OPERATOR SPACE VALUED HANKEL MATRICES

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ABSTRACT. If E is an operator space, the non-commutative vector valued L^p spaces $S^p[E]$ have been defined by Pisier for any $1 \leq p \leq \infty$. In this paper a necessary and sufficient condition for a Hankel matrix of the form $(a_{i+j})_{0 \leq i,j}$ with coefficients in E to be bounded in $S^p[E]$ is established. This extends previous results of Peller where $E = \mathbb{C}$ or $E = S^p$. This condition is that the series $\varphi(z) = \sum_{n \geq 0} a_n z^n$ belongs to some vector valued Besov space. In particular this condition only depends on the Banach space structure of E. We also show that the norm of the isomorphism $\varphi \mapsto (\widehat{\varphi}(i+j))_{i,j}$ grows as $\sqrt{p}$ as $p \to \infty$, and compute the norm on S^p of the natural projection onto the space of Hankel matrices.

INTRODUCTION

This paper is devoted to the study of Hankel matrices in the vector-valued non-commutative L^p-space $S^p[E]$ defined by Pisier [7]. A Hankel matrix is a matrix the entries of which are indexed by $(j,k) \in \mathbb{N} \times \mathbb{N}$ and depend only on the sum $j+k$. The celebrated theorem of Nehari characterizes the Hankel matrices that represent a bounded operator on $B(\ell^2)$, and states that the operator norm a such a matrix $(x_{i+j})_{i,j \geq 0}$ is equal to the smallest value of $\|\varphi\|_{L^\infty}$, for $\varphi \in L^\infty(\mathbb{T})$ such that $\widehat{\varphi}(n) = x_n$ for all $n \geq 0$. Peller [2] has characterized the Hankel matrices belonging to the Schatten class S^p for all $p > 0$ (see below). For a detailed exposition on Hankel matrices and applications, see [5].

The main result of this paper is a characterization, for any operator space E, of the norm of Hankel matrices in the vector-valued non-commutative L^p-space $S^p[E]$ in terms of vector-valued Besov spaces $B_p^s(E)_+$ defined in the second section. The surprising fact is that these norms only depend on the Banach-space structure of E. The main result is the following.

If $\varphi = \sum_{n \in \mathbb{N}} a_n z^n$ is a formal series with a_n belonging to an operator space E, we denote $a_n = \widehat{\varphi}(n)$ ($\widehat{\varphi}(n)$ coincides with the Fourier coefficient of φ when $\varphi \in L^1(\mathbb{T}; E)$), the Hankel matrix Γ_φ is defined by its matrix representation

$$\Gamma_\varphi = \left(\widehat{\varphi}(j+k)\right)_{j,k \geq 0}.$$

2000 *Mathematics Subject Classification.* 47B35, 46L07.

Partially supported by ANR-06-BLAN-0015.

Theorem 0.1. *Let $1 \leq p < \infty$. A Hankel matrix $(a_{j+k})_{j,k\geq 0}$ belongs to $S^p[E]$ if and only if the formal series $\sum_{n\geq 0} a_n z^n$ belongs to $B_p^{1/p}(E)_+$.*

More precisely there is a constant $C>0$ such that for any operator space E and any formal series $\varphi = \sum_{n\geq 0} a_n z^n$

$$C^{-1} \|\varphi\|_{B_p^{1/p}(E)_+} \leq \|\Gamma_\varphi\|_{S^p[E]} \leq C\sqrt{p}\, \|\varphi\|_{B_p^{1/p}(E)_+}.$$

Moreover the rate of growth as $\sqrt{p}$ is optimal already in the scalar case: there is a constant $c>0$ (independent of p) and $\varphi \in B_{p+}^{1/p}$ such that $\|\Gamma_\varphi\|_{S^p} \geq c\sqrt{p}\, \|\varphi\|_{B_{p+}^{1/p}}$.

As a consequence we also get that the norm of the natural projection onto the space of Hankel matrices grows as $\sqrt{p}$ as $p \to \infty$, and as $1/\sqrt{p-1}$ as $p \to 1$:

Theorem 0.2. *Let P_{Hank} be the natural projection from the space of infinite matrices to the subspace of Hankel matrices:*

$$P_{Hank}\left((a_{j,k})_{j,k\geq 0}\right) = \left(\frac{1}{j+k+1} \sum_{s+t=j+k} a_{s,t}\right)_{j,k\geq 0}.$$

Then, for $1<p<\infty$, P_{Hank} is bounded on S^p (and on $S^p[E]$ for any operator space E) and its norms satisfy the following inequality with a constant $C>0$ independent of E and p:

$$C^{-1}\sqrt{\frac{p^2}{p-1}} \leq \|P_{Hank}\|_{S^p \to S^p} \leq \|P_{Hank}\|_{S^p[E] \to S^p[E]} \leq C\sqrt{\frac{p^2}{p-1}}.$$

As often for results on non-commutative L^p spaces, Theorem 0.1 is proved using the complex interpolation method. For $p=1$ the above theorem can be proved directly. A first natural attempt to derive the Theorem for any p would be to get something for $p=\infty$. Bounded Hankel operators are well-known with Nehari's theorem and its operator valued version, which states that for $E \subset B(\ell^2)$ and $p=\infty$, Γ_φ belongs to $B(\ell^2) \otimes E$ if and only if there is a function $\psi \in L^\infty(\mathbb{T}; B(\ell^2))$ such that $\widehat{\psi}(k) = \widehat{\varphi}(k)$ for $k>0$. But for non-injective operator spaces, this seems very complicated (at least to me) to relate this function ψ to properties of E, and the results of Theorem 0.1 even seem quite disjoint from Nehari's theorem. Another natural attempt would be to interpolate between $p=2$ and $p=1$ since often for $p=2$ results are obvious. But it should be pointed out that here the Theorem is non trivial for $p=2$ as well. We are thus led to pass from a problem with only one parameter p to a problem with more parameters to "get room" in order to be able to use the interpolation method. This is done with the so-called generalized Hankel matrices.

For real (or complex) numbers α, β the generalized Hankel matrix with symbol φ is defined by

$$\Gamma_\varphi^{\alpha,\beta} = \big((1+j)^\alpha (1+k)^\beta \widehat{\varphi}(j+k)\big)_{j,k\geq 0}.$$

Our main theorem characterizes, for an operator space E and a $1 \leq p \leq \infty$, the generalized Hankel matrices that belong to $S^p[E]$ under the conditions that $\alpha + 1/2p > 0, \beta + 1/2p > 0$.

