In this part we assume that $\mathbf{F}$ is non-archimedean, and we denote by q the cardinal of the residue field $\mathcal{O}/e^{-1}\mathcal{O}$. Since $\mathcal{O}$ is a compact abelian group, it carries a Haar probability measure $\mathbb{P}$. Let $\widehat{\mathcal{O}}$ be the group of characters of $\mathcal{O}$, *i.e.* the group of continuous homomorphisms from $\mathcal{O}$ to the group of complex numbers of modulus one. By Pontryagin duality, the Fourier transform induces an isometry $\mathbf{F}\colon L_2(\mathcal{O}) \to \ell_2(\widehat{\mathcal{O}})$ given by

$$\mathcal{F}f(\chi) = \mathbb{E}_b \overline{\chi}(b) f(b)$$

and inverse

$$\mathcal{F}^{-1}f(a) = \sum_\chi \chi(a) g(a).$$

We will need the following elementary fact on characters of $\mathcal{O}$. If $\chi \in \widehat{\mathcal{O}}$, then for every integer d, $e^{-d}\mathcal{O}$ is a subgroup of $\mathcal{O}$, its image $\chi(e^{-d}\mathcal{O})$ is therefore a subgroup of $\mathbb{T}$. By continuity of χ, there is therefore a smallest $d = D_\chi$ such that $\chi(e^{-D_\chi}\mathcal{O}) = \{1\}$. Moreover, the characters such that $D_\chi \leq D$ are in bijections with the characters of the group $\mathcal{O}/e^{-D}\mathcal{O}$ and hence are of cardinality q^D.

For every $\delta \in \mathbf{F}$ with $|\delta| < 1$ we define $T_\delta\colon L_2(\mathcal{O}^2) \to L_2(\mathcal{O}^2)$ by

$$T_\delta f(a,b) = \mathbb{E}_x f(x, ax+b+\delta).$$

It is easy to see that T_δ has norm 1.

We shall show the estimate

$$\|T_0 - T_\delta\|_{B(L_2(\mathcal{O}^2))} \leq 2\sqrt{|\delta|}. \tag{1.8}$$

As an immediate consequence, if $\mathcal{H}$ is a Hilbert space and $\xi \in L_2(\mathcal{O}^2;\mathcal{H})$ and $\eta \in L_2(\mathcal{O}^2;\mathcal{H})$ are such that

$$\langle \xi(x,y), \eta(a,b)\rangle = \begin{cases} \lambda & \text{if } y = ax+b+\delta \\ \mu & \text{if } y = ax+b \end{cases}$$

then

$$|\lambda - \mu| = |\langle (T_\delta \otimes Id_\mathcal{H})\xi, \eta\rangle - \langle (T_0 \otimes Id_\mathcal{H})\xi, \eta\rangle| \leq 2|\delta|^{1/2} \|\xi\|_{L_2(\mathcal{O}^2;\mathcal{H})} \|\eta\|_{L_2(\mathcal{O}^2;\mathcal{H})}.$$

In particular if for $a, b, x, y \in \mathcal{O}$ we define matrices in K

$$\alpha(a,b) = \begin{pmatrix} 1 & -a & -b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; \beta(x,y) = \begin{pmatrix} y & 0 & -1 \\ x & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then

$$\alpha(a,b)\beta(x,y) = \begin{pmatrix} y-ax-b & -a & -1 \\ x & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

If $y - ax - b = \delta$, we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & x \end{pmatrix} \alpha(a,b)\beta(x,y) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & a \end{pmatrix} = k_\delta.$$

Therefore if we keep the notation from Proposition 1.2 and apply the preceding to $\xi(x,y) = \pi(\beta(x,y))\xi$ and $\eta(a,b) = \pi(\alpha(a,b)^{-1})\eta$ then we get exactly the conclusion of Proposition 1.2.

We are left to prove (1.8). This is done by an explicit diagonalization of the matrix T_δ. We can first compute that the operator $(1 \otimes \mathcal{F}^{-1})T_\delta(1 \otimes \mathcal{F})$ has the following form. For every $f \in L_2(\mathcal{O}) \otimes \ell_2(\widehat{\mathcal{O}})$ and $(a,\chi) \in \mathcal{O} \times \widehat{\mathcal{O}}$,

$$\begin{aligned}\Big((1 \otimes \mathcal{F}^{-1})T_\delta(1 \otimes \mathcal{F})f\Big)(a,\chi) &= \mathbb{E}_b\overline{\chi}(b)\mathbb{E}_x \sum_{\chi'} \chi'(ax+b+\delta)f(x,\chi') \\ &= \mathbb{E}_x \sum_{\chi'} \chi'(ax+\delta)f(x,\chi')\mathbb{E}_b\overline{\chi}(b)\chi'(b) \\ &= \chi(\delta)\mathbb{E}_x\chi(ax)f(x,\chi).\end{aligned}$$

Therefore, if $A_\chi \colon L_2(\mathcal{O}) \to L_2(\mathcal{O})$ is the operator $A_\chi f(a) = \mathbb{E}_x\chi(ax)f(x)$ then T_δ is unitarily equivalent to the direct sum over χ of $\chi(\delta)A_\delta$. In particular

$$\|T_\delta - T_0\| = \sup_{\chi \in \mathcal{O}} |\chi(\delta) - 1|\|A_\chi\|.$$

If we compute $A_\chi^* A_\chi$ we get

$$A_\chi^* A_\chi = (\mathbb{E}_x\overline{\chi}(ax)\chi(bx))_{a,b} = (1_{a-b\in e^{-D_\chi}\mathcal{O}})_{a,b},$$

from which we get that $q^{D_\chi}A_\chi^*A_\chi$ is a direct sum of q^{D_χ} rank one projections. In particular $\|A_\chi\| = q^{-D_\chi/2}$, and more generally $Tr(|A_\chi|^p) = q^{D_\chi(1-p/2)}$ for every $p > 0$. On the other hand, let D be the biggest integer such that $\delta \in e^{-D}\mathcal{O}$. Then $|\delta| = q^{-D}$, and $\chi(\delta) \neq 1$ only if $D > D_\chi$, so that we get

$$\|T_\delta - T_0\| \leq 2 \sup_{D_\chi > D} q^{-D_\chi/2} \leq 2\sqrt{|\delta|}.$$

This proves (1.8). Remark that we can also compute for every $p > 4$,

$$\begin{aligned} Tr(|T_\delta - T_0|^p) &\leq 2^p \sum_{\chi, D_\chi > D} \|A_\chi\|_p^p \\ &= 2^p \sum_{\chi, D_\chi > D} q^{D_\chi - pD_\chi/2} \\ &= 2^p \sum_{D' > D} q^{2D' - pD'/2} \\ &= \frac{2^p}{1 - q^{2-p/2}} |\delta|^{p(\frac{1}{2} - \frac{2}{p})}. \end{aligned}$$

This shows that $T_0 - T_\delta \in S_p$ if and only if $p > 4$ and that there exists $C(q) > 0$ such that

$$\frac{C(q)}{(p-4)^{1/p}} |\delta|^{\frac{1}{2} - \frac{2}{p}} \leq \|T_0 - T_\delta\|_{S_p} \leq \frac{C(q)}{(p-4)^{1/p}} |\delta|^{\frac{1}{2} - \frac{2}{p}}. \tag{1.9}$$

1.5 Comments on the proofs

Logically, the proof of Theorem 1.1 decomposes in two disctinct steps. The first step, Proposition 1.2, is analytic, and deals with harmonic analysis on a compact group, and more precisely on the pair $(U \subset K)$ of compact groups. The second step, the reduction from Proposition 1.2 to Theorem 1.1 is geometric/combinatorial. For example in the real case, what happens is that one studies the various embeddings of the sphere S^2 (identified with $K/U = \mathrm{SO}(3)/\mathrm{SO}(2)$) into the symmetric space $G/K = \mathrm{SL}_3(\mathbf{R})/\mathrm{SO}(3)$. The relative position between a pair of such embeddings gives rise to an embedding of $U\backslash K/U = [-1,1]$ inside the Weyl chamber $K\backslash G/K$ (the segments in Figure 1.1). Combining these embeddings allows to explore the whole Weyl chamber of $\mathrm{SL}(3,\mathbf{R})$. The crucial Lemma 1.3 expresses that this embedding is exponentially distorted in the distance to the origin in the Weyl chamber, and hence that this exploration allows to escape to infinity *in finite time*. If one makes similar computations for rank 1 simple Lie groups G which contain a subgroup isomorphic to $\mathrm{SO}(3)$ (for example for $\mathrm{SO}(3,1)$), one gets also lots of embeddings of $[-1,1]$ inside the Weyl chamber $[0,\infty)$ of G, and also enough to explore the whole Weyl chamber. The difference (and the reason why this does not create a contradiction by proving that $\mathrm{SO}(3,1)$ has property (T)!) is that these embeddings are almost isometric, and so it takes an infinite time to explore the whole Weyl chamber.

The fact that all the analysis is done at the level of the compact groups U, K is very important for my work presented in the following two chapters, because harmonic analysis for compact groups is much better understood than for arbitrary groups (see for example the very easy results in Lemma 2.4 and Proposition 3.8). This will allow to use a similar approach for other objects than coefficients of unitary representations, and to prove rigidity results in various other linear settings. A project I have is to try to use similar ideas in a non-linear setting, to understand actions of higher rank groups on low dimensional manifolds.

Chapter 2

Multipliers, Approximation properties for groups and operator algebras

This chapter presents my work on Fourier multipliers and approximation properties [8, 1, 9, 7].

2.1 Herz-Schur multipliers

2.1.1 Definitions

Let G be a locally compact group with a Haar measure. Its left-regular representation λ is the representation on $L_2(G)$ by left translation :

$$\lambda(g)f(h) = f(g^{-1}h).$$

The von Neumann algebra $\mathcal{L}(G)$ is the bicommutant $\lambda(G)'' \subset B(L_2(G))$, or equivalently the weak-* closure of the linear span of $\lambda(G)$ by the bicommutant theorem. According to a celebrated result of Bożejko and Fendler [BF84], for a function $\varphi\colon G \to \mathbf{C}$ the following conditions are equivalent :

1. $\lambda(g) \mapsto \varphi(g)\lambda(g)$ extends to a weak-* continuous completely bounded map on $\mathcal{L}(G)$.
2. $T = (T_{s,t})_{s,t\in G} \in S_2(L_2(G)) \mapsto (\varphi(st^{-1})T_{s,t})_{s,t}$ extends to a weak-* continuous bounded map on $B(L_2(G))$.
3. there are bounded continuous functions ξ, η from G to a Hilbert space H and such that $\varphi(st^{-1}) = \langle \xi_t, \eta_s\rangle$ for all $s, t \in G$.

If these properties hold, φ is a called Herz-Schur multiplier, and the cb norm of $\lambda(g) \mapsto \varphi(g)\lambda(g)$ is equal to norm (and the cb norm) of $T \mapsto (\varphi(st^{-1})T_{s,t})$ and to the infimum of $\sup_s \|\eta_s\| \sup_t \|\xi_t\|$ over all ξ, η satisfying $\varphi(st^{-1}) = \langle \xi_t, \eta_s\rangle$. This common quantity is denoted by $\|\varphi\|_{M_0A(G)}$. The map $\lambda(g) \mapsto \varphi(g)\lambda(g)$ is denoted m_φ and called the Fourier multiplier with symbol φ. The map of $T \mapsto (\varphi(st^{-1})T_{s,t})$ is denoted M_φ.

The reader not familiar with operator algebras and completely bounded maps can read this section by taking 3 as a definition of the elements of $M_0A(G)$, and accept that by its connection 1 with von Neumann algebras, this is an interesting object.

Remark 2.1. The reason for this notation is that the notion of Herz-Schur multiplier coincides with the notion of completely bounded multipliers of the Fourier algebra $A(G)$

of G. Recall that by regarding an element f in the predual of $\mathcal{L}(G)$ as the function $g \mapsto f(\lambda(g))$, one realizes $\mathcal{L}(G)_*$ as a space of continuous functions on G which coincides with $A(G)$, the set of coefficients of the left-regular representation λ of G on $L_2(G)$:

$$A(G) = \{g \mapsto \langle \lambda(g)\xi, \eta\rangle, \xi_n, \eta_n \in L_2(G)\}.$$

The identification of the set of coefficients of λ with $\mathcal{L}(G)_*$ implies that this set is a vector space and that

$$\|f\| = \inf\{\|\xi\|\|\eta\|, f = \langle \lambda(\cdot)\xi, \eta\rangle\}$$

is a norm (note that these two facts are not true if one replaces $L_2(G)$ by $L_p(G)$ for $p \neq 2$).

The preadjoint of m_φ is then the linear map of pointwise multiplication by φ, and (by definition of the operator space structure on $A(G)$) the cb norm of m_φ coincides with the cb norm of the multiplication by φ.

Since $\|\varphi\|_{M_0A(G)} \geq \|\varphi\|_\infty$, every $f \in L_1(G)$ defines a bounded linear form on $M_0A(G)$ by $\langle f, \varphi\rangle = \int f\varphi$. The closure of $L_1(G)$ in $M_0A(G)$ is denoted by $Q(G)$. It is easy to show [HK94] that $Q(G)^*$ identifies naturally with $M_0A(G)$. In particular this defines a weak-* topology on $M_0A(G)$.

Definition 2.2. [HK94] G has the Approximation Property of Haagerup and Kraus (or AP) if the constant function equal to 1 belongs to the weak-* closure of $A(G)$.

So G has the AP if there is a net $\varphi_i \in A(G)$ which converges weak-* to 1. Weak amenability corresponds to the particular case when the net is taken to be bounded in $M_0A(G)$ (and by the uniform boundedness principle this happens if the net is a sequence), and the best bound on $\sup_i \|\varphi_i\|_{M_0A(G)}$ is called the weak amenability constant (or Cowling-Haagerup constant) of G and denoted $\Lambda(G)$. So a weakly amenable group has the AP, but not conversely (for example, $\mathrm{SL}(2, \mathbf{Z}) \rtimes \mathbf{Z}^2$ has the AP because AP is stable by semidirect product [HK94] but is not weakly amenable [Haa86], see also [Oza12]).

A motivation for this definition is the following two theorems from [HK94], similar to (but more difficult than) parallel results for weak amenability and completely bounded approximation property [Haa86]) :

— If $\Gamma \subset G$ is a lattice, then Γ has the AP if and only if G has.

— If Γ is a discrete group, then Γ has the AP if and only if its reduced C^*-algebra has the operator space approximation property, if and only if its reduced C^*-algebra has the strong operator space approximation property if and only if its von Neumann algebra has the weak-* operator space approximation property etc.

2.1.2 Groups without the AP

The Cowling-Haagerup constant for connected real simple Lie groups has been computed : it is 1 for compact groups and groups locally isomorphic to $\mathrm{SO}(n, 1)$, $\mathrm{SU}(n, 1)$ [DCH85], $2n - 1$ for $\mathrm{Sp}(n, 1)$, 21 for $F_{4(-20)}$ [CH89] and ∞ for groups of real rank at least 2 [Haa86].

The problem of finding groups without the AP took more time to be settled. The first examples came with the first examples by Gromov of non-exact discrete groups, because AP implies exactness, see [BO08]. Then the main question was whether there exist exact groups without the AP, and specifically whether $\mathrm{SL}_3(\mathbf{Z})$ (or equivalently $\mathrm{SL}_3(\mathbf{R})$) lacks the AP, as conjectured by Haagerup and Kraus [HK94]. The reason why this question remained open was that the traditional tools to study representation theory of higher rank Lie groups failed to tackle the AP question, for example because $\mathrm{SL}(2,\mathbf{R}) \ltimes \mathbf{R}^2$ has the AP.

Vincent Lafforgue and I proved this conjecture in [8], elaborating on the techniques presented in Chapter 1. I present below a very simple proof of this result. By [HdL13, HdL16, Lia15] the same result is true for every connected simple Lie group of real rank at least 2 (real case) and every almost simple algebraic group with split rank at least 2 (non-archimedean case).

The original proof was much less direct. It relied on L_p multipliers and involved non-trivial operator space theory, see Section 2.2. A significantly simpler proof, based on the Krein–Schmulian theorem, was obtained in [HdL13]. I found the even simpler proof presented here while preparing a lecture series at the winter school on Isomorphism Conjectures and Geometry of Groups, held in Münster in january 2016.

Theorem 2.3 ([8]). *Let $\mathbf{F}$ be a local field. Then $\mathrm{SL}(3,\mathbf{F})$ does not have AP.*

The theorem will be an easy consequence of Chapter 1 and of the following well-known easy lemma, which expresses that, for compact groups, a converse holds to the general fact that every coefficient of a unitary representation of G (or more generally of a uniformly bounded representation of G) belongs to $M_0A(G)$.

Lemma 2.4. *If K is compact, then $M_0A(K) = A(K)$ with equal norms.*

Proof. We only have to prove the norm 1 inclusion $M_0A(K) \subset A(K)$. Let φ belong to the unit ball of $M_0A(K)$. There is a Hilbert space H and continuous functions ξ, η from K to the unit ball of H such that $\varphi(st^{-1}) = \langle \xi_t, \eta_s \rangle$ for all $s, t \in K$. We may regard ξ, η as elements in the unit ball of $L_2(K; H)$, and we have

$$\langle \lambda(k)\xi, \eta \rangle = \int \langle \xi_{k^{-1}s}, \eta_s \rangle ds = \varphi(k),$$

which shows that $\varphi \in A(K)$ with norm $\leq \|\xi\|_{L_2(K;H)} \|\eta\|_{L_2(K;H)} \leq 1$. □

Proof of Theorem 2.3. Let $G = \mathrm{SL}(3,\mathbf{F})$. Automorphisms of G and translations induce isometries on $M_0A(G)$. Moreover, if $\varphi \in M_0A(G)$ is U-biinvariant, its restriction to K is a U-biinvariant element of $M_0A(K)$. By Lemma 2.4 it is also a U-biinvariant coefficient of a unitary representation of K and by Proposition 1.2 it satisfies

$$|\varphi(k_\delta) - \varphi(k_0)| \leq 2|\delta|^{1/2} \|\varphi\|_{M_0A(G)}.$$

Proposition 1.5 implies that there is a constant C' such that every K-biinvariant $\varphi \in M_0A(G)$ has a limit l and satisfies

$$|\varphi(g) - l| \leq C' \max(\|g\|, \|g^{-1}\|)^{-\frac{1}{2}} \|\varphi\|_{M_0A(G)} = \varepsilon(g)\|\varphi\|_{M_0A(G)}.$$

If m_g is the measure on G given by $\int f dm_g = \iint_{K \times K} f(kgk')$, we therefore have that $m_g - m_{g'}$ has norm less that $\varepsilon(g) + \varepsilon(g')$ in $M_0A(G)^*$. So $(m_g)_{g \in G}$ is Cauchy in $M_0A(G)^*$ and hence has a limit m_∞. Then $\langle m_\infty, 1 \rangle = 1$, whereas $\langle m_\infty, \varphi \rangle = 0$ for every $\varphi \in C_0(G)$, and in particular every $\varphi \in A(G)$. Therefore m_∞ separates $A(G)$ from 1. It remains to prove that m_∞ is weak-* continuous, *i.e.* that $m_\infty \in Q(G)$. If $\mathbf{F}$ is nonarchimedean this is immediate because $m_g \in L_1(G)$. In the archimedean case m_g is not absolutely continuous, but $\frac{1}{|B(e,1)|} \int_{B(e,1)} m_{gh} dh$ is absolutely continuous and converges to m_∞. □

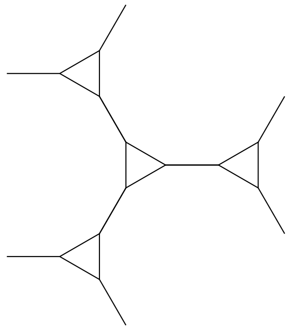

Figure 2.1 – A non-bipartite tree: the Cayley graph of $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$.

2.1.3 Radial Herz-Schur multipliers for hyperbolic groups

If Γ is a finitely generated group with a fixed finite generating set with associated word-length $|\cdot|$, a radial function $\Gamma \to \mathbf{C}$ is a function of the form $\gamma \mapsto f(|\gamma|)$ for a function $f\colon \mathbf{N} \to \mathbf{C}$.

If X is graph, a function $X \times X \to \mathbf{C}$ is called a radial kernel if it has the form $(x, y) \mapsto f(d(x, y))$ for a function $f\colon \mathbf{N} \to \mathbf{C}$. This generalizes the previous notion, as a function $\varphi\colon G \to \mathbf{C}$ is radial if and only if $(s, t) \mapsto \varphi(st^{-1})$ is a radial kernel on the Cayley graph of Γ.

It was proved by Haagerup and Szwarc in the 1980's (and published in [HSS10]) that a function $f\colon \mathbf{N} \to \mathbf{C}$ induces a radial Herz-Schur multiplier on F_d (the free group with $d \geq 2$ generators and standard generating set) if and only if the infinite matrix $(f(j + k) - f(j + k + 2))_{j,k\geq 0}$ is trace class. This condition does not depend on d, but Haagerup–Steenstrup–Szwarc are able to compute the exact value of the norm, and this depends on d.

This result was generalized by Wysoczański [Wys95] to cover all free products. In particular, he showed that a function $f\colon \mathbf{N} \to \mathbf{C}$ induces a radial Herz-Schur multiplier on $\Gamma = \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$ with generating set $\mathbf{Z}/2\mathbf{Z} \cup \mathbf{Z}/3\mathbf{Z}$ if and only if the infinite matrix $(f(j + k) - f(j + k + 1))_{j,k\geq 0}$ is trace class. This condition is stronger than the condition in [HSS10], and is sufficient for all free products with block-length. The difference reflects that trees are bipartite, but not the Cayley graph of $\Gamma = \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$, see Figure 2.1.

In the collaboration [9] which started from the mathoverflow discussion [Mei] we generalized these results to arbitrary Gromov-hyperbolic graphs, in particular to arbitrary generating sets in free groups [1].

Theorem 2.5 ([9])**.** *Let X be a hyperbolic graph with bounded degree. Then there is a constant C such that for every $\phi\colon \mathbf{N} \to \mathbf{C}$, the norm of the Schur multiplier with kernel $\phi(d(x, y))$ is bounded by $C\|\phi(|\cdot|)\|_{M_0A(\mathbf{Z}/2\mathbf{Z}*\mathbf{Z}/3\mathbf{Z})}$.*

If moreover the graph is bipartite, $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$ can be replaced by F_2.

This theorem gives a conceptual explanation why hyperbolic groups are weakly amenable [Oza08]: since the weak amenability of free groups can be proved by exhibiting a sequence of radial finitely supported multipliers with norm 1 and converging pointwise to 1 (this is essentially the proof in [Haa79], see [BO08]), the same holds with constant C on hyperbolic

1. This was our motivation : when we had proven (see Theorem 2.6) that the heat semigroup $\lambda(g) \mapsto e^{-t|g|^2}\lambda(g)$ with respect to the standard word-length was completely bounded on the free group von Neumann algebras, Ozawa asked us wether the same holds for other generating sets of free groups. To our surprise it turned out that the answer to this question was yes, and that this holds more generally on every hyperbolic group.

groups Γ which surject on $\mathbf{Z}/2\mathbf{Z}$, for example $\Gamma = \Gamma' \times \mathbf{Z}/2\mathbf{Z}$ for every hyperbolic group Γ' (because such groups admit bipartite Cayley graphs). The proof of this theorem is an elaboration from the proof in [Oza08].

The second main result in [9] is more technically involved and is a precise incarnation of the well-known fact that smooth symbols yield bounded Fourier multipliers.

Theorem 2.6 ([9])**.** *Let $f\colon [0,\infty) \to \mathbf{R}$ be a bounded continuous function of class C^2 on $(0,\infty)$, and $\frac{1}{2} \geq \alpha > 0$. Then the trace class norm of the matrix $(f(j+k)-f(j+k+1))_{j,k\geq 0}$ is less than $\frac{C}{\sqrt{\alpha}}\sqrt{\|x^{\frac{3}{2}-\alpha}f''\|_{L^2(\mathbf{R}_+)}\|x^{\frac{3}{2}+\alpha}f''\|_{L^2(\mathbf{R}_+)}}$ for some universal constant C.*

One application is that the Heat semigroup $S_t\colon \lambda(g) \mapsto e^{-t|g|^2}$ (and more generally the semigroup $S_t^r\colon \lambda(g) \mapsto e^{-t|g|^r}$ for $r > 0$) is uniformly completely bounded, and even analytic on hyperbolic group von Neumann algebras. This came as a surprise, as it was proved by Knudby [Knu14] that if a semigroup of radial Herz-Schur multipliers $e^{-t\varphi(|g|)}$ is completely *contractive* on the free group von Neumann algebra $\mathcal{L}F_2$, then φ grows sublinearly. The analyticity of the semigroups S_t^r opens the possibility to applications of methods of classical harmonic analysis to the study of hyperbolic group von Neumann algebras as in [JLMX06, JM12, JMP14]...

2.2 L_p multipliers

To every von Neumann algebra $\mathcal{M}$ one associates, following Segal and Dixmier (in the tracial case) or Haagerup (general case) a family of non-commutative L_p spaces $L_p(\mathcal{M})$ for $0 < p \leq \infty$. When $p = \infty$, $L_p(\mathcal{M}) = \mathcal{M}$. Such a non-commutative L_p space is said to have the completely bounded approximation property (CBAP) if there is a net of finite rank maps $T_\alpha\colon L_p(\mathcal{M}) \to L_p(\mathcal{M})$ such that $T_\alpha(x) \to x$ for all $x \in L_p(\mathcal{M})$ and $\sup_\alpha \|T_\alpha\|_{cb} < \infty$, where the completely bounded (cb) norm of T_α is

$$\|T_\alpha\|_{cb} = \sup_n \|T_\alpha \otimes \mathrm{id}\|_{L_p(\mathcal{M}\otimes M_n(\mathbf{C}))\to L_p(\mathcal{M}\otimes M_n(\mathbf{C}))}.$$

This section contains results on multipliers on $L_p(\mathcal{L}G)$.

2.2.1 Rigidity for L_p multipliers of higher rank groups

When G is a locally compact group and $\varphi\colon G \to \mathbf{C}$, there are two natural definitions of a non-commutative L_p version of Herz-Schur multipliers (where the Herz-Schur multiplier corresponds to $p = \infty$).

One is the notion of Fourier multiplier of $L_p(\mathcal{L}G)$ of symbol φ, still denoted m_φ, which generalizes the multipliers $\lambda(g) \mapsto \varphi(g)\lambda(g)$. Since when G is not discrete and $p < \infty$, $\lambda(g)$ does not belong to $L_p(\mathcal{L}G)$, the definition of m_φ needs a bit of care, in particular when G is not unimodular because we have to work with the Plancherel weight on $\mathcal{L}G$, and therefore work as if $\mathcal{L}G$ was of type III [1].

Another is the Schur multiplier $M_\varphi\colon (T_{s,t})_{s,t\in G} \in S_p(L_2G) \mapsto (\varphi(st^{-1})T_{s,t})_{s,t}S_p(L_2G)$.

The Fourier multipliers are the interesting objects as far as von Neumann algebras as concerned. For example, by the same averaging argument as the one used in [Haa86], one sees that, for a discrete (hyperlinear) group, if $L_p(\mathcal{L}G)$ has the CBAP then the net approximating the identity can be taken as finitely supported Fourier multipliers. But it is difficult to work directly with Fourier multipliers. On the other hand, Schur multipliers M_φ have very good properties : it is easy to restrict them to subgroups, to induce from lattice to a bigger group [8]. The following question, which has also been asked by Quanhua Xu, is natural in view of the properties of Herz-Schur multipliers.

Question 2.7. Is it true that $\|M_\varphi\colon S_p(L_2G) \to S_p(L_2G)\|_{cb} = \|m_\varphi\colon L_p(\mathcal{L}G) \to L_p(\mathcal{L}G)\|_{cb}$ for every locally compact group G and every group continuous function $\varphi\colon G \to \mathbf{C}$?

The inequality $\|M_\varphi\|_{cb} \leq \|m_\varphi\|_{cb}$ is rather easy (although a bit technical in full generality, see [1, Theorem 4.2]). Apart for the extreme cases $p \in \{1, 2, \infty\}$, the converse inequality is known only for amenable groups : see [NR11] for discrete groups and [1] for general groups. The proof goes as follows : we observe that, in the Connes–Hilsum spatial realization of $L_p(\mathcal{L}G)$ as unbounded operators on $L_2(G)$, the corners $P_F L_p(\mathcal{L}G) P_F$ (obtained by cutting by the projection on $L_2(F) \subset L_2(G)$ for a finite Haar measure subset $F \subset G$) lie in $S_p(L_2(F))$, and that when F_n is a Følner sequence these corners, correctly normalized, converge to $L_p(\mathcal{L}G)$.

In [8] we make the following definition.

Definition 2.8. If G is a locally compact second countable group and $1 \leq p \leq \infty$, we say that G has the *property of completely bounded approximation by Schur multipliers on* S^p ($\mathrm{AP}^{Schur}_{pcb}$) if there is a constant C, a net of functions $\varphi_\alpha \in A(G)$ such that $\varphi_\alpha \to 1$ uniformly on compact subsets of G and such that $\|m_{\varphi_\alpha}\|_{cb(S_p(L_2G))} \leq C$ for all α.

In [8] we proved that if $1 < p < \infty$ and if a discrete group has ($\mathrm{AP}^{Schur}_{pcb}$), then it has ($\mathrm{AP}^{Schur}_{pcb}$) with constant 1. So there is no need to introduce a p-version of the Cowling–Haagerup constant.

It follows from the preceding discussion that ($\mathrm{AP}^{Schur}_{pcb}$) is equivalent for a group and a lattice in it, and that a discrete group has ($\mathrm{AP}^{Schur}_{pcb}$) if $L_p(\mathcal{L}G)$ has the CBAP. Using some operator space theory we also proved in [8] that AP implies ($\mathrm{AP}^{Schur}_{pcb}$) for every $1 < p < \infty$.

The main result in [8] was the following, which implied Theorem 2.3.