Theorem 0.3. *Let $1 \leq p \leq \infty$ and $\alpha, \beta > -1/2p$. Then for a formal series $\varphi = \sum_{n\geq 0} \widehat{\varphi}(n) z^n$ with $\widehat{\varphi}(n) \in E$, $\Gamma_\varphi^{\alpha,\beta} \in S^p[E]$ if and only if $\varphi \in B_p^{1/p+\alpha+\beta}(E)_+$.*

More precisely, for all $M > 0$, there is a constant $C = C_M$ (depending only on M, not on p, E) such that for all such φ, all $1 \leq p \leq \infty$ and all $\alpha, \beta \in \mathbb{R}$ such that $-1/2p < \alpha, \beta < M$,

$$C^{-1} \|\varphi\|_{B_p^{1/p+\alpha+\beta}(E)_+} \leq \left\|\Gamma_\varphi^{\alpha,\beta}\right\|_{S^p[E]} \leq \frac{C}{\sqrt{\min(\alpha,\beta)+1/2p}^{\,1+1/p}} \|\varphi\|_{B_p^{1/p+\alpha+\beta}(E)_+}. \tag{1}$$

The usual convention is to define $S^\infty[E]$ as $\mathcal{K} \otimes_{min} E$. However in the previous Theorem one has to (abusively) understand $\|\cdot\|_{S^\infty[E]}$ as $\|\cdot\|_{B(\ell^2)\otimes_{min} E}$ (if E is finite dimensional) or even as $\|\cdot\|_{B(\ell^2\otimes H)}$ if $E \subset B(H)$.

Note that surprisingly, this theorem shows that the condition $\Gamma_\varphi^{\alpha,\beta} \in S^p[E]$ only depends on the Banach space structure of E (whereas the Banach space structure of $S^p[E]$ depends on the operator space structure of E).

These results extend results of Peller in the scalar case or in the case when $E = S^p$ ([2],[4],[3],[5]). In the scalar case Peller's theorem indeed shows that the space $Hank_p$ of Hankel matrices in S^p is isomorphic to a Besov space $B_{p+}^{1/p}$. The case when $E = S^p$ shows that this isomorphism is in fact a complete isomorphism. The results stated above show that this isomorphism has the stronger property of being *regular* as well as its inverse in the sense of [6]. In this paper the choice was made to use the vocabulary of regular operators, but one could easily avoid this notion (replacing, in the proof of Lemma 3.1, the use of Pisier's Theorem 1.3 by Stein's interpolation method).

Remark. If E and F are subspaces of commutative or non-commutative L^p-spaces (on finite hyperfinite von Neumann algebras), one can define the *regular distance* between E and F as the least value of $\|T\|_{reg}\|T^{-1}\|_{reg}$, over all regular isomorphisms $T : E \to F$ (see section 1 for definitions). Thus Theorem 0.1 states that the regular distance of $Hank_p$ to the set of subspaces of *commutative* L^p spaces is less than $C^2\sqrt{p}$. It can be shown that this rate of growth is also optimal. See Proposition 3.3. To summarize, there is a constant C' such that this distance d_p satisfies

$$C'^{-1}\sqrt{p} \leq d_p \leq C'\sqrt{p}. \tag{2}$$

The natural projection P_{Hank} was also studied by Peller (Chapter 6 of [5]) who proved that it is bounded on S^p if $1 < p < \infty$ and unbounded if $p = 1$ or ∞. Here we prove that it is even regular, and show that its norm as well as its regular norm behaves as $\sqrt{p}$ ($p \geq 2$) or as $1/\sqrt{p-1}$ ($p \leq 2$). This seems to be new even in the scalar case.

These results should be considered as remarks on Peller's proof rather than new theorems, since the steps presented here are all close to one of Peller's proofs ([5], sections 8 and 9 of Chapter 6). There are still some adaptations to make since for example the result for $p = 2$ is non-trivial here whereas it is obvious in Peller's case. Moreover as far as the constants in the isomorphisms are concerned, our results are more precise and optimal in some sense (if one follows Peller's proofs, one is led to constants growing at least as fast as p in the right-hand side of the inequality of the Theorem 0.1). For completeness we provide a detailed proof. We would also like to mention here the fact that Éric Ricard has found a much shorter and elementary proof of Theorem 0.1 (which is in particular a new simpler proof of Peller's results), but it leads to constants of order p instead of $\sqrt{p}$. It is also worth mentioning that (at least one direction of) his proof also works for $p < 1$ (in the scalar and S^p-valued case).

Peller's classical results also have an extension to the case $0 < p < 1$. Here there are some obstructions: we should first of all clarify the notion of vector-valued non-commutative L^p spaces for $p < 1$. But even then, since the proof given here really lies on duality and interpolation, some new ideas would be needed.

This paper is organized as follows: in the first section we recall briefly definitions and facts on regular operators. In the second section we give definitions and classical results on Besov spaces of analytic functions $B^s_{p,q+}$ that will be used later. All results are proved. In the third and last section we prove the main result.

Notation. We will use the following notation: if X and Y are two Banach spaces (resp. operator spaces), we write $X \simeq Y$ if X and Y are isomorphic (resp. completely isomorphic). Most of the time the isomorphism will not be explicitly stated since it is natural. If A and B are two nonnegative numerical expressions (depending on some parameters), we will write $A \approx B$ if there is a constant c such that $c^{-1}A \leq B \leq cA$. In the whole paper $\mathbb{N}$ will stand for the set of non-negative integers:

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}.$$

1. Background on regular operators

1.1. **Commutative case.** We start by recalling the definition of regular operators in the commutative setting.

Definition 1.1. A linear operator $u : \Lambda_1 \to \Lambda_2$ between Banach lattices is said to be regular if for any Banach space X, $u \otimes id_X : \Lambda_1(X) \to \Lambda_1(X)$ is bounded. Equivalently (taking for $X = \ell_n^\infty$), if there is a constant C such that for any n and $f_1, \ldots, f_n \in \Lambda_1$,

$$\left\| \sup_k |u(f_k)| \right\|_{\Lambda_2} \leq C \left\| \sup_k |f_k| \right\|_{\Lambda_1}.$$

The smallest such C is denoted by $\|u\|_r$.

This theory applies in particular if $\Lambda_1 = \Lambda_1$ are (commutative) L^p spaces: when $p = 1$ or $p = \infty$ a map is regular if and only if it is bounded. Similarly, a map that is simultaneously bounded $L^1 \to L^1$ and $L^\infty \to L^\infty$ is regular on L^p. This is not far from being a characterization since it is known that the set of regular operators: $L^p \to L^p$ coincides with the interpolation space (for the second complex interpolation method) between $B(L^\infty, L^\infty)$ and $B(L^1, L^1)$.

We refer to [1] for facts on the complex interpolation method.

1.2. **Non-commutative case.** Let S be a subspace of a non-commutative L^p space constructed on a hyperfinite von Neumann algebra. In the sequel for an operator space E we will denote by $S[E]$ the (closure of) the subspace $S \otimes E$ of the vector valued non-commutative L^p-space $L^p(\tau; E)$ defined in [7].

Definition 1.2. A linear map $u : S \to T$ between subspaces S and T of non-commutative L^p spaces as above is said to be regular if for any operator space E, $u \otimes id_E : S[E] \to T[E]$ is bounded. As in the commutative case $\|u\|_r$ will denote the best constant C such that $\|u \otimes id_E\|_{S[E] \to T[E]} \leq C$ for all E.

The set of regular operators equipped with this norm will be denoted by $B_r(S, T)$.