Theorem 2.9 ([8])**.** *Let* $\mathbf{F}$ *be a local field. Then* $\mathrm{SL}(3, \mathbf{F})$ *does not have* ($\mathrm{AP}^{Schur}_{pcb}$) *if* $p \in [1, \frac{4}{3}) \cup (4, \infty]$.

We now prove this theorem. By duality we can assume $p > 4$. By Proposition 1.5, the theorem follows from the following proposition, which is itself a consequence of (1.6) and (1.9).

Proposition 2.10. *Let* $p > 4$. *With the notation of Chapter 1, for every* U*-biinvariant function* $\varphi\colon K \to \mathbf{C}$ *we have*

$$|\varphi(k_\delta) - \varphi(k_0)| \leq C|\delta|^{\frac{1}{2} - \frac{2}{p}} \|m_\varphi\|_{cb(S_p(L_2K))}.$$

Afterwards, the following generalizations have been obtained. Namely we know now that G does not have ($\mathrm{AP}^{Schur}_{pcb}$) in the following cases :

— [8] $G = \mathrm{SL}(2n+1, \mathbf{F})$ for a non-archimedean $\mathbf{F}$ and $p \notin [2 - \frac{2}{n+2}, 2 + \frac{2}{n}]$.
— [7] $G = \mathrm{SL}(2n+1, \mathbf{R})$ and $p \notin [2 - \frac{2}{n+2}, 2 + \frac{2}{n}]$.
— [Lia15] $G = \mathrm{Sp}(2, \mathbf{F})$ and $p \notin [4/3, 4]$.
— [dL13, HdL16] $G = \mathrm{Sp}(2, \mathbf{R})$ or its universal cover and $p \notin [\frac{12}{11}, 12]$ (improved to $p \notin [\frac{10}{9}, 10]$ in [6]).

It is also known ([dL13] [6, 7]) [2] that every connected simple Lie group of real rank $r \geq 2$ does not have ($\mathrm{AP}^{Schur}_{pcb}$) if $p \notin [p'_r, p_r]$ for a sequence $p_r \leq 10$ satisfying that $p_r \leq 2 + O(r^{-1})$ as $r \to \infty$.

2. I should mention that we were a bit sloppy in deriving this in [7], as we only considered the groups whose Lie algebra has a complex structure. The result is however correct, but one needs also to go, case-by-case, through Cartan's list of non-split and non-compact Lie simple Lie algebras [Kna02, Theorem 6.105].

The proof of all these results consist in generalizing the method of Chapter 1 to other groups. There are two independant parts. The first part of the generalization is to replace the operators T_δ of section 1.3 and 1.4 by other (higher-dimensional) averaging operators with better summability properties in the sense that they belong to the Schatten p class for more values of p.

For $G = \mathrm{SL}(2n+1, \mathbf{F})$, the operators are $T_\delta\colon L_2(\mathcal{O}^{n+1}) \to L_2(\mathcal{O}^{n+1})$ given by

$$T_\delta f(a_1, \dots, a_n, b) = \mathbb{E}_{x_1,\dots,x_n} f(x_1, \dots, x_n, \sum_i a_i x_i + b + \delta)$$

and are p-summable for $p > 2 + \frac{2}{n}$.

For $G = \mathrm{SL}(2n+1, \mathbf{R})$ the operators are $T_\delta\colon L_2(S^{n+1}) \to L_2(S^{n+1})$ given by

$$T_\delta f(x) = \text{average of } f \text{ on } \{y \in S^{n+1}, \langle x, y\rangle = \delta\}$$

and are p-summable for $p > 2 + \frac{2}{n}$.

For $G = \mathrm{Sp}(2, \mathbf{F})$ the averaging operators are the same as for $\mathrm{SL}(3, \mathbf{F})$ and are p-summable for $p > 4$.

For $G = \mathrm{Sp}(2, \mathbf{R})$ there are two families averaging operators : the same as for $\mathrm{SL}(3, \mathbf{R})$, and the operators $S_\theta\colon L_2(S^1_{\mathbf{C}}) \to L_2(S^1_{\mathbf{C}})$ (for $\theta \in \mathbf{R}/2\pi\mathbf{Z}$), given by

$$S_\theta f(x) = \text{average of } f \text{ on } \{y \in S^1_{\mathbf{C}}, \langle x, y\rangle = \frac{e^{i\theta}}{\sqrt{2}}\},$$

where $S^1_{\mathbf{C}}$ is the unit sphere in $\mathbf{C}^2$ equipped with its usual Hermitian product.

In the non-archimedean case the spectral analysis of the averaging operators is essentially the same as in section 1.4 : the conjugation by a partial Fourier transform gives an explicit diagonalization of T_δ. In the real case also the analysis is similar to the one in section 1.3, except that the eigenfunctions are (functions of) more complicated spherical functions, for which the Bernstein type inequalities were not known. An important part of the proofs is to establish such inequalities. For the Jacobi polynomials involved in the diagonalization of the operators S_θ, Bernstein type inequalities were obtained in [HS14], yielding to $S_\theta \in S_p$ for all $p > 12$ [dL13]. These Bernstein type inequalities were improved in the appendix [6] to prove $S_\theta \in S_p$ for all $p > 10$. The polynomials involved in the spectral analysis of the averaging operators on S^{n+1} are Gegenbauer polynomials, and the corresponding Bernstein type inequalities were obtained in [7].

The second part in the proofs is geometric/combinatorial. The idea is to construct a path in the Weyl chamber of G which goes to infinity and which is made of intervals given by these averaging operators as in Figure 1.1. This is done by a careful analysis of all the natural embeddings of the space of the averaging operators into the symmetric space (or building) G/K. For example for $\mathrm{SL}(2n+1, \mathbf{R})$ this amounts to analyzing the combinatorial self-intersection structures of the $n+1$-dimensional spheres inside the symmetric space $\mathrm{SL}(2n+1, \mathbf{R})/\mathrm{SO}(2n+1)$. For $\mathrm{SL}(2n+1, \mathbf{F})$ this amounts to analyzing the combinatorial self-intersection structures of the $n+1$-dimensional flats inside the Bruhat-Tits building of $\mathrm{PGL}(2n+1, \mathbf{F})$. Of course, there are already many $n+1$-dimensional spheres (respectively $n+1$-dimensional flats) inside $\mathrm{SL}(r, \mathbf{R})/\mathrm{SO}(r)$ (respectively the Bruhat-Tits building of $\mathrm{PGL}(r, \mathbf{F})$) for $r \geq n+2$, but $r - 1 = 2n$ is the critical rank where there starts to be enough such spheres/flats to construct a connected path going to infinity.

One sees from this list that for $\mathrm{SL}(n)$, although the proofs in the real and non-archimedean case are quite different, the numerology at the end is the same. This is no longer the case for $\mathrm{Sp}(2)$. I wonder whether there is anything to say about this symmetry breaking.

2.2.2 Consequence for the operator space approximation property

A consequence of Theorem 2.9 is that, for every lattice Γ in $\mathrm{SL}(3,F)$, $L_p(\mathcal{L}\Gamma)$ does not have the CBAP for $p \in [1,\frac{4}{3}) \cup (4,\infty]$.

It is tempting to introduce an invariant of $\mathcal{M}$ given by the set of values of $p \in [1,\infty]$ such that $L_p(\mathcal{M})$ has the completely bounded approximation property. So far this invariant has not been so useful because in all cases when it has been computed, it is either (1) $[1,\infty]$, or (2) $(1,\infty)$ or (3) $\{2\}$. For example, among discrete group von Neumann algebras, the von Neumann algebras of weakly amenable groups (e.g. hyperbolic groups) fall into case (1), whereas the von Neumann algebra of $\mathrm{SL}(2,\mathbf{Z}) \ltimes \mathbf{Z}^2$ falls into case (2), and (by the preceding) the von Neumann algebra of groups containing $\mathrm{SL}(n,\mathbf{Z})$ for all n falls into case (3).

The following tabular summarizes, in terms of the group G, what is known about the above invariant for the von Neumann algebra of a lattice in G. In the following $\mathbf{F}$ denotes a *non-archimedean local field.*

G	$\{p, L_p(\mathcal{L}\Gamma)$has CBAP$\}$ for $\Gamma \subset G$ lattice	Reference
$\mathrm{SL}(3,\mathbf{R})$	$\subset [4/3,4]$	[8]
$\mathrm{SL}(3,\mathbf{F})$	$\subset [4/3,4]$	[8]
$\mathrm{SL}(2n+1,\mathbf{F})$	$\subset [2-\frac{2}{n+2}, 2+\frac{2}{n}]$	[8]
$\mathrm{SL}(2n+1,\mathbf{R})$	$\subset [2-\frac{2}{n+2}, 2+\frac{2}{n}]$	[7]
$\mathrm{Sp}(2,\mathbf{R})$	$\subset [\frac{10}{9}, 10]$	[dL13][6]
$\mathrm{Sp}(2,\mathbf{F})$	$\subset [4/3,4]$	[Lia15]

The first line says for example that the non-commutative L_p space of the von Neumann algebra of $\mathrm{SL}(3,\mathbf{Z})$ does not have the CBAP for $p > 4$ or $p < 4/3$, but does not say anything if $p \in [\frac{4}{3},4]$. Nothing is known about the reverse inclusions and it might be that for every lattice in a higher rank group and ever $p \neq 2$, $L_p(\mathcal{L}\Gamma)$ lacks the CBAP. However, by (1.7), the interval $[\frac{4}{3},4]$ is optimal for the proofs to work.

Question 2.11. Has $L_p(\mathrm{SL}(3,\mathbf{Z}))$ the CBAP for some $p \in (2,4]$?

A related question is

Question 2.12. Has $\mathrm{SL}(3,\mathbf{Z}))$ (equivalently $\mathrm{SL}(3,\mathbf{R})$) the property $(\mathrm{AP}_{pcb}^{Schur})$ for some $p \in (2,4]$?

Both questions are still wide open, and the tiny hope that they could have a positive answer has been my main motivation for all my work on L_p-multipliers after [8]. From [7] we know that a positive answer to question 2.11 would distinguish the von Neumann algebras of $\mathrm{SL}(3,\mathbf{Z})$ and $\mathrm{PSL}(n,\mathbf{Z})$ for large n [Con82].

Answering positively question 2.12 seems more accessible that question 2.11, and of course they would be equivalent if question 2.7 had a positive answer. [1, Theorem 2.1] however suggests that 2.7 has a negative answer for every nonamenable group. The reason why we made so much effort in [1] to cover also non-unimodular groups is because $G = \mathrm{SL}(3,\mathbf{R})$, as every connected simple Lie group with finite center, can be written $G = PK$ with P the amenable (non-unimodular) group of triangular matrices and K the compact group $\mathrm{SO}(3)$. Together with the main result from [1] and the results from [CPPR15], this suggests that perhaps $\|M_\varphi\|_{cb} = \|m_\varphi\|_{cb}$ for every p and every $\mathrm{SO}(3)$-biinvariant function $\varphi\colon \mathrm{SL}(3,\mathbf{R}) \to \mathbf{C}$.

Chapter 3

Group representations on Banach spaces

This chapter presents my work on Banach space representations of locally compact groups, from [10, 6, 7, 5, 11].

3.1 Versions of property (T) for Banach space representations

Property (T) has several equivalent characterizations. Most of them make sense also for Banach space representations, but they are not equivalent. I first recall some of these notions, and the relations between them. In the following, $\mathcal{E}$ is a class of Banach spaces.

If one adopts the definition in terms of almost invariant vectors, one gets property (T_X) as defined in [BFGM07]. The following is a variant of the original definition, which agrees in most cases (at least when $X^{\pi(G)}$ has a $\pi(G)$-invariant complement subspace, for example when X is reflexive). I prefer this definition to the original one, because it is only for this definition that I am conviced that the implication $(\mathrm{F}_{\mathcal{E}}) \implies (\mathrm{T}_{\mathcal{E}})$ holds (see below for the terminology).

Definition 3.1. (Bader, Furman, Gelander, Monod [BFGM07]) A locally compact group G has property $(\mathrm{T}_{\mathcal{E}})$ if for every isometric representation $\pi\colon G \to O(X)$ on a space X in $\mathcal{E}$, there is a compact subset $Q \subset G$ and $\varepsilon > 0$ such that $\sup_{g\in Q} \|\pi(g)x - x\|_X \geq \varepsilon \|x\|_{X/X^{\pi(G)}}$ for every $x \in X$. Here, $X^{\pi(G)}$ denotes the closed subspace of X consisting of vectors that are fixed by π.

An equivalent characterization of property (T) is in cohomological terms. Adapting this definition yields:

Definition 3.2. ([BFGM07]) A locally compact group G has property $(\mathrm{F}_{\mathcal{E}})$ if every action of G by affine isometries on a space in $\mathcal{E}$ has a fixed point.

Another equivalent characterization of property (T) is in terms of the existence of a Kazhdan projection in the full C^*-algebra of G. The approach of Vincent Lafforgue was to adapt this notion [1].

1. the proof of (T) for $\mathrm{SL}(3, \mathbf{F})$ that I presented in Chapter 1 actually constructed (without explicitely saying so) the Kazhdan projection in $C^*(G)$, and then deduced that a unitary representation without invariant vectors has no almost invariant vectors.

Let $\mathcal{F}$ be a class of representations of a locally compact group G such that

$$\sup_{(\pi,X)\in\mathcal{F}} \|\pi(g)\|_{B(X)}$$

is bounded on compact subsets of G. A Banach algebra $\mathcal{C}_{\mathcal{F}}(G)$ is defined as the completion of $C_c(G)$ for the norm

$$\|f\|_{\mathcal{F}} = \sup\{\|\pi(f)\|_{B(X)}, (\pi, X) \in \mathcal{F}\}.$$

For example, if $\mathcal{F}$ is the class of unitary representations of G then $\mathcal{C}_{\mathcal{F}}(G)$ is the full C^*-algebra of G.

Definition 3.3. A Kazhdan projection in $\mathcal{C}_{\mathcal{F}}(G)$ is an element p belonging to the closure of $\{f \in C_c(G), \int f = 1\}$ such that $\pi(p)$ is a projection on X^π for every $(\pi, X) \in \mathcal{F}$.

Equivalently, a Kazhdan projection in $\mathcal{C}_{\mathcal{F}}(G)$ is an element belonging to the closure of $\{f \in C_c(G), \int f = 1\}$ that is invariant by left translation by G [11]. It is sometimes interesting to request that the Kazhdan projection belong to the center of $\mathcal{C}_{\mathcal{F}}(G)$, or equivalently that it be also invariant by right translations. This is investigated in [11], where it is shown that in many interesting cases but not all, a Kazhdan projection is always central. In the case G admits a compact generating set S, I also proved the following characterization of the existence of Kazhdan projections:

Theorem 3.4 ([11])**.** *$\mathcal{C}_{\mathcal{F}}(G)$ contains a Kazhdan projection if and only if there exists a compactly supported measure m on G with $\int 1dm = 1$ such that*

$$\delta_S^\pi(\pi(m)x) \le \frac{1}{2}\delta_S^\pi(x) \text{ for all } (\pi, X) \in \mathcal{F} \text{ and } x \in X,$$

where $\delta_S^\pi(x) = \sup_{s\in S} \|\pi(s)x - x\|$ measures how much x is moved by the generators.

To simplify some arguments in this chapter, we will modify slightly the definition of $\mathcal{C}_{\mathcal{F}}(G)$ and of Kazhdan projections by replacing $C_c(G)$ in both definitions by the space of compactly supported regular measures on G. This does not affect the definitions nor results, but is convenient because in the case of simple Lie groups, the Kazhdan projection appears naturally as a limit of probability measures which are not absolutely continuous.

Lafforgue [Laf08] defines [2] *Banach property (T) with respect to a class $\mathcal{E}$ of Banach spaces* by requesting that $\mathcal{C}_{\mathcal{F}}(G)$ has a Kazhdan projection, where $\mathcal{F}$ is the class of all isometric strongly continuous representations on a space in $\mathcal{E}$. To avoid confusion with the notion of [BFGM07], we denote this property by ($\mathrm{T}_{\mathcal{E}}^{proj}$), where proj stands for projection.

If $m\colon G \to (0, \infty]$ is a function, we denote $\mathcal{F}(\mathcal{E}, m)$ all representations (π, X) such that $X \in \mathcal{E}$ and $\|\pi(g)\| \le e^{m(g)}$ for all g. In the next definition, assume for simplicity that G is compactly generated and denote by ℓ the word-length function with respect to some compact symmetric generating set. The definition does not depend on the generating set.

Definition 3.5. (Lafforgue [Laf08]) If $\mathcal{E}$ is a class of Banach spaces, one says that G has strong property (T) with respect to $\mathcal{E}$ if [2] there there is $s > 0$ such that for all $C > 0$, $\mathcal{C}_{\mathcal{F}(\mathcal{E},s\ell+C)}(G)$ has a Kazhdan projection.

2. actually Lafforgue only considers this property when $\mathcal{E}$ is stable by duality, subspaces and complex conjugation, and requests that the Kazhdan projection be self-adjoint and real, but this is automatic in this case, see [11, §3.2].

If "for all $C > 0$" is replaced by "there exists $C > 0$", one gets robust $(\mathrm{T}_{\mathcal{E}})$ considered by Oppenheim [Opp15]. The following diagram summarizes the known implications between all these Banach space variants of property (T).

$$\begin{tikzcd}
 & & (F_{\mathcal{E}}) \arrow[dl, Rightarrow, shift left, "3"] \arrow[dr, Rightarrow] & \\
\text{Strong } (\mathrm{T}_{\mathcal{E}}) \arrow[r, Rightarrow] & \text{Robust } (\mathrm{T}_{\mathcal{E}}) \arrow[ur, Rightarrow, shift left, "1"] \arrow[dr, Rightarrow] & & (T_{\mathcal{E}}) \arrow[dl, Rightarrow, shift left, "2"] \\
 & & (T_{\mathcal{E}}^{proj}) \arrow[ur, Rightarrow, shift left] &
\end{tikzcd} \tag{3.1}$$

where

- — the implication 1 holds if $\mathcal{E}$ is stable by $X \mapsto X \oplus_p \mathbf{C}$ for some $1 \leq p \leq \infty$ [Laf08, Opp15].
- — the implication 2 has to be understood as follows: if $\mathcal{E}$ is a superreflexive class of Banach spaces, property $(\mathrm{T}_{\mathcal{E}}^{\mathrm{proj}})$ is equivalent to "uniform" version of property $(\mathrm{T}_{\mathcal{E}})$ (see [7] for discrete groups and [DN15] for arbitrary locally compact groups).
- — the very recent implication 3 holds if $\mathcal{E}$ is the class of L_p spaces for some $1 < p < \infty$, or if $\mathcal{E}$ is superreflexive and stable by finite representability (or ultraproducts if G is discrete). The proof is based on Theorem 3.4, and uses the implication $(\mathrm{F}_{\mathcal{E}}) \Longrightarrow (\mathrm{T}_{\mathcal{E}}^{\mathrm{proj}})$ (a strengthening of implication 2) and an ultraproduct argument, see [11]. A consequence of this implication is that, to prove that a discrete group does not have (F_X) with respect to spaces isomorphic to a Hilbert space [3], it is enough to construct a family of representations on a Hilbert space satisfying $\|\pi_n(g)\| \leq Ce^{\ell(g)/n}$ for some C independant from n in which the space of invariant vectors has no $\pi_n(G)$-invariant complement subspace. Representations with this property have been constructed by Lafforgue for hyperbolic groups, with $\|\pi(g)\| \leq C(1 + \ell(g))^2$.

For example, if $\mathcal{E}$ is the class of Hilbert spaces, all the above properties are equivalent except strong (T), which is strictly stronger.

For completeness, we recall that a class of Banach spaces is superreflexive $\mathcal{E}$ if all its ultraproducts are reflexive. By a theorem of Enflo this is equivalent to the existence of an equivalent uniformly convex norm on every space in $\mathcal{E}$ (uniformly in $\mathcal{E}$).

Each of these forms of Banach space property (T) becomes stronger if $\mathcal{E}$ increases, and a non-compact group cannot have (T) with respect to all Banach spaces (indeed, the left-regular representation of G on the space of continuous functions on G modulo constants has almost invariant vectors but no nonzero invariant vectors). The game is, given a group G, to find a class $\mathcal{E}$ as large as possible with respect to which G has (some of) the above forms of property (T). There is some challenge since, for example, hyperbolic groups may have property (T) but do not have strong (T) with respect to Hilbert spaces [Laf08], nor (F_{L_p}) for p large enough [Yu05]. On the opposite, it is expected, and in some sense rather well understood, that higher rank groups are much more rigid than rank 1 groups or other hyperbolic groups. For example, it was conjectured in [BFGM07] that higher rank simple algebraic groups over local fields and their lattices have (F_X) with respect to all superreflexive Banach spaces. This conjecture implies another conjecture, namely that if Γ is such a lattice with finite generating set S and Γ_n is a sequence of finite quotients of Γ with cardinality going to ∞, then the Cayley graphs of Γ_n with respect to the generating set S (which are expanders by Kazhdan and Margulis) do not coarsely embed in a superreflexive Banach space.

3. It is a conjecture of Shalom that this holds for every infinite hyperbolic group.

The ultimate result in this perspective is [Laf09, Lia14] where Lafforgue and Liao prove that higher rank groups over non-archimedean local fields have strong (T_X) with respect to all spaces X of type > 1. A space X has type $p > 1$ if there exists C such that for every n and every $x_1, \dots, x_n \in X$,

$$\left(\mathbb{E}\|\sum_{i=1}^{n} \varepsilon_i x_i\|^p\right)^{1/p} \leq C(\sum_{i=1}^{n} \|x_i\|^p)^{1/p} \tag{3.2}$$

where ε_i are independant random variables uniformly distributed in $\{-1, 1\}$. There are many equivalent characterizations of the Banach spaces X with type > 1. One is that X does not contain uniformly the family ℓ_1^n of n-dimensional ℓ_1 spaces : there is n and $c > 0$ such that every subspace $E \subset X$ of dimension n is at Banach-Mazur distance at least $1 + c$ from ℓ_1^n. From this characterization it is clear that superreflexive Banach spaces have type > 1, so the above conjecture holds over non-archimedean fields. Another characterization of type > 1, due to Milman and Wolfson, is that $d_i(X) = o(i^{\frac{1}{2}})$ as $i \to \infty$, where $d_i(X)$ is the supremum, over all subspace $E \subset X$ of dimension i, of the Banach-Mazur distance between E and ℓ_2^i [MmW78].

I have tried hard to prove the same result over the field of reals. The next result summarizes the partial results I have obtained, mainly in collaboration with Tim de Laat:

Theorem 3.6 ([10, 6, 7, 5])**.** *For every connected simple Lie group G of real rank at least 2, there is $\beta(G) \in (0, \frac{1}{2})$ such that G has strong property (T_X) for every Banach space X which is finitely representable in a subquotient of complex interpolation space $X_\theta = [X_0, X_1]_\theta$ ($\theta \in (0, 1)$) between a space satisfying $d_i(X_0) = O(i^{\beta(G)})$ and an arbitrary space X_1.*

Moreover $\beta(G)$ goes to $\frac{1}{2}$ as the real rank of G goes to ∞.

As a corollary, if G and X are as above and Γ is a lattice in G, then G and Γ have (F_X) [4], and the expanders constructed from Γ do not embed in X.

In particular, if one believes that every space of type > 1 not only satisfies $d_i(X) = o(i^{\frac{1}{2}})$ as proved by Milman–Wolfson, but also satisfies $d_i(X) = O(i^{\frac{1}{2} - \varepsilon})$ for some $\varepsilon > 0$ [5], this theorem says that for every space of type > 1, $\mathrm{SL}(r, \mathbf{Z})$ has (F_X) and the expanders $(\mathrm{SL}(r, \mathbf{Z}/n\mathbf{Z}))_{n \geq 1}$ do not coarsely embed in X for every n large enough. Also note that it is not known whether the condition on X in Theorem 3.6 really depends on r. It does not for Banach lattices [Pis79].

By the classification of real simple Lie algebras and Lie groups, every simple Lie group of real rank at least 2 contains a closed subgroup which is isomorphic either to a finite extension of $\mathrm{PSL}(3, \mathbf{R})$, or to a finite extension of $\mathrm{PSp}(2, \mathbf{R})$, or to $\widetilde{\mathrm{Sp}}(2, \mathbf{R})$. Similarly, every simple Lie group or rank $10n$ contains a subgroup isomorphic to a finite extension of $\mathrm{PSL}(n, \mathbf{R})$. Therefore the main part in Theorem 3.6 is to consider the cases of $G = \mathrm{SL}(r, \mathbf{R})$ for $r \geq 3$, of $G = \mathrm{Sp}(2, \mathbf{R})$ or its universal cover $\widetilde{\mathrm{Sp}}(2, \mathbf{R})$. And if one is not interested in the convergence of $\beta(G)$ to $\frac{1}{2}$, it suffices to consider $\mathrm{SL}(3, \mathbf{R})$, $\mathrm{Sp}(2, \mathbf{R})$ and $\widetilde{\mathrm{Sp}}(2, \mathbf{R})$. The rest of this chapter is devoted to a discussion of the proof of these cases and to the differences with the non-archimedean case.

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4. this is immediate if Γ is cocompact because strong property (T) passes to cocompact lattices, otherwise we have to use that Γ is p-integrable.

5. this is an open question from the 80's [TJ89, Problem 27.6], mentioned as a conjecture in [Bou82], and known to be true for spaces of type 2, or more generally for spaces of type p and cotype q satisfying $\frac{1}{p} - \frac{1}{q} < \frac{1}{2}$ [TJ89].

3.2 Norm estimates and Banach space valued spectral gap

We introduce an important ingredient in the proof of Theorem 3.6.

If E is a subspace of an L_p space $L_p(\Omega, \mu)$ and $T\colon E \to L_p(\Omega', \mu')$ is a linear map, we denote by $\|T_X\| \in \mathbf{R}^+ \cup \{\infty\}$ the norm of $T \otimes \mathrm{Id}$ from the subspace $E \otimes X$ of the space of X-valued p-integrable functions $L_p(\Omega, \mu; X)$ to $L_p(\Omega', \mu'; X)$.

Estimating $\|T_X\|$ in terms of the properties of T and the geometric properties of X is a central aspect in the geometry of Banach spaces, and most natural geometric classes of Banach spaces are characterized in terms of such quantities. For example:

— Hilbert spaces are characterized by the parallelogram inequality, *i.e.* the property $\|T_X\| \leq 1$ where $T\colon \ell_2^2 \to \ell_2^2$ has matrix $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

— A Banach space X has type > 1 if and only if it is K-convex [Pis82]: $\|T_X\| < \infty$, where $T \in B(L_2(\{-1,1\}^{\mathbf{N}}))$ is the orthogonal projection on the space spanned by the coordinates $\varepsilon_i\colon \omega = (\omega_n)_{n \in \mathbf{N}} \mapsto \omega_i$.

— [Bou82] A Banach space X has type > 1 if and only if there is $p > 1$ such that for every discrete abelian group Γ, the Fourier transform $\mathcal{F}\colon \ell_2(\Gamma) \to L_2(\hat{\Gamma})$ is bounded from $\ell_p(\Gamma; X)$ to $L_2(\hat{\Gamma}; X)$.

— Similarly (this is due to Pisier but written in [7]) up to a factor 2, $d_i(X)$ is equal to $\sup \|T_X\|$, where the sup is taken over all $T\colon L_2 \to L_2$ of norm 1 and rank i. Therefore X has type $p > 1$ if and only if $\|T_X\| = o(\|T\| \mathrm{rk}(T)^{\frac{1}{2}})$ as $\mathrm{rk}(T) \to \infty$.

— More abstractly [Her84] a class $\mathcal{E}$ of Banach spaces is stable by ℓ_p-direct sum and finite representability if and only if there is a closed subspace $E \subset L_p([0,1])$ and a linear map $T\colon E \to L_p([0,1])$ such that

$$\mathcal{E} = \{X, \|T \otimes \mathrm{Id_X}\| \leq 1\}.$$

$\mathcal{E}$ is in addition stable by duality if and only if T can be taken as an operator $L_p \to L_p$.

A motivation for studying the quantity $\|T_X\|$ is its well-known connection with Poincaré inequalities and embeddability of expanders in X. If (V, E) is a finite graph, we denote by M_V^0 the operator of the random walk on V, but restricted to $\ell_2^0(V)$, the orthogonal in $\ell_2(V)$ of the constant functions on V. A sequence of bounded degree connected graphs (V_n, E_n) is an expander sequence if $\lim_n |V_n| = \infty$ and $\sup_n \|M_n^0\| < 1$.

Proposition 3.7. *Let X be a Banach space. Let (V_n, E_n) be a sequence of graphs such that*

$$\sup_n \|(M_n^0)_X\| < 1.$$

There is a constant R such that for every n and every 1*-Lipschitz map $f_n\colon V_n \to X$, there is a ball of radius R in X that contains the image of at least $\frac{|V_n|}{2}$ vertices.*

In particular, if (V_n, E_n) is a sequence of d-regular expanders with $\sup_n \|(M_n^0)_X\| < 1$, *then the sequence V_n does not coarsely embed into X.*

There is a kind of converse to the previous Proposition (see also [MN14, Lemma 6.6]). By an easy combinatorial argument, by adding a bounded number of self-loops at each vertex, every d-regular graph can be turned to a Schreier graph of a group. And if (V_n, E_n) is a sequence of Schreier graphs of a group by the (proof of) implication 2 in (3.1), for a uniformly convex space X, the assumption $\sup_n \|(M_n^0)_X\| < 1$ is equivalent to the validity of the following Poincaré type inequality: there exists $\lambda > 0$ such that for all n and all

$f: V_n \to X$,

$$\frac{\lambda}{|V_n|^2} \sum_{x,y \in V_n} \|f(x) - f(y)\|^2 \leq \frac{1}{|E_n|} \sum_{(x,y) \in E_n} \|f(x) - f(y)\|^2.$$

By [Tes09] there is equivalence between non-coarse embeddability into families of Banach spaces closed under ultraproducts and ℓ_p direct sums and some *other* forms of Poincaré inequalities.