Since classical L^p spaces are special cases of non-commutative L^p spaces, this notion applies also for commutative L^p spaces (but fortunately the two notions coincide). This notion was defined and studied in [6]. In particular the following result was proved:

Theorem 1.3 (Pisier). *Let $(\mathcal{M}, \tau)$ and $(\mathcal{N}, \widetilde{\tau})$ be hyperfinite von Neumann algebras with normal semi-finite faithful traces. Then a map $u : L^p(\tau) \to L^p(\widetilde{\tau})$ is regular is and only if it is a linear combination of bounded completely positive operators. Moreover isomorphically (with constant not depending on p or on $\mathcal{M}, \mathcal{N}$)*

$$B_r(L^p, L^p) \simeq \left[CB(L^\infty, L^\infty), CB(L^1, L^1) \right]^\theta \ \textit{for } \theta = 1/p.$$

We will only apply this fact in the case of von Neumann algebras that are either commutative or equal $B(\ell^2)$ equipped with the usual trace. The following result was also proved:

Theorem 1.4. *Let $1 \leq p < \infty$. Then $u : L^p(\tau) \to L^p(\widetilde{\tau})$ is regular if and only if $u^* : L^{p'}(\widetilde{\tau}) \to L^{p'}(\tau)$ is regular, and $\|u\|_r = \|u^*\|_r$.*

2. Vector valued Besov spaces

In this section we introduce the Besov spaces of analytic functions $B^s_{p,q+}$. Before that we need some facts on Fourier multipliers. Everything in this section is classical (the results are stated in [5], and they are proved for the real line instead of the unit circle in [1]), but we give precise proofs in order to get quantitative bounds on the norms of the different isomorphisms.

2.1. **Fourier Multipliers on the circle.**

Here $\mathbb{T}$ will denote the unit circle: $\mathbb{T} = \{z \in \mathbb{C}, |z| = 1\}$ and will be equipped with its Haar probability measure.

The Fourier multiplier with symbol $(\lambda_k)_{k\in\mathbb{Z}}$ ($\lambda_k \in \mathbb{C}$) is the linear map on the polynomials in z and $\overline{z}$ denoted by $M_{(\lambda_k)_k}$ and mapping $\sum_{k\in\mathbb{Z}} a_k z^k$ to $\sum_{k\in\mathbb{Z}} \lambda_k a_k z^k$. For $1 \le p \le \infty$ we say that the Fourier multiplier is bounded on L^p if the map $M_{(\lambda_k)_k}$ can be extended to a bounded operator on $L^p(\mathbb{T})$ such that for $f \in L^p(\mathbb{T})$, $g = M_{(\lambda_k)_k}(f)$ satisfies $\widehat{g}(k) = \lambda_k \widehat{f}(k)$.

Similarly if X is a Banach space the multiplier $M_{(\lambda_k)_k}$ is said to be bounded on $L^p(\mathbb{T};X)$ if $M_{(\lambda_k)_k} \otimes id_X$ extends to a continuous map on $L^p(\mathbb{T};X)$ (which we still denote by $M_{(\lambda)_k)}$), such that for $f \in L^p(\mathbb{T};X)$, $g = (M_{(\lambda_k)_k} \otimes id_X)(f)$ satisfies $\widehat{g}(k) = \lambda_k \widehat{f}(k)$.

In the vocabulary of section 1 a multiplier $M_{(\lambda_k)_k}$ is said to be regular on L^p if it is bounded on $L^p(\mathbb{T};X)$ for any Banach space X.

For example if $\lambda_k = \widehat{\mu}(k)$ for some complex Borel measure μ on $\mathbb{T}$ then $M_{(\lambda_k)_k}$ is bounded on $L^p(\mathbb{T};X)$ ($1 \le p \le \infty$) for any Banach space X since it corresponds to the convolution map $f \mapsto \mu \star f$. Its regular norm on L^p is therefore equal to the total variation of μ.

The following Lemma will be essential.

Lemma 2.1. *Let $\lambda = (\lambda_k)_{k\in\mathbb{Z}} \in \mathbb{C}^\mathbb{Z}$ satisfying $\|\lambda\|_2 < \infty$. Then the Fourier multiplier with symbol λ is bounded on every L^p and*

$$\left\|M_{(\lambda_k)_k}\right\|_{L^p \to L^p} \le \frac{2}{\sqrt{\pi}} \sqrt{\|\lambda\|_2 \|(\lambda_{k+1} - \lambda_k)_k\|_2}.$$

It is even regular and its regular norm on L^p is less than or equal to

$$2/\sqrt{\pi}\sqrt{\|\lambda\|_2 \|(\lambda_{k+1} - \lambda_k)_k\|_2}.$$

Proof. Since $\|(\lambda_k)\|_2 < \infty$, the function $f : z \mapsto \sum_{k\in\mathbb{Z}} \lambda_k z^k$ is in L^2 and $\|f\|_2 = \|(\lambda_k)\|_2$. Similarly, the function $g : z \mapsto (1 - z)f(z)$ satisfies $\|g\|_2 = \|(\lambda_k - \lambda_{k+1})_{k\in\mathbb{Z}}\|_2$.

Since the multiplier with symbol (λ_k) corresponds to the convolution by f, by the remark preceding the Lemma we only have to prove that $\|f\|_1^2 \lesssim$

$\|f\|_2\|g\|_2$. But for any $0 < s < 1/2$:

$$\begin{aligned}\|f\|_1 &= \int_0^1 |f(e^{2i\pi t})|dt \\ &= \int_{-s}^{s} |f(e^{2i\pi t})|dt + \int_s^{1-s} \frac{1}{|1-e^{2i\pi t}|}|(1-e^{2i\pi t})f(e^{2i\pi t})|dt \\ &\le \sqrt{2s}\|f\|_2 + \sqrt{\int_s^{1-s} \frac{1}{|1-e^{2i\pi t}|^2}dt}\|g\|_2\end{aligned}$$

by the Cauchy-Schwarz inequality. The remaining integral can be computed:

$$\begin{aligned}\int_s^{1-s} \frac{1}{|1-e^{2i\pi t}|^2}dt &= 2\int_s^{1/2} \frac{1}{4\sin^2(\pi t)}dt \\ &= \frac{1}{2}\left[\frac{-\cos(\pi t)}{\pi\sin(\pi t)}\right]_s^{1/2} = \frac{1}{2\pi\tan(\pi s)} \le \frac{1}{2\pi^2 s}\end{aligned}$$

where we used that $\tan x \ge x$ for all $0 \le x \le \pi/2$. Taking $s = \|g\|_2/2\pi\|f\|_2 \le 1/2$ we get the desired inequality. ☐

The following consequence will be also used a lot:

Lemma 2.2. *Let $I = [a,b] \subset \mathbb{Z}$ be an interval of size N and take $(\lambda_k)_{k\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$.*

Then for any $1 \le p \le \infty$, any Banach space X and any $f \in L^p(\mathbb{T};X)$ such that $\widehat{f}$ is supported in I,

(3)

$$\left\|M_{(\lambda_k)_k} f\right\|_{L^p(\mathbb{T};X)} \le 2\|f\|_p \max\left(\sup_{k\in I}|\lambda_k|, \sqrt{N\sup_{k\in I}|\lambda_k| \sup_{a\le k<b}|\lambda_k-\lambda_{k+1}|}\right).$$

In other words, the restriction of the multiplier M_λ to the subspace of $L^p(\mathbb{T})$ of functions with Fourier transform vanishing outside of I has a regular norm less than the right-hand side of this inequality.