Another motivation comes from representation theory. Recall that a locally compact group G is amenable if and only if its full and reduced C^*-algebras coincide. The same proof gives:

Proposition 3.8. *Assume that G is amenable, and let $\lambda\colon G \to \mathcal{U}(L^2(G))$ be the left regular representation. If $\mathcal{F}$ is the class of isometric representations on X, then for every $f \in C_c(G)$,*

$$\|f\|_{\mathcal{C}_{\mathcal{F}}(G)} \leq \|(\lambda(f))_X\|.$$

In the particular case of compact groups, the result lies at the heart of the proofs of Theorem 3.6.

3.3 Case of SL_3

Take $G = \mathrm{SL}_3(\mathbf{F})$ for a local field $\mathbf{F}$, $K \subset G$ its maximal compact subgroup, $U \subset K$ the subgroup of block-diagonal matrices and T_δ the averaging operators as in Chapter 1. Lafforgue proved that G has strong property (T) with respect to X provided that there exists $\theta > 0$ and C such that for all δ

$$\|(T_0 - T_\delta)_X\| \leq C|\delta|^\theta. \tag{3.3}$$

In the non-archimedean case, the averaging operators are very much related to the Fourier transform on the compact abelian group $\mathcal{O}$, and using Bourgain's work [Bou82] mentioned above together with a variant of the fast Fourier transform, Lafforgue was able to prove that (3.3) holds for every space of type > 1 (this condition is clearly necessary, because if X does not have type > 1, then $\|(T_0 - T_\delta)_X\| = \|(T_0 - T_\delta)_{L_\infty}\| = 2$).

In the real case it is expected, but still open, that (3.3) holds for every space of type > 1. As explained above, this would prove that $\mathrm{SL}(3, \mathbf{R})$ has strong (T) with respect to all spaces with type > 1. Motivated by this question, Lafforgue asked under which condition on X there exists $\alpha \in (0,1)$ and C such that $\|T_X\| \leq C\|T\|^\alpha$ for every operator $T\colon L_2 \to L_2$ with the property that $\sup_Y \|T_Y\| = \|T_{\ell_\infty}\| = 1$. This was answered by Pisier [Pis10], and the (not at all obvious) answer is that this holds only in the essentially obvious case when X is finitely representable is a subquotient of a generalized α-Hilbertian space. In particular, such a space is superreflexive. Hence, additional properties of the averaging operators T_δ have to be exploited in order to establish (3.3) for spaces that are not superreflexive. In [10] I used the p-summability properties (1.6) to prove if $d_i(X) = O(i^{\frac{1}{4}-\varepsilon})$ for some $\varepsilon > 0$, then (3.3) holds with $\theta < 2\varepsilon$. This leaded to Theorem 3.6 for $G = \mathrm{SL}(3, \mathbf{R})$ with $\beta(G) = \frac{1}{4} - \varepsilon$.

3.3.1 On the proof that (3.3) implies strong (T) with respect to X

With the material from Chapter 1, it is easy to explain the proof.

Step 1 First, since compact groups are amenable, it follows from Proposition 1 and the discussions in §1.3 and 1.4 that for every representation of K by isometries π on a Banach space X, and every U-invariant unit vectors $\xi \in X, \eta \in X^*$,

$$|\langle \pi(k_\delta)\xi, \eta\rangle - \langle \pi(k_0)\xi, \eta\rangle| \leq \|(T_\delta - T_0)_X\|.$$

So if (3.3) holds and if π is a representation of G on X such that the exponential growth s of $\|\pi(g)\|$ is not too large compared to θ, Proposition 1.4 implies that K-biinvariant coefficients of π converge uniformly to a limit. Equivalently, the measures $m_g = \iint_{K\times K} \delta_{kgk'} dkdk'$ satisfy the Cauchy criterion and converge as $g \to \infty$ to an element $P \in \mathcal{C}_{\mathcal{F}(X, s\ell + C)}$, the candidate for the required Kazhdan projection.
Step 2 P is a projection and $m_g * P = P$ for every $g \in G$. This is obvious :

$$m_g * P = \lim_{g'\to\infty} m_g * m_{g'} = \lim_{g'\to\infty} \int_K m_{gkg'} dk = P.$$

Step 3 The image of $\pi(P)$ is made of invariant vectors. This relies on a study of the coefficients of π with respect to vectors of arbitrary K-type. We recall that if V is an irreducible representation of K, a vector ξ in an arbitrary representation (π, X) of K is of K-type V if the space spanned by $\pi(K)\xi$, as a linear representation of K, is isomorphic to $V^{\oplus d}$ for some integer d (necessary bounded by the dimension of V). Then Lafforgue shows for every K-invariant vector $\xi \in X$ and every $\eta \in X$ of nontrivial K-type V, the coefficients $\langle \pi(g)\xi, \eta\rangle$ tends to 0 as $g \to \infty$. The conclusion follows from step 2 and the Peter-Weyl theorem.

In [10] I extended this step 3 to prove a quantitative form of decay of matrix coefficients of (non-necessarily isometric nor uniformly bounded) Banach space representations, generalizing the results from [Oh02]:

Theorem 3.9 ([10])**.** *Denote $G = SL(3, \mathbf{R})$ and $K = \mathrm{SO}_3$. Let X be a space such that* (3.3) *holds and π a representation of G on X without invariant vectors and with small exponential growth. There is a function $\varepsilon \in C_0(G)$ such that for every $\xi \in X$, $\eta \in X^*$ of finite K-type*[6]

$$|\langle \pi(g)\xi, \eta\rangle| \leq C\varepsilon(g)\|\xi\|\|\eta\|$$

for some constant C depending only on the dimensions of $\mathrm{span}(\pi(K)\xi)$ *and* $\mathrm{span}({}^t\pi(K)\eta)$.

The function ε is an explicit exponentially decreasing function, see [10, Theorem 4.2].

When ξ and η have K-type V, V' and V' is the trivial representation, this theorem is the content of Step 3, and was proved by Lafforgue. When $V \neq V'$ the proof of this theorem follows the same lines as in Lafforgue's proof. The case when $V = V'$ requires a new idea.

3.4 Case of other groups

3.4.1 Case of $\mathrm{Sp}(2, \mathbf{R})$ and $\widetilde{\mathrm{Sp}}(2, \mathbf{R})$

I obtained Theorem 3.6 for $G = \mathrm{Sp}(2, \mathbf{R})$ or $G = \widetilde{\mathrm{Sp}}(2, \mathbf{R})$ with Tim de Laat in [6] with $\beta(G) = \frac{1}{10} - \varepsilon$. This is the same 10 as in §2.2.1. Although more technically involved, the proof is very similar to $G = \mathrm{SL}(3, \mathbf{R})$.

6. ξ (η) is of finite K-type if the vector space generated by the K-orbit of ξ (respectively η) is finite dimensional

Let me us discuss first $\mathrm{Sp}(2, \mathbf{R})$. In that case the maximal compact subgroup, still denoted K, is isomorphic to $U(2)$. The averaging operators T_δ from §1.3 and S_θ from §2.2.1 are both p-summable for $p > 10$, and therefore by the same argument as for $\mathrm{SL}(3, \mathbf{R})$ the maps $\delta \mapsto (T_\delta)_X$ and $\theta \mapsto (S_\theta)_X$ are Hölder-continuous for every Banach space satisfying $d_i(X) = O(i^{\frac{1}{10}-\varepsilon})$. On the other hand, we are able to combine these two averaging operators in the same way as for $\mathrm{SL}(3, \mathbf{R})$ to explore the whole Weyl chamber $\Lambda = U(2)\backslash \mathrm{Sp}(2, \mathbf{R})/U(2)$, and we can adapt the three steps of the preceding section similarly.

We move to the case of the universal covering $\widetilde{\mathrm{Sp}}(2, \mathbf{R})$. This case is treated by elaborating on the ideas developed in [HdL16]. Recall that it is a central extension

$$0 \to \mathbf{Z} \to \widetilde{\mathrm{Sp}}(2, \mathbf{R}) \to \mathrm{Sp}(2, \mathbf{R}) \to 0. \tag{3.4}$$

The preimage of $K = U(2)$ is $\tilde{K} = \mathbf{R} \times \mathrm{SU}(2)$. This group being noncompact, we cannot average with respect to it. What plays the role of the Weyl chamber is now the quotient of G by the subgroup $\{((t, k), (t, k')| t = t'\} \simeq \mathbf{R} \times \mathrm{SU}(2)^2$ of $(\mathbf{R} \times \mathrm{SU}(2))^2$ acting by left/right multiplication (since this action factors through the compact group $\mathbf{R}/2\mathbf{Z} \times \mathrm{SU}(2)^2$). This "modified Weyl chamber" is naturally homeomorphic to the direct product $\Lambda \times \mathbf{R}$ of the Weyl chamber of $\mathrm{Sp}(2, \mathbf{R})$ with $\mathbf{R}$. The same moves as for $\mathrm{Sp}(2, \mathbf{R})$ using the averaging operators T_δ and S_θ still allow to explore this whole "modified Weyl chamber" above the Weyl chamber of $\mathrm{Sp}(2, \mathbf{R})$. What is crucial is that the exploration can be performed fast enough in the sense that the moves, although unbounded in the Λ direction, remain bounded in the $\mathbf{R}$ direction. This holds because the central extension (3.4) is given by a bounded cohomology class.

3.4.2 The case of $\mathrm{SL}(r, \mathbf{R})$

In the case of $\mathrm{SL}(r, \mathbf{R})$ for $r > 3$, a difficulty arises from the fact that the rank is large.

For $n \geq 1$, we can consider the averaging operators T_δ on the $n+1$-dimensional sphere S^{n+1} already considered in §2.2.1. Using that they are p-summable for $p > 2 + \frac{2}{n}$, it can be shown that they satisfy (3.3) for every Banach space for which $d_i(X) = O(i^{\frac{1}{2}-\frac{1}{2n+2}-\varepsilon})$ for some ε.

Also, as in §2.2.1, for $r = 2n+1$, the various embeddings of S^{n+1} in the symmetric space $\mathrm{SL}(r, \mathbf{R})/\mathrm{SO}(r)$ can be combined to explore a connected unbounded part $\mathcal{D}$ of the Weyl chamber $\Lambda = \mathrm{SO}(r)\backslash \mathrm{SL}(r, \mathbf{R})/\mathrm{SO}(r)$. This allows to perform an analogous of Step 1 in the case of $\mathrm{SL}(3, \mathbf{R})$ and prove that the measures $m_g = \iint_{\mathrm{SO}(r)\times \mathrm{SO}(r)} \delta_{kgk'} dk dk'$ converge, as $\mathrm{SO}(r) g \mathrm{SO}(r)$ goes to infinity staying in $\mathcal{D}$, to a candidate for the Kazhdan projection. The difficulty is to prove that this limit is a Kazhdan projection: the region $\mathcal{D}$ of the Weyl chamber that we are able to explore is very thin : it is contained in the union of the faces of codimension $n-1$. This does not allow to perform Step 2 if $n > 1$.

We could not overcome this difficulty in [7], where we only dealt with the situation where we knew *a priori* that $\pi(g)$ converges in the weak operator topology to a projection on the invariant vectors (Howe-Moore property). This happens for example if π is a uniformly bounded representation on a reflexive space [Vee79], of π is of the form $\pi_0 \otimes 1_X$ by permutations on $L_2(\Omega; X)$ for a representation π_0 on $L_2(\Omega)$ by the usual Howe-Moore property for unitary representations. This is enough to prove property (T_X^{proj}) on reflexive spaces satisfying $d_i(X) = O(i^{\frac{1}{2}-\frac{1}{2n-2}-\varepsilon})$, or non-embeddability of expanders [7] on any space satisfying $d_i(X) = O(i^{\frac{1}{2}-\frac{1}{2n-2}-\varepsilon})$.

It is only in [5] that we found a way to overcome this difficulty, at the cost of enlarging r to $r = 3n$. Still using the same averaging operators on S^{n+1}, we significantly improved our exploration of the Weyl chamber of $\mathrm{SL}(3n, \mathbf{R})$, and found a thick enough path going to infinity, which allowed to perform an analogous of Step 2 and Step 3.

Chapter 4

Extensive amenability

This chapter presents my work on amenability for discrete groups [4, 3, 2].

4.1 Motivation, Definitions and first properties

Throughout this chapter all the groups and sets we consider will be equipped with the discrete topology (even those which carry a natural non-discrete topology).

Definition 4.1. A mean on a set X is a finitely additive probability measure defined on all subsets of X.

If m is a mean on X, we can integrate every bounded function on X with respect to m and get a state on $\ell_\infty(X)$, denoted $f \mapsto \int f dm$. Conversely, a state on $\ell_\infty(X)$, when restricted to indicator functions, gives rise to a mean. This establishes a one-to-one bijection between means on X and states on $\ell_\infty(X)$. In practice it is much more convenient to work with the state than the mean.

Definition 4.2. An action of a group G on a set X is amenable if there is a mean on X that is invariant under the action of G.

When specializing to the action of G on itself by left-translation, one gets the definition of amenable groups.

Definition 4.3. A group G is amenable if the action of G on itself by left-translation is amenable.

This definition 4.2 of amenable actions is due to Greenleaf [Gre69], and should not be confused with the somewhat opposite notion defined later by Zimmer : an action is amenable in both senses if and only if the group is amenable.

The class of amenable groups contains all finite and abelian groups, and is stable under subgroups, quotients, direct limits and extensions. The class EG of elementary amenable groups is the smallest class containing all finite and abelian groups, and stable under all these properties. These are in a sense the groups that are amenable for “an obvious reason”. Let us call non-elementary amenable groups the groups which are amenable but not elementary amenable.

The first example of a non-elementary amenable group was the Grigorchuk group of intermediate growth [Gri84]. An example of a group which cannot be constructed from groups of sub-exponential growth (which, in some sense, can be also considered as an “easy case” of amenability) is the Basilica group introduced by Grigorchuk and Zuk and whose amenability was proved in [BV05] using asymptotic properties of random walks on groups. These methods were then generalized in [BKN10] and [AAV13] for a big class of groups acting on rooted trees. Finally, a striking recent family of examples of non-elementary amenable groups is given by the topological full groups of minimal subshifts, whose commutator subgroup is simple [JM13].

It is known [Cho80] that every group in EG can be obtained from finite and abelian groups by taking successive direct limits and extensions. So typically, to prove the amenability of a finitely generated group G by proving that it belongs to EG, one reduces the problem in two simpler problems by considering a normal subgroup $N \subset G$: the simpler problems are N (respectively G/N) are amenable. Of course in the extreme case of a simple group as in [JM13], such a reduction cannot be performed. However, if one accepts to leave the category of groups and work instead with the category of group actions, there is no a priori reason why such a task cannot always work, and the following obvious lemma says that there is some hope.

Lemma 4.4. *Let $G \curvearrowright X$ be an action. Then G is amenable if and only if $G \curvearrowright X$ is amenable and the action has amenable stabilizers.*

In particular, if H is a subgroup of G, then G is amenable if and only if H is amenable and $G \curvearrowright G/H$ is amenable.

A first naive strategy would be to consider the amenable group actions, start from the "obvious examples" and exploit the permanence properties of amenable actions. This is not such a good idea, because the amenability of a group action has bad permanence properties. Quite easily, it does not pass to subactions, and neither to extensions [MP03].

The solution is to work with a stronger notion called extensive amenability for an action. As the name is intended to suggest, this is a much stronger form of amenability and it has an intimate connection with extensions of groups and of actions. This property was introduced (without a name) in [JM13] as a tool to prove amenability of groups, a role it continued to play in [3]. Some of its properties were studied in [4]. It was only in [2] that the terminology was coined, and that most properties have been proved. In order to give its formal definition, we denote by $\mathscr{P}_{\mathrm{f}}(X)$ the set of all finite subsets of X.

Definition 4.5 ([2])**.** The action of a group G on a set X is *extensively amenable* if there is a G-invariant mean m on $\mathscr{P}_{\mathrm{f}}(X)$ so that for every $A \in \mathscr{P}_{\mathrm{f}}(X)$ the mean m gives full weight to the collection of subsets that contain A.

Extensive amenability has many stability properties. Most of them have been established in [2], but some already appeared in [JM13] or in [4].

Locality An action $G \curvearrowright X$ is extensively amenable if and only the action of G on every orbit is extensively amenable. If N is the kernel of the action, then $G \curvearrowright X$ is extensively amenable if and only if $G/N \curvearrowright X$ is extensively amenable.

Stable by subactions If $G \curvearrowright X$ is an extensively amenable action, if $H \subset G$ is a subgroup and $Y \subset X$ is a H-invariant subset, then $H \curvearrowright Y$ is extensively amenable.

Stable by direct unions If $G_i \curvearrowright X_i$ is an increasing net of extensively amenable actions, then its direct union (the action of $\cup_i G_i$ on $\cup_i X_i$) is extensively amenable.

Stable by quotients If $G \curvearrowright X$ is extensively amenable and $G \curvearrowright Y$ is another action with $q\colon Y \to X$ a surjective G-map, then $G \curvearrowright Y$ is extensively amenable.

Stable by extensions Let G be a group acting on two sets X, Y and let $q\colon X \to Y$ be a G-map. If $G \curvearrowright Y$ is extensively amenable and if for every $y \in Y$ the action of the stabilizer $G_y \curvearrowright q^{-1}(y)$ is extensively amenable, then $G \curvearrowright X$ is extensively amenable.

Stable by Functor-extensions The most general statement of functor-extensions (for which we refer to Theorem 4.13) is a bit technical to state. Here are two simple particular representative cases :

— If $G \curvearrowright X$ is extensively amenable and A is an amenable group, then $A^{(X)} \rtimes G \curvearrowright A^{(X)}$ is extensively amenable.

— If $G \curvearrowright X$ is extensively amenable and $\mathrm{Sym}(X)$ denotes the group of permutations of X with finite support, then $\mathrm{Sym}(X) \rtimes G \curvearrowright \mathrm{Sym}(X)$ is extensively amenable.

4.2 Examples and non-examples

It is easy to see that every action of an amenable group is extensively amenable, and that the action of G on itself by left-multiplication is extensively amenable if and only if G is amenable. The main source of extensively amenable actions of nonamenable groups is given by the next result. We say that an action $G \curvearrowright X$ is recurrent if for every finitely supported symmetric probability measure μ on G and every $x_0 \in X$, the random walk on X starting at x_0 and which walks according to μ is recurrent. If $G = \langle S \rangle$ is finitely generated, it is enough to check this property for the uniform probability measure on $S \cup S^{-1}$ (or any other symmetric probability measure with finite and generating support).

Theorem 4.6 ([3]). *Recurrent actions are extensively amenable.*

It is also an easy fact that extensive amenability of an action implies amenability of the action. Since extensive amenability is preserved by subactions, we get that extensively amenable actions are hereditarily amenable in the sense that all their subactions are amenable. This condition was studied in [4]. By [2, Theorem 6.1] the converse is not true : the action on $\mathbf{R}$ of the group of piecewise $\mathrm{PSL}_2(\mathbf{R})$-homeomorphisms of $\mathbf{R}$ is hereditarily amenable but not extensively amenable.

4.3 Applications of extensive amenability

Starting from the recurrent actions and using all the stability properties above, we get a class of extensively amenable actions, that we could call *elementary extensively amenable action* and denote EEA. The class of groups for which $G \curvearrowright G$ is elementary extensively amenable could be called the class of *elementary extensively amenable groups* and denoted EEG. The main result from [3], that I present now, gives a general criterion for amenability of groups of homeomorphisms of topological spaces. At the time when [3] was written, this criterion allowed to recover the amenability of all known non-elementary amenable groups, thereby proving that every known example of amenable group belongs to EEG.

4.3.1 Amenability of groups of homeomorphisms and groupoids

Let $\mathcal{X}$ be a topological space. A *germ* of homeomorphism of $\mathcal{X}$ is an equivalence class of pairs (g, x) where $x \in \mathcal{X}$ and g is a homeomorphism between a neighborhood of x and a neighborhood of $g(x)$, where two germs (g_1, x_1) and (g_2, x_2) are equal if $x_1 = x_2$, and if g_1 and g_2 coincide on a neighborhood of x_1. The set of all germs of the action of G on $\mathcal{X}$ is a *groupoid*. Denote by $\mathsf{o}(g, x) = x$ and $\mathsf{t}(g, x) = g(x)$ the *origin* and the *target* of the germ.

A composition $(g_1, x_1)(g_2, x_2)$ is defined if $g_2(x_2) = x_1$, and then it is equal to $(g_1 g_2, x_2)$, which makes sense because $g_1 g_2$ is defined on a neighborhood of x_2. The inverse of a germ (g, x) is the germ $(g, x)^{-1} = (g^{-1}, g(x))$.

A groupoid of germs of homeomorphisms on $\mathcal{X}$ is a set of germs of homeomorphisms of $\mathcal{X}$ that is closed under composition and inverse, and that contains all germs $(Id_{\mathcal{X}}, x)$ for $x \in \mathcal{X}$.

If a group G acts by homeomorphisms on $\mathcal{X}$, its groupoid of germs is the groupoid of all germs (g, x) for $g \in G$ and $x \in \mathcal{X}$.

For a given groupoid $\mathcal{G}$ of germs of homeomorphisms on $\mathcal{X}$, and for $x \in \mathcal{X}$, the *isotropy group*, or *group of germs* $\mathcal{G}_x$ is the group of all germs $\gamma \in \mathcal{G}$ such that $\mathsf{o}(\gamma) = \mathsf{t}(\gamma) = x$. If $\mathcal{G}$ is the groupoid of germs of the action of a group G on $\mathcal{X}$, then the group of germs $\mathcal{G}_x$ is the quotient of the stabilizer G_x of x by the subgroup of elements of G that act trivially on a neighborhood of x.

The *topological full group* of a groupoid of germs $\mathcal{G}$, denoted $[[\mathcal{G}]]$ is the set of all homeomorphisms $F : \mathcal{X} \to \mathcal{X}$ such that all germs of F belong to $\mathcal{G}$.

Theorem 4.7 ([3])**.** *Let G be a group of homeomorphisms of a topological space $\mathcal{X}$, and $\mathcal{G}$ be its groupoid of germs. Let $\mathcal{H}$ be a groupoid of germs of homeomorphisms of $\mathcal{X}$. Suppose that the following conditions hold.*

1. *The group $G \cap [[\mathcal{H}]]$ is amenable.*
2. *For every generator g of G the set of points $x \in \mathcal{X}$ such that $(g, x) \notin \mathcal{H}$ is finite. We say that $x \in \mathcal{X}$ is* singular *for g if $(g, x) \notin \mathcal{H}$.*
3. *The action of G on $\{gx, g \in G, x \in X$ is singular for some element of $G\} \subset \mathcal{X}$ is extensively amenable.*
4. *The groups of germs $\mathcal{G}_x$ are amenable.*

Then the group G is amenable.

4.3.2 Interval exchanges

An interval exchange transformation is a permutation of a circle obtained by cutting this circle into finitely many intervals (arcs), and reordering them. More precisely, an interval exchange transformation is a right-continuous permutation g of $\mathbf{R}/\mathbf{Z}$ such that the set $\{gx - x, x \in \mathbf{R}/\mathbf{Z}\}$, called the set of angles of g, is finite. Interval exchanges form a group of permutations of $\mathbf{R}/\mathbf{Z}$ that we denote IET.

A basic question on IET was raised by Katok, namely whether or not this group contains a non-abelian free group. This problem attracted some attention recently; for instance, Dahmani–Fujiwara–Guirardel [DFG13] showed that free subgroups in IET are *rare* in the sense that a group generated by a generic pair of interval exchange transformations is not free. A related open question raised in [dC14, p.4] is whether IET is amenable. Although IET can be naturally realized as a group of homeomorphisms, Theorem 4.7 cannot be used to attack this question. In [2] we developed an approach for this problem and proved:

Theorem 4.8 ([2])**.** *If the action of a subgroup $G \subset$ IET on $\mathbf{R}/\mathbf{Z}$ is extensively amenable, then G is amenable.*

The rank of a group G of interval exchange transformations is the supremum of the integers d such that $\mathbf{Z}^d$ embeds in the group generated by the set of angles of all elements of G. By using Theorem 4.6, every group of interval exchanges of rank ≤ 2 is amenable. This gives again new examples of amenable groups (still in EEG).

The following natural question remains open.

Problem 4.9. Find groups which are amenable but not extensive elementary amenable.

To prove the amenability of IET it would be enough to answer positively the following question

Question 4.10 ([2])**.** Let $G \curvearrowright X$ be an action of a finitely generated group such that the orbits grow polynomially. Is this action extensively amenable?

4.4 Functor-extensions

The aim of this section is to present, for the first time, the most general version of the stability property of extensive amenability through function-extensions.

Definition 4.11. An *amenable category of actions* is a small category $\mathcal{C}$ of group actions [1], and such that for any object $H \curvearrowright Y$ in $\mathcal{C}$, there is a mean on Y invariant under $H \rtimes Isom_{\mathcal{C}}(H \curvearrowright Y)$.

We say that $\mathcal{C}$ is an extensively amenable category of actions if moreover the action of $H \rtimes Isom_{\mathcal{C}}(H \curvearrowright Y)$ on Y is extensively amenable.

Remark 4.12. In an amenable category of actions, for every object $H \curvearrowright Y$ in $\mathcal{C}$, there is an H-invariant mean $m_{H \curvearrowright Y}$ on Y with the property that the push-forward of $m_{H \curvearrowright Y}$ by any isomorphism $H \curvearrowright Y \to H' \curvearrowright Y'$ is $m_{H' \curvearrowright Y'}$. Indeed, we can pick a representative $H \curvearrowright Y$ in each isomorphism class in $\mathcal{C}$, then pick an-$H \rtimes Isom_{\mathcal{C}}(H \curvearrowright Y)$-invariant mean on Y for each such representative, and finally define $m_{H' \curvearrowright Y'}$ as the push-forward of $m_{H \curvearrowright Y}$ by any isomorphism $H \curvearrowright Y \to H' \curvearrowright Y'$ since this does not depend on the isomorphism.

Assume that $G \curvearrowright X$. Consider the small category where the objects are the elements of $\mathscr{P}_{\mathrm{f}}(X)$ and the set of arrows between A and B is the set of g such that $g(A) \subset B$. We will be interested in "almost functors" from this category to $\mathcal{C}$, which are functors except that we want the axioms of a functor to be satisfied only "almost everywhere" with respect to the order on $\mathscr{P}_{\mathrm{f}}(X)$.

Formally an almost everywhere G-equivariant functor from $\mathscr{P}_{\mathrm{f}}(X)$ to $\mathcal{C}$ is a partial assignement of objects and arrows in $\mathcal{C}$ as follows :

— there is $A_0 \in \mathscr{P}_{\mathrm{f}}(X)$, objects $F(A)$ in $\mathcal{C}$ for every $A_0 \subset A \in \mathscr{P}_{\mathrm{f}}(X)$ and arrows $F(\subset) \in \mathrm{Mor}_{\mathcal{C}}(F(A), F(B))$ for every $A_0 \subset A \subset B \in \mathscr{P}_{\mathrm{f}}(X)$ satisfying that

$$\begin{tikzcd} F(A) \arrow[r, "F(\subset)"'] \arrow[rr, bend left, "F(\subset)"] & F(B) \arrow[r, "F(\subset)"'] & F(C) \end{tikzcd}$$

commutes for every $A_0 \subset A \subset B \subset C \in \mathscr{P}_{\mathrm{f}}(X)$.

— for every $g, h \in G$ there are $A_g, A_h, A_{g,h} \in \mathscr{P}_{\mathrm{f}}(X)$ containing A_0 and isomorphisms $F(g) \in \mathrm{Mor}_{\mathcal{C}}(F(A), F(gA))$ for every $A_g \subset A \in \mathscr{P}_{\mathrm{f}}(X)$ satisfying that the diagrams

$$\begin{tikzcd} F(A) \arrow[r, "F(g)"] \arrow[d, "F(\subset)"] & F(gA) \arrow[d, "F(\subset)"] \\ F(B) \arrow[r, "F(g)"] & F(gB) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F(C) \arrow[r, "F(h)"'] \arrow[rr, bend left, "F(gh)"] & F(hC) \arrow[r, "F(g)"'] & F(ghC) \end{tikzcd}$$

1. the objects are group actions and the morphisms $(G \curvearrowright X) \to (H \curvearrowright Y)$ are pairs of a map $X \to Y$ and a group homomorphism $G \to H$ which intertwine the actions

commute for every $A_g \subset A \subset B$ and $A_{g,h} \subset C$.
The notion of almost everywhere G-equivariant functor from $\mathscr{P}_{\mathrm{f}}(X)$ to $\boldsymbol{\mathcal{C}}$ makes sense for an arbitrary category $\boldsymbol{\mathcal{C}}$, but if $\boldsymbol{\mathcal{C}}$ is a category of group actions (or any subcategory of a category admitting direct limits) this notion yields an interesting action of G. Namely if F is an almost everywhere G-equivariant functor from $\mathscr{P}_{\mathrm{f}}(X)$ to $\boldsymbol{\mathcal{C}}$, $(F(A))_{A_0 \subset A}$ is an inductive system and we can define $F(X)$ as its direct limit, and the compatible morphisms $F(g)$ yield to an action on $F(X)$ by isomorphisms. Explicitely if $F(A) = F_{\mathbf{Grp}}(A) \curvearrowright F_{\mathbf{Set}}(A)$, then $F(X)$ is the action $F_{\mathbf{Grp}}(X) \curvearrowright F_{\mathbf{Set}}(X)$ where $F_{\mathbf{Grp}}(X) = \lim_A F_{\mathbf{Grp}}(A)$ and $F_{\mathbf{Set}}(X) = \lim_A F_{\mathbf{Grp}}(A)$, and the action of G is a pair of actions (one of $F_{\mathbf{Set}}(X)$ by permutation and one on $F_{\mathbf{Grp}}(X)$ by group automorphisms) which yield an action of $G \ltimes F_{\mathbf{Grp}}(X)$ on $F_{\mathbf{Set}}(X)$.