Proof. Consider the multiplier M_μ with symbol $(\mu_k)_{k\in\mathbb{Z}}$ where $\mu_k = \lambda_k$ if $k \in I$, $\mu_k = 0$ if $k \le a-N$ or if $k \ge b+N$, and μ_k is affine on the intervals $[a-N,a]$ and $[b,b+N]$.

Since M_μ and M_λ coincide on the space of functions such that $\widehat{f}(k) = 0$ for $k \notin I$, the claim will follow from the fact that the regular norm of M_μ is less that the right-hand side of (3). For this we use Lemma 2.1, so we have to dominate $\|(\mu_k)\|_2$ and $\|(\mu_{k+1}-\mu_k)\|_2$. Since both sequences $(\mu_k)_k$ and $(\mu_{k+1}-\mu_k)_k$ are supported in $]a-N,b+N]$ which is of size less than $3N$, their ℓ^2-norm is less than $\sqrt{3N}$ times their ℓ^∞ norm. The inequality $\sup_k|\mu_k| \le \sup_{k\in I}|\lambda_k|$ is obvious by definition of μ_k. On the other hand we have $|\mu_{k+1}-\mu_k| = |\lambda_{k+1}-\lambda_k|$ if $k \in [a,b[$, and $|\mu_{k+1}-\mu_k| \le \sup_{k\in I}|\lambda_k|/N$ otherwise since μ_k is affine on the intervals of size N $[a-N,a]$ and $[b,b+N]$.

Thus by Lemma 2.1,

$$\|M_\mu\|_{L^p(\mathbb{T};X)\to L^p(\mathbb{T};X)} \leq \frac{2\sqrt{3}}{\sqrt{\pi}} \max\left(\sup_{k\in I} |\lambda_k|, \sqrt{N \sup_{k\in[a,b[} |\lambda_k| \sup_{k\in I} |\lambda_k - \lambda_{k+1}|}\right).$$

This concludes the proof since $3 \leq \pi$. □

For all $n \in \mathbb{N}$, $n > 0$ we define the function W_n on $\mathbb{T}$ (see Figure 1) by

$$\widehat{W_n}(k) = \begin{cases} 2^{-n+1}(k-2^{n-1}) & \text{if } 2^{n-1} \leq k \leq 2^n \\ 2^{-n}(2^{n+1}-k) & \text{if } 2^n \leq k \leq 2^{n+1} \\ 0 & \text{otherwise.} \end{cases}$$

We also define $W_0(z) = z + 1$.

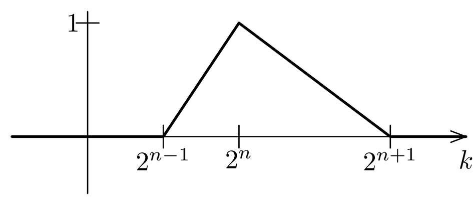


FIGURE 1. $\widehat{W_n}(k)$

Note that for all $k \in \mathbb{N}$, $\sum_{n\in\mathbb{N}} \widehat{W_n}(k) = 1$ (finite sum).

Since for $n > 0$, $\|(\widehat{W}_n(k))_k\|_2 \leq \sqrt{2^n}$ and $\|(\widehat{W}_n(k) - \widehat{W}_n(k+1))_k\|_2 = \sqrt{3/2^n}$, Lemma 2.1 implies that the multiplier $f \mapsto W_n \star f$ has regular norm less than $2\sqrt{3/\pi} \leq 2$ on $L^p(\mathbb{T})$ any $1 \leq p \leq \infty$. The same is obvious for W_0.

2.2. **Besov spaces of vector-valued analytic functions.** We define the X-valued weighted ℓ_p spaces $\ell_p^s(\mathbb{N};X)$ for $p > 0$, $s \in \mathbb{R}$ and a Banach space X as the space of sequences $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ such that $\|(x_n)_n\|_{\ell_p^s(\mathbb{N};X)} = \left\|(2^{ns}\|x_n\|_X)_{n\in\mathbb{N}}\right\|_p < \infty$.

We will deal in this paper with Besov spaces of "analytic functions", which are defined in the following way. First note that the reader should take the term "analytic" with care. Elements of the Besov spaces are indeed defined as formal series $\sum_{k\geq 0} x_k z^k$ with $z \in \mathbb{T}$. The term analytic means that the formal series are indexed by $\mathbb{N}$ and not $\mathbb{Z}$ (in particular this has nothing to do with analytic functions defined on the real analytic manifold $\mathbb{T}$).

Let X be a Banach space; $p, q > 0$ and s real numbers. The Besov space $B_{p,q}^s(X)_+$ is defined as the space of formal series $f(z) = \sum_{k\in\mathbb{N}} x_k z^k$ with $x_k \in X$ such that $(2^{ns}\|W_n \star f\|_p)_{n\in\mathbb{N}} \in \ell_q$, with the norm $\left\|(2^{ns}\|W_n \star f\|_p)_{n\in\mathbb{N}}\right\|_q$. Here by $W_n \star f$ we mean the (finite sum) $\sum_{k\geq 0} \widehat{W}_n(k) x_k z^k$, and this coincides with the classical notion when $f \in L^1(\mathbb{T};X)$. When $X = \mathbb{C}$ the Besov space $B^s_{p,q}(X)_+$ is simply denoted by $B^s_{p,q+}$.

Remark (Elements of $B^s_{p,q}(X)_+$ as functions). It is easy to see that when $s > 0$, any $f \in B^s_{p,q}(X)_+$ corresponds to a function belonging to $L^p(\mathbb{T};X)$ (and therefore also to $L^1(\mathbb{T};X)$). In this case the series $\sum_{n\geq 0} W_n \star f$ indeed converges in $L^p(\mathbb{T};X)$ (because $\sum_{n\geq 0}\|W_n \star f\|_p < \infty$). It is also immediate to see that for any real s, $\|x_k\|_X \leq C\|f\|_{B^s_{p,q}(X)_+} k^{-s}$ for some constant $C = C(s) > 0$, and thus that for any $f \in B^s_{p,q}(X)_+$, $\sum_{k\geq 0} x_k z^k$ converges for all z in the unit ball $\mathbb{D}$ of $\mathbb{C}$.

On the opposite when $s < 0$ there are elements $f = \sum_{k\geq 0} x_k z^k \in B^s_{p,q}(X)_+$ such that the sequence x_k is not even bounded (and thus cannot represent a function in $L^1(\mathbb{T};X)$).