Theorem 4.13. *Let G be a group with an extensively amenable action on a set X, and F an almost-everywhere G-equivariant functor from $\mathscr{P}_{\mathrm{f}}(X)$ to an amenable category of actions $\boldsymbol{\mathcal{C}}$. Then the action $G \ltimes F_{\mathbf{Grp}}(X)$ on $F_{\mathbf{Set}}(X)$ is amenable.*

If $\boldsymbol{\mathcal{C}}$ is an extensively amenable category of actions, then the action of $F_{\mathbf{Grp}}(X) \rtimes G$ on $F_{\mathbf{Set}}(X)$ is extensively amenable.

Proof. Let us first show that the action is amenable. Let m be a G-invariant mean on $\mathscr{P}_{\mathrm{f}}(X)$ giving full weight to the subsets containing any given finite subset of X. By remark 4.12, for each $A \in \mathscr{P}_{\mathrm{f}}(X)$ there is a $F_{\mathbf{Grp}}(A)$-invariant mean m_A on $F_{\mathbf{Set}}(A)$, with the property that $m_{A'}$ is the push-forward of m_A by every isomorphism $F(A) \to F(A')$, and in particular m_{gA} is the push-forward of m_A by g for each $g \in G$ for which $A_g \subset A$. Define a $F_{\mathbf{Grp}}(A)$-invariant mean $\widetilde{m}_A$ on $F_{\mathbf{Set}}(X)$ as the push-forward of m_A under $F(\subset)$. Then $\widetilde{m}_{gA}$ is the push-forward of $\widetilde{m}_A$ by g provided that $A_g \subset A$. Therefore the mean $\int_{\mathcal{P}_f(X)} \widetilde{m}_A dm(A)$ is G-invariant and $F_{\mathbf{Grp}}(X)$-invariant.

Let us now assume that $\boldsymbol{\mathcal{C}}$ is an extensively amenable category of actions. In the proof that the action of $F_{\mathbf{Grp}}(X) \rtimes G$ on $F_{\mathbf{Set}}(X)$ is extensively amenable, we shall use twice (see [JM13]) that for an action $\Gamma \curvearrowright Z$,

$$\Gamma \curvearrowright Z \text{ is extensively amenable} \iff (\mathbf{Z}/2\mathbf{Z})^{(Z)} \rtimes \Gamma \curvearrowright (\mathbf{Z}/2\mathbf{Z})^{(Z)} \text{ is amenable .} \quad (4.1)$$

Note that the direction $\Longrightarrow$ is a special case of what we just proved.

The image $\widetilde{\boldsymbol{\mathcal{C}}}$ of $\boldsymbol{\mathcal{C}}$ by the functor $H \curvearrowright Y \to (\mathbf{Z}/2\mathbf{Z})^{(Y)} \rtimes H \curvearrowright (\mathbf{Z}/2\mathbf{Z})^{(Y)}$ is a small subcategory of the category of actions. Moreover for every object $x = (\mathbf{Z}/2\mathbf{Z})^{(Y)} \rtimes H \curvearrowright (\mathbf{Z}/2\mathbf{Z})^{(Y)}$, the action of $((\mathbf{Z}/2\mathbf{Z})^{(Y)} \rtimes H) \rtimes Isom_{\widetilde{\boldsymbol{\mathcal{C}}}}(x) = (\mathbf{Z}/2\mathbf{Z})^{(Y)} \rtimes (H \rtimes Isom_{\boldsymbol{\mathcal{C}}}(H \curvearrowright Y))$ is amenable by the direction $\Longrightarrow$ in (4.1) and the assumption that $H \rtimes Isom_{\boldsymbol{\mathcal{C}}}(H \curvearrowright Y) \curvearrowright Y$ is extensively amenable. Therefore $\widetilde{\boldsymbol{\mathcal{C}}}$ is an amenable category of actions. Moreover we have an almost-everywhere G-equivariant functor from $\mathscr{P}_{\mathrm{f}}(X)$ to $\widetilde{\boldsymbol{\mathcal{C}}}$ which sends A to the action of $(\mathbf{Z}/2\mathbf{Z})^{(F_{\mathbf{Set}}(A))} \rtimes F_{\mathbf{Grp}}(A)$ on $(\mathbf{Z}/2\mathbf{Z})^{(F_{\mathbf{Set}}(A))}$. By the first part of the Theorem we get that the action of $((\mathbf{Z}/2\mathbf{Z})^{(F_{\mathbf{Set}}(X))} \rtimes F_{\mathbf{Grp}}(X)) \rtimes G$ on $(\mathbf{Z}/2\mathbf{Z})^{(F_{\mathbf{Set}}(X))}$ is amenable. By (4.1) this is equivalent to extensive amenability of the action of $F_{\mathbf{Grp}}(X) \rtimes G$ on $F_{\mathbf{Set}}(X)$. ☐

Let us explain how to recover Theorem 4.7.

Proof of Theorem 4.7. Denote by $X \subset \mathcal{X}$ the subset

$$X = \{gx, g \in G, x \in \mathcal{X} \text{ is singular for some element of } G\}.$$

We work with $\mathcal{G}/\mathcal{H}$, and notice that the target map t is well-defined on $\mathcal{G}/\mathcal{H}$. We call an element of $x\mathcal{H} \in \mathcal{G}/\mathcal{H}$ trivial if x belongs to $\mathcal{H}$.

We consider the extensively amenable category of action $\mathcal{C}$ defined as follows. The objects are indexed by $\mathcal{P}_f(X)$, and the object corresponding to $A \in \mathcal{P}_f(X)$ is the action of the trivial group on $Y_A := \prod_{a\in A} Y_a$, where $Y_a = \{f \in \mathcal{G}/\mathcal{H}, t(f) = a\}$. Since all objects are actions of the trivial group on a set, to define a morphism we just have to give a map between sets. We have a morphism $Y_A \to Y_B$ for each A-uple $(g_a)_{a\in A}$ satisfying that $o(g_a) = a$ and $\sigma\colon a \mapsto t(g_a)$ is an injection of A to B, given by $(y_a)_{a\in A} \mapsto (\widetilde{y}_b)_{b\in B}$ by setting $\widetilde{y}_{\sigma(a)} = g_a y_a$ and $\widetilde{y}_b$ is trivial if $b \notin \sigma(A)$. For every object Y_A, the isomorphism group in $\mathcal{C}$ of Y_A has a natural homomorphism on $Sym(A)$, whose kernel is $\prod_{a\in A} \mathcal{G}_a$, which is amenable by 4. This proves that $\mathcal{C}$ is an extensively amenable category of actions, because every action of an amenable group is extensively amenable.

For $A \subset B \in \mathcal{P}_f(X)$, we can realize $Y_A \subset Y_B$ by sending $(y_a)_{a\in A} \to (y_b)_{b\in B}$ by setting y_b is trivial if $b \in B \setminus A$. Moreover if for $g \in G$ we denote by A_g the set of points that are singular for g, there is an isomorphism $Y_A \to Y_{gA}$ defined by $(y_a)_{a\in A} \in Y_A \mapsto ((g, g^{-1}a)y_{g^{-1}a})_{a\in gA} \in Y_{gA}$. It is straightforward to check that all the axioms of an almost everywhere G-equivariant functor from $\mathscr{P}_{\mathrm{f}}(X)$ to $\mathcal{C}$ are satisfied for the map $F\colon A \mapsto Y_A$. Moreover, $F(X)$ ideitifies with the action of the trivial group on

$$Z = \{(y_x)_{x\in X} \in \prod_{x\in X} Y_x, y_x \text{ is trivial for all but finitely many } x\},$$

and the action of G on Z is given by $g \cdot (y_x) = ((g, g^{-1}x)y_{g^{-1}x})_{x\in X}$.

By Theorem 4.13 the action of G on Z is extensively amenable. In particular it is hereditarily amenable. In order to prove the amenability of G, it remains to find an element of Z with amenable stabilizer. This will be the trivial element. Indeed, the stabilizer of the trivial element in Z is $\{g \in G, (g, g^{-1}x) \in \mathcal{H} \forall x \in X\}$, which coincides with $G \cap [[\mathcal{H}]]$, which is amenable by 1. □

Let us explain how to recover (and generalize) Theorem 4.8.

A bijection g of $\mathbf{R}/\mathbf{Z}$ is called piecewise orientation-preserving and continuous if there is a partition of $\mathbf{R}/\mathbf{Z}$ into finitely many infinite intervals $I_1, \dots, I_n$ such that the restriction of g to I_k is strictly increasing and continuous. In particular, g has left and right limits at each point, and we can consider the functions g_- and g_+ defined by $g_\pm(x) = \lim_{\varepsilon\to 0, \varepsilon>0} g(x\pm\varepsilon)$. The assumption that g is increasing on the interior of each I_k implies that g_+ and g_- are both bijections of $\mathbf{R}/\mathbf{Z}$. The assumption that the I_k's are infinite implies that $g_+ = g_-$ if and only if g is a homeomorphism (otherwise it would imply that g is the composition of a homeomorphism and a permutation with finite support).

Theorem 4.14. *Let G be a group of piecewise orientation-preserving and continuous bijections of $\mathbf{R}/\mathbf{Z}$. Assume that $G \curvearrowright \mathbf{R}/\mathbf{Z}$ is extensively amenable, and that $G \cap Homeo(\mathbf{R}/\mathbf{Z})$ is amenable. Then G is amenable.*

Proof. For a set X consider $Sym(X)$, the set of all bijections of X with finite support.

We have an action of G on the set $Sym(\mathbf{R}/\mathbf{Z})$ given by $g \cdot \sigma = g_- \circ \sigma \circ g_+^{-1}$. We claim that this action is hereditarily amenable if $G \curvearrowright \mathbf{R}/\mathbf{Z}$ is extensively amenable. Before we prove the claim, we observe that this will prove the Theorem. Indeed, the stabilizer of the identity of $\mathbf{R}/\mathbf{Z}$ for this action is $\{g \in G, g_+ = g_-\} = G \cap Homeo(\mathbf{R}/\mathbf{Z})$. We conclude that G is amenable by the observation that a hereditarily amenable action with one amenable stabilizer is amenable.

The claim follows from Theorem 4.13, which will actually prove that the action of G on $Sym(\mathbf{R}/\mathbf{Z})$ is extensively amenable. The group is G acting on $\mathbf{R}/\mathbf{Z}$. The category $\mathcal{C}$ is the full subcategory of **Set** whose objects are $\{Sym(A), A \in \mathcal{P}_f(\mathbf{R}/\mathbf{Z})\}$. We see it as a category of actions by identifying a set with the action of the trivial group on this set. Since $Sym(A)$ is a finite set for all finite set A, $\boldsymbol{\mathcal{C}}$ is clearly an extensively amenable category of actions. The functor is Sym, and the notation is coherent in the sense that $Sym(\mathbf{R}/\mathbf{Z})$ is indeed the direct limit (the union) of $Sym(A)$ for $A \in \mathcal{P}_f(X)$. For $g \in G$ the set A_g is the points of discontinuity of g, and for $A_g \subset A$ the morphism $Sym(A) \to Sym(gA)$ is the restriction of the action of g on the subset $Sym(A)$ of $Sym(X)$. This makes sense because if $\sigma \in Sym(A)$, then $g_-\sigma g_+^{-1} \in Sym(gA)$: for all $x \in \mathbf{R}/\mathbf{Z} \setminus gA$, using that $g_+^{-1}(x) = g_-^{-1}x = g^{-1}x \notin A$ because $gx \notin A_g$, we have $g_-\sigma g_+^{-1}(x) = g_-\sigma(g^{-1}x)g_-(g^{-1}x) = x$. $\square$

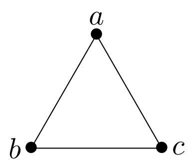
Chapter 5

On the topological structure of the space of large-scale simply connected graphs

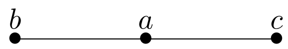
This chapter presents my work with Romain Tessera on the topological structure of the space of large-scale simply connected graphs [12, 13].

By a graph X we always mean a locally finite connected unoriented graph without multiple edges or loops, and when we write $x \in X$ we always mean that x is a vertex in X. Each edge is endowed with the metric of $[0, 1]$, which makes the graph into a geodesic space. Since there is no multiple edge or loop, a graph is entirely determined by the restriction of the graph distance to the vertex set. Conversely, every metric space (X, d) where the distance takes integer values and is 1-geodesic comes from a graph structure (put an edge between any two points at distance 1). Here 1-geodesic means that for every $x, y \in X$ there is a sequence $x = x_0, x_1, \ldots, x_n = y$ where $d(x_i, x_{i+1}) = 1$ and $n = d(x, y)$.

We adopt the following convention: if $x \in X$ and $R > 0$, $B(x, R)$ denotes the set of points in the geodesic space X at distance $\leq R$ from x. For its intrinsic metric, it becomes a geodesic space and also, if R is an integer, a graph. For example, if X is a complete graph on 3 vertices a, b, c



then $B(a, 1)$ is the not the complete graph on a, b, c but



Definition 5.1. Two graphs X, Y are said to be R-close if for every $x \in X$ there is $y \in Y$ (and conversely for every $y \in Y$ there exists $x \in X$) such that the balls $B(x, R)$ and $B(y, R)$ are isometric.

We denote

$$d(X, Y) = \inf\{2^{-R}, X, Y \text{ are } R\text{-close}\}.$$

This is not a distance on the isometry classes of graphs (two graphs might be non-isometric but be at distance 0). However, in the sequel, we will mainly consider this notion when one of the graphs (say X) is a vertex-transitive graph, and in that case by compactness $d(X,Y)=0$ if and only if Y is isometric to X. Also, in that case and if $x_0 \in X$, X and Y are R-close if and only if $B(y,R)$ is isometric to $B(x_0,R)$ for every $y \in Y$.

For example, X and the d-regular tree are 1-close if and only if X is a d-regular graph.

Definition 5.2. A transitive graph X is LG-rigid (for Local-to-Global rigid) if there exists $R \in \mathbf{N}$ such that every graph R-close to X is covered by X.

X is LG-rigid among a class C of graphs if there exists $R \in \mathbf{N}$ such that every graph in the class C which is R-close to X is covered by X.

In words, X is LG-rigid (among C) if the only way for a sequence of graphs (in C) to converge to X is by below. So a necessary condition for X to be LG-rigid is that it cannot be approached "by above" by other graphs. A graph that cannot be approached by above is called *large-scale simply connected*.

Definition 5.3. If $k \in \mathbf{N}$, a graph X is k-simply connected if the isometries are the only graph coverings $f\colon Y \to X$ which are injective on all balls of radius $k/2$. We say that X is large scale simply connected if there is $k \in \mathbf{N}$ such that X is k-simply connected.

By a universal cover argument, X is k-simply connected if and only if $P_k(X)$ is simply connected, where $P_k(X)$ is the polygonal 2-complex whose 1-skeleton is X, and whose 2-cells are m-gons for $0 \le m \le k$, defined by cycles $(x_0,\dots,x_m = x_0)$ of length m in X, up to cyclic permutations.

This is a purely combinatorial notion : a graph X is k-simply connected if and only if the quotient of the fundamental groupoid of X by the normal groupoid generated by the cycles of length $\le k$ is the trivial groupoid $X \times X$. Here the fundamental groupoid is the set of all non-backtracking paths (*i.e.* all sequences $x_0,\dots,x_n$ of vertices with $d(x_i,x_{i+1})=1$ and $x_i \neq x_{i+2}$) for the obvious operation of concatenation and reduction, defined when one path ends where the other starts. In particular the Cayley graph of G with respect to S is k-simply connected if and only if G has a presentation $G = \langle S|R\rangle$ with relations of length $\le k$.