The space can be equivalently defined as a subspace of $\ell^s_q(\mathbb{N};L^p(\mathbb{T};X))$ with the isometric injection

$$\begin{array}{rcl} B^s_{p,q}(X)_+ & \longrightarrow & \ell^s_q(\mathbb{N};L^p(\mathbb{T};X)) \\ f & \mapsto & (W_n \star f)_{n\in\mathbb{N}} \end{array}$$

Moreover the image of $B^s_{p,q}(X)_+$ in the isometric injection is a complemented subspace. The projection map is given by

$$\begin{array}{rcl} P : \ell^s_q(\mathbb{N};L^p(\mathbb{T};X)) & \longrightarrow & B^s_{p,q}(X)_+ \\ (a_n) & \mapsto & (W_0 + W_1)\star a_0 + \displaystyle\sum_{n\geq 1}(W_{n-1} + W_n + W_{n+1})\star a_n \end{array}$$

and has norm less than $C2^{2|s|}$ for some constant $C \leq 20$. Indeed, if $V_n = W_{n-1} + W_n + W_{n+1}$ if $n \geq 1$ and $V_0 = W_0 + W_1$, then $W_m \star V_n = 0$ if $|n-m| > 2$, and moreover if $|n-m| \leq 2$, $\|(W_m \star V_n)\star a_n\|_p \leq 4\|a_n\|_p$ by Lemma 2.1. This implies that

$$\begin{aligned} \left\| \sum_{n\geq 0} V_n \star a_n \right\|_{B^s_{p,q}(X)_+} &\leq \sum_{-2\leq\epsilon\leq 2} 4\left\|(2^{ns}\|a_{n+\epsilon}\|_p)_{n\in\mathbb{N}}\right\|_q \\ &\leq 4\left(2^{-2s} + 2^{-s} + 1 + 2^s + 2^{2s}\right)\left\|(2^{ns}\|a_n\|_p)_{n\in\mathbb{N}}\right\|_q . \end{aligned}$$

When $p = q$, the Besov space $B^s_{p,q}(X)_+$ is also denoted by $B^s_p(X)_+$. In this case B^s_{p+} is a subspace of $\ell^s_p(\mathbb{N};L^p(\mathbb{T}))$ which is just the L^p space on $\mathbb{N}\times\mathbb{T}$ with respect to the product measure of the Lebesgue measure on $\mathbb{T}$ and the measure on $\mathbb{N}$ giving mass 2^{nsp} to $\{n\}$. Moreover (at least for $p < \infty$) $B^s_p(X)_+$ is the closure of $B^s_{p+}\otimes X$ in the vector-valued L^p space $L^p(\mathbb{N}\times\mathbb{T};X)$. This will allow to speak of regular operators between B^s_{p+} and another (subspace of a) non-commutative L^p space. Note in particular that the above remark shows that B^s_{p+} is a complemented subspace of $L^p(\mathbb{N}\times\mathbb{T})$ and that the projection map P (which does not depend on p) is regular.

As a consequence of the complementation, we have the following property of Besov spaces:

Theorem 2.3. *The properties of the Besov spaces with respect to duality are: if $p, q < \infty$*

$$B^s_{p,q}(X)^*_+ \simeq B^{-s}_{p',q'}(X^*)_+$$

isomorphically for the natural duality $\langle f, g\rangle = \sum_{n\geq 0}\langle \widehat{f}(n), \widehat{g}(n)\rangle$. Moreover for $M > 0$ and any $|s| < M$ the constants in this isomorphism depend only on M.

Proof. The boundedness of P formally implies that the dual of $B^s_{p,q}(X)_+$ is isomorphically identified with the set of formal series $g(z) = \sum_k \widehat{g}(k) z^k$ ($\widehat{g}(k) \in X^*$) equipped with the norm coming from the embedding $P^* : g \mapsto (V_n \otimes g)_n \in \ell^{-s}_{q'}(\mathbb{N}; L^{p'}(\mathbb{T}; X^*))$. But the same argument as in the proof of the boundedness of P shows that (up to constants depending only on M if $|s| < M$)

$$\|(V_n \otimes g)_n\|_{\ell^{-s}_{q'}(\mathbb{N};L^{p'}(\mathbb{T};X^*))} \approx \|(W_n \otimes g)_n\|_{\ell^{-s}_{q'}(\mathbb{N};L^{p'}(\mathbb{T};X^*))} = \|g\|_{B^{-s}_{q'}(X^*)_+}.$$

□

For a real (or complex) number α and an integer n, we define the number D^α_n by $D^\alpha_0 = 1$ and for $n \geq 1$,

$$D^\alpha_n = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!} = \prod_{j=1}^{n}\left(1 + \frac{\alpha}{j}\right).$$

For any $t \in \mathbb{R}$, we define the maps I_t and $\widetilde{I}_t$ by

$$I_t(\sum_{k\geq 0} a_k z^k) = \sum_{k\geq 0}(1+k)^t a_k z^k.$$

$$\widetilde{I}_t(\sum_{k\geq 0} a_k z^k) = \sum_{k\geq 0} D^t_k a_k z^k.$$

The boundedness properties of the maps I_t and $\widetilde{I}_t$ are described by the following result:

Theorem 2.4. *Let $M > 0$ be a real number. There is a constant $C = C_M$ (depending only on M) such that for any $1 \leq p, q \leq \infty$, any $|t| \leq M$, any $s \in \mathbb{R}$, and any Banach space X,*

$$\|I_t : B^s_{p,q}(X)_+ \to B^{s-t}_{p,q}(X)_+\|, \|I_t^{-1} : B^{s-t}_{p,q}(X)_+ \to B^s_{p,q}(X)_+\| \leq C.$$

Moreover if $-1/2 \leq t \leq M$,

$$\|\widetilde{I}_t : B^s_{p,q}(X)_+ \to B^{s-t}_{p,q}(X)_+\|, \|\widetilde{I}_t^{-1} : B^{s-t}_{p,q}(X)_+ \to B^s_{p,q}(X)_+\| \leq C.$$

Proof. Fix $M > 0$ (and even $M \geq 1$) and take $|t| \leq M$. Let us treat the case of I_t. Let $f = \sum_{k\geq 0} a_k z^k \in B^s_{p,q}(X)_+$. Since the maps $f \mapsto W_n \star f$ and $f \mapsto I_t f$ are both multipliers, they commute, and we have that

$$\|I_t f\|_{B^{s-t}_{p,q}(X)_+} = \left\|(2^{ns}\|I_t/2^{nt}(W_n \star f)\|_p)_{n\in\mathbb{N}}\right\|_q .$$

To show that $\|I_t\| \leq C$, it is therefore enough to show that the multiplier $I_t/2^{nt}$ (the symbol of which is $((1+k)/2^n)^t$) is bounded by some constant C on the subspace of $L^p(\mathbb{T}, X)$ consisting of functions whose Fourier transform is supported in $]2^{n-1}, 2^{n+1}[$. This follows from Lemma 2.2. We indeed have $((1+k)/2^n)^t \leq 2^{|t|}$ for $k \in]2^{n-1}, 2^{n+1}[$. To dominate the difference $|((2+k)/2^n)^t - ((1+k)/2^n)^t|$ for $2^{n-1} < k < 2^{n+1} - 1$, just dominate the derivative of $x \mapsto (x/2^n)^t$ on the interval $[2^{n-1}, 2^{n+1}]$ by $|t|2^{|t-1|}/2^n \leq M2^{M+1}/2^n$. The multiplier $I_t/2^{nt}$ is thus bounded by $4\sqrt{M}2^M$.

This shows that

$$\|I_t : B^s_{p,q}(X)_+ \to B^{s-t}_{p,q}(X)_+\| \leq 4\sqrt{M}2^M$$

Since $I_{-t} = {I_t}^{-1}$, the inequality for I_{-t} follows.

By the same argument, to dominate the norms of $\widetilde{I}_t$ (resp. its inverse), we have to get a uniform bound on $\sup_k |\lambda_k|$ and $2^n \sup_k |\lambda_{k+1} - \lambda_k|$ where $\lambda_k = D^t_k/2^{nt}$ (resp. $\lambda_k = 2^{nt}/D^t_k$). This amounts to showing that there is a constant $C(M)$ (depending on M only) such that $1/C(M) \leq |D^t_k/2^{nt}| \leq C(M)$ and $|D^t_{k+1}/2^{nt} - D^t_k/2^{nt}| \leq C(M)/2^n$ for $2^{n-1} \leq k < 2^{n+1}$ (the inequality $|2^{nt}/D^t_{k+1} - 2^{nt}/D^t_k| \leq C(M)^3/2^n$ will follow from the formula $|1/x - 1/y| = |y-x|/|xy|$). The first inequality can be proved by taking the logarithm, noting that $\log(1+t/j) = t/j + O(1/j^2)$ up to constants depending only on M if $-1/2 \leq t \leq M$, and remembering that $\sum_1^N 1/j = \log N + O(1)$. The second inequality follows easily since $D^t_{k+1} - D^t_k = t/(k+1)D^t_k$. □

We also use the following characterization of Besov spaces of analytic vector-valued functions. In this statement as well as in the rest of this section we will identify a function (or distribution) $f : \mathbb{T} \to X$; $f(z) = \sum_{n\geq 0} z^n a_n$ with its analytic extension to the disc.

Theorem 2.5. *Let $M > 0$. Then there is a constant $C = C_M$ (depending only on M) such that for all $0 < s < M$, for all Banach spaces X, all $1 \leq p \leq \infty$ and all $f : \mathbb{T} \to X$,*

$$C^{-1}\|f\|_{B^{-s}_{p,p}(X)_+} \leq \left\|(1-|z|)^{s-1/p} f\right\|_{L^p(\mathbb{D},dz;X)} \leq \frac{C}{s}\|f\|_{B^{-s}_{p,p}(X)_+}.$$

Proof. The left-hand side inequality is easier. For any $0 < r < 1$, let f_r denote the function $f_r(\theta) = f(re^{i\theta})$. Then

$$\left\|(1-|z|)^{s-1/p} f\right\|_{L^p(\mathbb{D},dz;X)} = (2\pi)^{1/p}\left(\int_0^1 (1-r)^{ps-1}\|f_r\|_p^p r dr\right)^{1/p}.$$

Let $1-2^{-n} \leq r \leq 1-2^{-n-1}$ with $n \geq 1$. Then $\|f_r\|_p \geq \|W_n \star f_r\|_p/2$. But f is the image of f_r by the multiplier with symbol $(r^{-k})_{k\in\mathbb{Z}}$. Note that for $2^{n-1} \leq k \leq 2^{n+1}$, $r^{-k} \leq 2^4$, and for $2^{n-1} \leq k < 2^{n+1}$, $r^{-k-1} - r^{-k} = (1-r)r^{-k-1} \leq 2^{-n+1}2^4 = 2^{-n+5}$. Thus since multipliers commute and since the Fourier transform of $W_n \star f$ vanishes outside of $]2^{n-1}, 2^{n+1}[$, Lemma 2.2 implies

$$\|W_n \star f\|_p \leq 2\|W_n \star f_r\|_p 2^5 \leq 2^7 \|f_r\|_p.$$

Moreover $(1-r)^{ps-1} \geq 2^{-ps}2^{-nsp+n}$. Integrating over r, we thus get that for $n \geq 1$:

$$2^{-nsp}\|W_n \star f\|_p^p \leq C^p \int_{1-2^{-n}}^{1-2^{-n-1}} (1-r)^{ps-1}\|f_r\|_p^p r dr$$

where C depends only on M. For $n=0$ the same inequality is very easy. Summing over n and taking the p-th root, we get the first inequality

$$\|f\|_{B^{-s}_{p,p}(X)_+} \leq C \left\|(1-|z|)^{s-1/p} f\right\|_{L^p(\mathbb{D},dz;X)}.$$

For the right-hand side inequality, note that since $\sum_n \widehat{W}_n(k) = 1$ for all $k \geq 0$, we have that for any $r > 0$

$$\|f_r\|_p \leq \sum_{n\geq 0} \|W_n \star f_r\|_p.$$

Then as above since $W_n \star f_r$ is the image of $W_n \star f$ by the Fourier multiplier of symbol r^k, Lemma 2.2 again implies than

$$\|W_n \star f_r\|_p \leq 2r^{2^{n-1}} \max(1, \sqrt{2^{n+1}(1-r)})\|W_n \star f\|_p.$$

If m is such that $1-2^{-m} \leq r \leq 1-2^{-m-1}$ then

$$r^{2^{n-1}} \leq \left((1-2^{-m-1})^{2^{m+1}}\right)^{2^{n-m-2}} \leq e^{-2^{n-m-2}}$$

and

$$\max(1, \sqrt{2^{n+1}(1-r)}) \leq \max(1, \sqrt{2}^{n+1-m}).$$

If for $k \in \mathbb{Z}$ one denotes $b_k = 2e^{-2^{k-2}} \max(1, \sqrt{2}^{k+1})2^{ks}$ one thus has

$$\|W_n \star f_r\|_p \leq 2^{ms} b_{n-m} 2^{-ns} \|W_n \star f\|_p.$$

If $a_n = 2^{-ns}\|W_n \star f\|_p$ for $n \geq 0$ and $a_n = 0$ if $n < 0$, summing the previous inequality over n we thus get

$$\|f_r\|_p \leq 2^{ms} \sum_{n\geq 0} b_{n-m} a_n = 2^{ms}(a \star b)_m.$$

Let us raise this inequality to the power p, multiply by $r(1-r)^{ps-1} \leq 2^{-mps}2^{m+1}$ and integrate on $[1-2^{-m}, 1-2^{-m-1}]$. One gets

$$\int_{1-2^{-m}}^{1-2^{-m-1}} (1-r)^{ps-1}\|f_r\|_p^p r dr \leq (a \star b)_m^p.$$