So a necessary condition for a Cayley graph to be LG-rigid is that the group be finitely presented. It was conjectured by Itai Benjamini and Agelos Georgakopoulos [Ben13, BE14] that this is also a sufficient condition.

Let us denote by $(\mathcal{G}_{ksc}, d)$ the (pseudo)metric space of all k-simply connected graphs for the distance d, and $\mathcal{G}_{ksd}^t$ the subspace of the vertex-transitive graphs. By a universal cover argument ([12, Proposition 1.5]), a k-simply connected vertex-transitive graph X is LG-rigid (respectively LG-rigid among transitive graphs) if and only if it is isolated in $\mathcal{G}_{ksd}$ (respectively $\mathcal{G}_{ksd}^t$). Therefore the conjecture mentioned above is equivalent to

Conjecture 5.4 (Benjamini–Georgakopoulos)**.** *For every k, the k-simply connected Cayley graphs are isolated points in $\mathcal{G}_{ksd}$.*

A related (stronger) question would be whether every graph in $\mathcal{G}_{ksd}^t$ is isolated in $\mathcal{G}_{ksd}$. A weaker question would be whether the Cayley graphs are isolated in $\mathcal{G}_{ksd}^t$ (this is equivalent to every Cayley graphs being LG-rigid among transitive graphs). An intermediate question would be whether $(\mathcal{G}_{ksd}^t, d)$ is discrete. By compactness, $(\mathcal{G}_{ksd}^t, d)$ is discrete if and only if for every $d \in \mathbf{N}$, there are finitely many isometry classes of k-simply connected transitive graphs of degree d. We shall see that all these questions have a negative answer.

Since for $k \le 2$ trees are the only k-simply connected graphs, the conjecture holds for $k \le 2$. More serious evidence for this conjecture were results of Benjamini–Ellis [BE14] that the usual grid in $\mathbf{Z}^d$ is LG-rigid, and of Georgakopoulos [Geo15] that one-ended planar graphs are LG-rigid. Also, since there are only finitely many k-simply connected Cayley graphs of fixed degree, every Cayley graph of a finitely presented group if LG-rigid among Cayley graphs [12, Corollary 1.7].

5.1 A first counterexample

With the discussion so far it is an immediate consequence of the work of Tits that $(\mathcal{G}^t_{ksd}, d)$ is not discrete already for $k = 3$. Indeed, for a non-archimedean local field F [1], the one-skeleton of the Bruhat-Tits building of $\mathrm{PGL}_3(F)$ is a 3-simply connected transitive graph which characterizes F and whose degree is an explicit function of the cardinality of the residue field of F. Since, for every (power of a) prime number q there are infinitely many non-archimedean local fields with residue field $\mathbf{F}_q$ (for example $\mathbf{Q}_q[q^{1/R}], R \in \mathbf{N}$), this implies that $(\mathcal{G}^t_{3sc}, d)$ is not discrete.

Hence there is at least one building which is not isolated in $\mathcal{G}^t_{3sc}$. By working a bit more and using the results from [CMSZ93], we can see that this transitive graph is a Cayley graph, providing a counterexample to Conjecture 5.4. Actually with Romain Tessera in [13] we give an almost complete characterization of which Bruhat-Tits buildings of type $\widetilde{A}_d$ are LG-rigid: if $d \geq 3$ these are exactly those constructed from a group of characteristic zero.

Theorem 5.5. *[13] If $d \geq 2$, and F has positive characteristic, then the Bruhat-Tits building $X_d(F)$ of $\mathrm{PGL}_{d+1}(F)$ is not LG-rigid.*

If $d \geq 3$ and F has characteristic zero, then the Bruhat-Tits building of $\mathrm{PGL}_{d+1}(F)$ is LG-rigid.

The case of $d = 2$ and of other reductive groups has been announced by Thierry Stulemeijer (personal communication).

The proof of Theorem 5.5 is a combination of several known or easy facts. First there is a classical locally compact topology on the set of (isomorphism classes of) non-archimedean local fields given by the distance

$$d(F_1, F_2) = \inf\{2^{-R}, F_1, F_2 \text{ are } R\text{-close}\}$$

where (F_1, v_1) and (F_2, v_2) are R-close if the residue rings

$$\{x \in F_1, v_1(x) \geq 0\} \Big/ \{x \in F_1, v_1(x) \geq R\} \text{ and } \{x \in F_2, v_2(x) \geq 0\} \Big/ \{x \in F_2, v_2(x) \geq R\}$$

are isomorphic. Moreover, for this topology, the isolated points are exactly the fields of characteristic zero. The simplest illustration of this phenomenon is for example the convergence of $\mathbf{Q}_p[p^{1/R}]$ to $\mathbf{F}_p((t))$ by the computation that $d(\mathbf{F}_p((t)), \mathbf{Q}_p[p^{1/R}]) = 2^{-R}$.

Theorem 5.5 is immediate from the following three facts and a compactness argument.

— ([13]) for every $d \geq 1$, $F \mapsto X_d(F)$ is 1-Lipschitz (hence continous).

— (Tits) if $d \geq 2$, $F \mapsto X_d(F)$ is injective.

— (Tits and [13]) if $d \geq 3$, $\{X_d(F), F\}$ is open in $\mathcal{G}_{3sc}$.

The compactness argument is used as follows: if $d \geq 3$, the map $F \mapsto X_d(F)$ is injective, continuous and proper, and hence is a homeomorphism on its image. This leaves open the following question:

1. in this section by non-archimedean local field we should mean a discrete valuation (not necessarily commutative) field (K, v) which is locally compact. The difference with Chapter 1 is that, in order to use Tit's work, we have to allow skew fields. In that case also [Wei74] non-archimedean local fields coincide with finite extensions of $\mathbf{Q}_p$ or of $\mathbf{F}_p((t))$ for some prime number p. For simplicity of the exposition, we write everything as if the field was commutative.

Question 5.6. What is the regularity of the inverse of the map $F \mapsto X_d(F)$?

In other words, the compactness argument implies that for every q and d, there is a function $f_q\colon \mathbf{N} \to \mathbf{N}$ such that, for every field F with residue field $\mathbf{F}_q$, the residue ring $\{v_F \geq 0\}/\{v_F \geq R\}$ is determined by the ball of radius $f_{q,d}(R)$ in $X_d(F)$. We only know that $f_{q,d}(R) \geq R$. The question asks for an upper bound on $f_{q,d}$. Is there an upper bound independant from q, d? Is R an upper bound?

Finally, let me mention the following result, which came out from discussions with Pierre-Emmanuel Caprace and Romain Tessera, and which will probably be included in the revised version of [13]. This is particularily interesting in view of the results in [12] (see Corollary 5.9 and Theorem 5.14 below).

Proposition 5.7. *There is a Cayley graph of a torsion-free finitely presented group which is not LG-rigid.*

Proof. The group will be a cocompact lattice in $\mathrm{PGL}_{d+1}(\mathbf{F}_p((t)))$ for an arbitrary $d \geq 2$.

Recall that in the 1-skeleton $X_d(F)$ of the Bruhat-Tits building of $\mathrm{PGL}_{d+1}(F)$, the vertices are partitionned according to their *type* in $\mathbf{Z}/(d+1)\mathbf{Z}$ (when $\mathcal{O}$ is the ring of units of F and when the building is identified with $\mathrm{PGL}_{d+1}(F)/\mathrm{PGL}_{d+1}(\mathcal{O})$, the type of $\gamma F^*\mathrm{PGL}_{d+1}(\mathcal{O})$ is $v(det(\gamma))+(d+1)\mathbf{Z}$). Consider the graph X, with vertex set the vertices of type 0 in $X_2(\mathbf{F}_p((t)))$, and an edge between two vertices at distance 2 in $X_2(\mathbf{F}_p((t)))$. It is a large-scale simply connected graph (because it is quasi-isometric to $X_2(\mathbf{F}_p((t)))$, and it is not LG-rigid (because if Y is a building R-close to $X_2(\mathbf{F}_p((t)))$, its set of vertices of type 0 is $R/2$-close to X).

It remains to see that X is the Cayley graph of a torsion free lattice in $\mathrm{PGL}_{d+1}(\mathbf{F}_p((t)))$. By [CMSZ93] there is a cocompact lattice Γ in $\mathrm{PGL}_{d+1}(\mathbf{F}_p((t))$ which acts simply transitively on $X_2(\mathbf{F}_p((t)))$. We do not know whether Γ is torsion-free, but its subgroup $\Lambda = \{\gamma, v(\det \gamma) \in (d+1)\mathbf{Z}\}$ satisfies

— Λ is an index $d+1$ subgroup of Λ which acts simply transitively by isometries on X (and therefore X is a Cayley graph of Λ).

— Λ is torsion free. Indeed, if $g \in \Lambda$ has finite order, then the circumcenter of any of its orbits would be a point in the Bruhat-Tits building fixed by g. If C is the cell of minimal dimension which contains this fixed point, then g induces a permutation of the vertices of this cell. As g preserves the type, this permutation is the identity. In particular, g fixes one vertex of the building, hence g is the identity by [CMSZ93]. □

5.2 LG-rigid groups

Our first main result in [12] gives a sufficient condition for X to be LG-rigid.

Theorem 5.8 ([12])**.** *Let X be a large-scale simply connected vertex-transitive graph. If X has a discrete isometry group, then X is LG-rigid.*

This allows to recover the results of Benjamini-Ellis and Georgakopoulos as particular cases. Using some structural results due to Furman (for lattices) and Trofimov (for groups with polynomial growth), we obtain, as a corollary,

Corollary 5.9 ([12])**.** *Under the assumption that they are torsion-free, every Cayley graphs of the following groups are LG-rigid:*

— *lattices in connected simple Lie groups with finite center;*

— *groups of polynomial growth.*

Actually the previous result is a corollary of Theorem 5.8 only for groups that are not virtually free. For virtually free group (which occur as lattices in $\mathrm{SL}(2, \mathbf{R})$), the Corollary follows from the very different result

Theorem 5.10 ([12])**.** *Let X be a large-scale simply connected vertex-transitive graph. If X is quasi-isometric to a tree, then X is LG-rigid.*

In a different direction, the following theorem implies that most finitely presented groups have an LG-rigid Cayley graph.

Theorem 5.11 ([12])**.** *Every finitely generated group with an element of infinite order has a Cayley graph with discrete isometry group.*

We were not able to settle the general case, that we left as a conjecture

Conjecture 5.12 ([12])**.** *Every finitely generated group has a Cayley graph with a discrete isometry group.*

We will see below that the torsion-free assumption in Corollary 5.9 is necessary for lattices. It might be that this is not the case for groups with polynomial growth.

Question 5.13. Are the Cayley graphs of every group with polynomial growth LG-rigid?

5.3 Non-LG-rigid groups and uncountably many large-scale simply connected graphs

Apart from Bruhat-Tits buildings, a second class of counterexamples for Conjecture 5.4 arises from the situation when G is a finitely presented group admitting a finitely generated non-finitely presented subgroup H. In fact one needs a bit more: that the second cohomology group $H^2(H, \mathbf{Z}/2\mathbf{Z})$ (which classifies the extensions $0 \to H \to E \to \mathbf{Z}/2\mathbf{Z} \to 0$) is infinite. The simplest example of such group if $G = F_2 \times F_2$ and H the kernel of the homomorphism $G \to \mathbf{Z}$ sending the 4 standard generators of G to 1. Other examples, including the case of G the product of two surfaces groups, are presented in an Appendix by Jean-Claude Sikorav in [12].

Theorem 5.14 ([12])**.** *If a finitely presented group G contains a finitely generated group H with $H^2(H; \mathbf{Z}/2\mathbf{Z})$ infinite, then $G \times \mathbf{Z}/2\mathbf{Z}$ has a Cayley graph which is not LG-rigid.*

Theorem 5.15 ([12])**.** *If moreover G splits as a semidirect product $H \rtimes Q$ and H contains an element of infinite order, then $G \times \mathbf{Z}/2\mathbf{Z}$ has a Cayley graph X such that for every R, the set of transitive graphs R-locally X and 4-Lispchitz isomorphic to X has the cardinality of the continuum.*

Remark 5.16. After [12] was submitted, we realized that the proof of the preceding theorem can be adapted to prove that, under the same hypotheses as in Theorem 5.14 and if H has an element of infinite order (respectively under the same hypotheses as in Theorem 5.15), every extension of G by $\mathbf{Z}/2\mathbf{Z}$ has a Cayley graph which is not LG-rigid (respectively not LG-rigid among transitive graphs). This improvement implies for example that $\mathrm{SL}(4, \mathbf{Z})$ has a Cayley graph which is not LG-rigid. Indeed it is an extension of $\mathrm{PSL}(4, \mathbf{Z})$ which contains a subgroup isomorphic to $F_2 \times F_2$ because $\mathrm{PSL}(2, \mathbf{Z})$ contains a subgroup F_2. Since $\mathrm{SL}(4, \mathbf{Z})$ is a lattice in the connected simple Lie group $\mathrm{SL}(4, \mathbf{R})$, this shows that the torsion-free assumption is crucial in Corollary 5.9.

The basic idea in both theorems and in the remark above is quite simple : the assumption that the compact abelian group $H^2(H;\mathbf{Z}/2\mathbf{Z})$ is infinite implies that it has the cardinality of the continuous, and hence that we can construct a continuum of 2-covers of the Cayley graphs of H. If $Y \to H$ is such a 2-cover, by partitionning G into H-cosets, we can construct a 2-cover of an appropriate Cayley graph of G by putting above every H-coset a copy of Y and adding appropriately edges between different copies. For the trivial covering, we get a Cayley graph X of $G \times \mathbf{Z}/2\mathbf{Z}$. It is rather easy to convince oneself that if Y is trivial when restricted to sufficiently large balls, then the resulting graph is R-close to X, and that it is always 4-Lipschitz equivalent to X. The difficulty in Theorem 5.15 is to make sure that for every Y, the constructed graph is transitive, and (more difficult) to see that different elements of $H^2(H;\mathbf{Z}/2\mathbf{Z})$ yield to non-isometric graphs. This is not quite true, but (by a variant of Theorem 5.11 which uses that H has an element of infinite order) we can arrange that the construction has the following feature: the map $H^2(H;\mathbf{Z}/2\mathbf{Z}) \to \mathcal{G}^t_{ksc}$ is finite-to-one.

In particular, since there are groups satisfying the assumption of the previous theorem (for example $F_2 \times F_2$) we have the following corollary, which answers a question raised by Yves de Cornulier.

Corollary 5.17 ([12])**.** *There exists k such that $\mathcal{G}^t_{ksc}$ has the cardinality of the continuum.*

The proof is effective, but we did not make an effort to estimate k (the formula would anyway be ugly because our proof of Theorem 5.11 and its variant is quite involved). In particular we do not know if the corollary is already true for $k = 3$.

Also, notice that we get a continuum many graphs in a 4-Lipschitz neighbourhoud of the countable family of large-scale simply connected Cayley graphs. The natural remaining question is

Question 5.18. Are there a continuum of quasi-isometry classes of large-scale simply connected graphs?

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