Summing over m this leads to

$$\left\|(1-|z|)^{s-1/p}f\right\|_{L^p(\mathbb{D},dz;X)} \leq \left(\sum_{m\geq 0}(a\star b)_m^p\right)^{1/p} \leq \|a\star b\|_{\ell^p(\mathbb{Z})}.$$

Now note that $\|a\star b\|_{\ell^p(\mathbb{Z})} \leq \|a\|_p\|b\|_1 = \|f\|_{B^{-s}_{p,p}(X)_+}\|b\|_1$. We are just left to prove that $b\in\ell^1(\mathbb{Z})$ and $\|b\|_1 \leq C/s$ with some constant C depending only on M. If $k\geq 0$, we have $|b_k| \leq 2\sqrt{2}e^{-2^{k-2}}2^{k(M+1/2)}$ which proves that $\sum_{k\geq 0} b_k \leq C_1$ for some constant depending only on M. If $k<0$, $|b_k|\leq 2^{ks+1}$, which proves that $\sum_{k<0}|b_k| \leq 2/(2^s-1) \leq C_2/s$ for some universal constant. This concludes the proof. □

When $p=2$ and X is a Hilbert space, the preceding result can be made more precise and more accurate (as $s\to 0$). This will be used later and was mentioned to the author by Quanhua Xu:

Theorem 2.6. *Let $M>0$ and X be a Hilbert space. Then for $-M\leq s\leq M$ and for all $f=\sum_k a_k z^k \in B^{-s}_{2,2}(X)_+$,*

$$\|f\|_{B^{-s}_{2,2}(X)_+} \approx \left(\sum_{k=0}^{\infty}\|a_k\|^2(1+k)^{-2s}\right)^{1/2} \approx \sqrt{s}\left\|(1-|z|)^{s-1/2}f\right\|_{L^2(\mathbb{D},dz;X)}$$

up to constants depending only on M.

Proof. The first inequality is obvious: indeed, since X is a Hilbert space, for any integer n we have

$$\|W_n\star f\|^2_{L^2(\mathbb{T};X)} = \sum_k \widehat{W_n}(k)^2\|a_k\|^2.$$

For the second inequality everything can be computed explicitly:

$$\begin{aligned}\left\|(1-|z|)^{s-1/2}f\right\|^2_{L^2(\mathbb{D},dz;H)} &= \int_0^1(1-r)^{2s-1}\sum_{k\geq 0}\|a_k\|^2 r^{2k+1}dr\\ &= \sum_{k\geq 0}\|a_k\|^2\int_0^1(1-r)^{2s-1}r^{2k+1}dr.\end{aligned}$$

Integrating by parts $2k+1$ times, one gets

$$\int_0^1(1-r)^{2s-1}r^{2k+1}dr = \frac{(2k+1)2k(2k-1)\dots 1}{2s(2s+1)\dots(2s+2k+1)} = \frac{1}{2sD^{2s}_{2k+1}}.$$

Note that $D^{2s}_{2k+1}\approx(1+k)^{2s}$ uniformly in k and s as long as $|s|<M$. This implies

$$\left\|(1-|z|)^{s-1/2}f\right\|^2_{L^2(\mathbb{D},dz;H)} \approx \frac{1}{s}\sum_k(1+k)^{-2s}\|a_k\|^2,$$

which concludes the proof. □

The following also holds (here if $f(z) = \sum_{n\geq 0} z^n a_n$, we denote $f'(z) = \sum_{n\geq 0} n z^{n-1} a_n$) :

Theorem 2.7. *Let $M > 0$. Then there is a constant $C = C_M$ (depending only on M) such that for all $-1 < s < M$, for all Banach spaces X, all $1 \leq p \leq \infty$ and all $f : \mathbb{T} \to X$,*

$$C^{-1}\|f\|_{B^{-s}_{p,p}(X)_+} \leq \|f(0)\|_X + \left\|(1-|z|)^{1+s-1/p} f'\right\|_{L^p(\mathbb{D},dz;X)} \leq \frac{C}{1+s}\|f\|_{B^{-s}_{p,p}(X)_+}.$$

Proof. By Theorem 2.5, it is enough to show that

$$\|f\|_{B^{-s}_{p,p}(X)_+} \approx \|f(0)\|_X + \|f'\|_{B^{-s-1}_{p,p}(X)_+}$$

up to constants depending only on M if $|s| < M$.

Since $\|f\|_{B^{-s}_{p,p}(X)_+} \approx \|f(0)\|_X + \|f - f(0)\|_{B^{-s}_{p,p}(X)_+}$, one can assume that $f(0) = 0$.

But since $I_1 g = (zg)'$ for any g, Theorem 2.4 implies that $\|g\|_{B^{-s}_{p,p}(X)_+} \approx \|(zg)'\|_{B^{-s-1}_{p,p}(X)_+}$. Applied to $g(z) = f(z)/z$ (recall that $f(0) = 0$) this inequality becomes $\|f'\|_{B^{-s-1}_{p,p}(X)_+} \approx \|z \mapsto f(z)/z\|_{B^{-s}_{p,p}(X)_+}$. The inequality

$$\|z \mapsto f(z)/z\|_{B^{-s}_{p,p}(X)_+} \approx \|f\|_{B^{-s}_{p,p}(X)_+}$$

is easy and concludes the proof. ☐

3. Operator space valued Hankel matrices

In this section we finally prove the main results stated in the Introduction, Theorem 0.3. In the particular case when $\alpha = \beta = 0$, we recover Theorem 0.1. We prove the two sides of (1) separately.

For the right-hand side, we first recall a proof for the cases when $p = 1$ or $p = \infty$ (this was contained in Peller's proof since for non-commutative L^1 or L^∞ spaces, regularity and complete boundedness coincide; we will still provide a proof which is more precise as far as constants are concerned). Then we derive the case of a general p by an interpolation argument.

The left-hand side inequality is then derived from the right-hand side for $\alpha = \beta = 1$ by duality.

We study the optimality of the bounds in Theorem 0.1, and finally derive Theorem 0.2.

3.1. Right hand side of (1) for $p = 1$.

We first prove that for a formal series $\varphi = \sum_{k\geq 0} \widehat{\varphi}(k) z^k$ with $\widehat{\varphi}(k) \in E$, it is sufficient that φ belongs to $B_p^{1/p+\alpha+\beta}(E)_+$ to ensure that $\Gamma_\varphi^{\alpha,\beta} \in S^p[E]$. We first treat the case when $p = 1$.

Let E be an arbitrary operator space. Since (formally) $\varphi = \sum_0^\infty W_n \star \varphi$, and $\|\varphi\|_{B_1^{1+\alpha+\beta}(E)_+} = \sum_{n\geq 0} 2^{n(1+\alpha+\beta)}\|W_n \star \varphi\|_1$, by the triangle inequality replacing φ by $W_n \star \varphi$ it is enough to prove that, if $\varphi = \sum_{k=0}^m a_k z^k$ with $a_k \in E$,

$$\left\|\Gamma_\varphi^{\alpha,\beta}\right\|_{S^1[E]} \leq C\frac{(1+m)^{1+\alpha+\beta}}{\sqrt{(\alpha+1/2)(\beta+1/2)}}\|\varphi\|_{L^1(\mathbb{T};E)}.$$

But we can write

$$\Gamma_\varphi^{\alpha,\beta} = \int_{\mathbb{T}} \left(\varphi(z)(1+j)^\alpha(1+k)^\beta \overline{z}^{j+k}\right)_{0\leq j,k\leq m} dz$$

and compute, for $z \in \mathbb{T}$,

$$\left\|\left(\varphi(z)(1+j)^\alpha(1+k)^\beta \overline{z}^{j+k}\right)_{0\leq j,k\leq m}\right\|_{S^1[E]} = \|\varphi(z)\|_E \left\|\left((1+j)^\alpha(1+k)^\beta \overline{z}^{j+k}\right)_{0\leq j,k\leq m}\right\|_{S^1},$$

with

$$\left\|\left((1+j)^\alpha(1+k)^\beta \overline{z}^{j+k}\right)_{0\leq j,k\leq m}\right\|_{S^1} = \left\|\left((1+j)^\alpha\right)_{j=0\ldots m}\right\|_{\ell^2} \left\|\left((1+k)^\beta\right)_{k=0\ldots m}\right\|_{\ell^2}.$$

Thus the lemma follows from the fact that

$$\left\|\left((1+j)^\alpha\right)_{j=0\ldots m}\right\|_{\ell^2}^2 \leq C\frac{(1+m)^{2\alpha+1}}{2\alpha+1}$$

for a constant C which depends only on $M = \max\{\alpha,\beta\}$ as long as $\alpha, \beta > -1/2$.

3.2. **Right hand side of** (1) **for** $p = \infty$**.** The sufficiency for $p=\infty$ is very similar to the easy direction in the classical proof of Nehari's Theorem that uses the factorization $H^1 = H^2 \cdot H^2$, which we first recall. Remember that Nehari's Theorem states that for any (polynomial function) $\varphi(z) = \sum_{n\geq 0} a_n z^n$ with $a_n \in \mathbb{C}$, $\|\Gamma_\varphi\|_{B(\ell^2)} = \|\varphi\|_{H^{1*}}$ for the duality $\langle \varphi, f\rangle = \sum_n a_n \widehat{\widehat{f}}(n)$ for $f \in H^1(\mathbb{T})$. With the notation $f_\xi(z) = \sum_n \xi_n z^n$ for $\xi = (\xi_n) \in \ell^2$, the inequality $\|\Gamma_\varphi\|_{B(\ell^2)} \leq \|\varphi\|_{H^{1*}}$ easily follows from the following elementary facts:

a. For any $\xi = (\xi_n), \eta = (\eta_n) \in \ell^2$,

$$\langle \Gamma_\varphi \xi, \eta\rangle_{\ell^2} = \sum_{n\geq 0} \widehat{\varphi}(n)\widehat{f_\xi f_\eta}(n) = \langle \varphi, f_\xi f_\eta\rangle.$$

b. The map $\xi \mapsto f_\xi$ is an isometry between ℓ^2 and $H^2(\mathbb{T})$.

c. For any $f_1, f_2 \in H^2(\mathbb{T})$, $f_1 f_2 \in H^1(\mathbb{T})$ with norm less than $\|f_1\|_{H^2}\|f_2\|_{H^2}$.

Let us now focus on the right-hand side of inequality (1) for $p=\infty$. We fix $\alpha,\beta>0$ and assume that $E\subset B(H)$ for a Hilbert space H. In this proof we use the fact that $H\hat{\otimes}\overline{H}\simeq B(H)_*$ isometrically through the duality $\langle T,\xi\otimes\overline{\eta}\rangle=\langle T\xi,\eta\rangle$. For a sequence $x=(\xi_n)$ with ξ_n in some vector space we also use the notation $f_\xi^\alpha(z)$ for the formal series $\sum_{n\geq 0}(1+n)^\alpha z^n\xi_n$.

Let $\varphi\in B_\infty^{\alpha+\beta}(E)_+$. We wish to prove that

$$\|\Gamma_\varphi^{\alpha,\beta}\|_{B(\ell^2(H))}\leq C/\sqrt{\min(\alpha,\beta)}\|\varphi\|_{B_\infty^{\alpha+\beta}(E)_+}.$$

Since $B_\infty^{\alpha+\beta}(E)_+$ is naturally isometrically contained in $B_\infty^{\alpha+\beta}(B(H))_+$ which is (by Theorem 2.3 and the identification $H\hat{\otimes}\overline{H}\simeq B(H)_*$) isomorphic to the dual space of $B_1^{-\alpha-\beta}\left(H\widehat{\otimes}\overline{H}\right)_+$, we are are left to prove that

$$\|\Gamma_\varphi^{\alpha,\beta}\|_{B(\ell^2(H))}\leq C/\sqrt{\min(\alpha,\beta)}\|\varphi\|_{B_1^{-\alpha-\beta}\left(H\widehat{\otimes}\overline{H}\right)_+^*}.$$

As above this inequality follows immediately from the following three facts:

a'. For any $\xi=(\xi_n)\in\ell^2(H),\eta=(\eta_n)\in\ell^2(H)$,

$$\langle\Gamma_\varphi^{\alpha,\beta}\xi,\eta\rangle_{\ell^2(H)}=\sum_{n\geq 0}\langle\widehat{\varphi}(n),\widehat{f_\xi^\beta\otimes f_{\overline{\eta}}^\alpha}(n)\rangle_{B(H),H\widehat{\otimes}\overline{H}}=\langle\varphi,f_\xi^\beta\otimes f_{\overline{\eta}}^\alpha\rangle.$$

b'. The map $\xi\in\ell^2(H)\mapsto f_\xi^\beta$ (resp. $\overline{\eta}=(\overline{\eta}_n)\in\ell^2(\overline{H})\mapsto f_{\overline{\eta}}^\alpha$) is an isomorphism between $\ell^2(H)$ and $B_2^{-\beta}(H)_+$ (resp. between $\ell^2(\overline{H})$ and $B_2^{-\alpha}\left(\overline{H}\right)_+$). Moreover the constants in these isomorphisms depend only on $M=\max(\alpha,\beta)$.

c'. For any $f\in B_2^{-\beta}(H)_+$ and $g\in B_2^{-\alpha}\left(\overline{H}\right)_+$, the series $f\otimes g$ belongs to $B_1^{-\alpha-\beta}\left(H\widehat{\otimes}\overline{H}\right)_+$ and moreover there is a constant C depending only on M such that

$$\|f\otimes g\|_{B_1^{-\alpha-\beta}\left(H\hat{\otimes}\overline{H}\right)_+}\leq\frac{C}{\min(\sqrt{\alpha},\sqrt{\beta})}\|f\|_{B_2^{-\beta}(H)_+}\|g\|_{B_2^{-\alpha}\left(\overline{H}\right)_+}.$$

The facts (a') and (b') are again elementary while fact (c') is not and follows from the properties of Besov spaces stated in the previous section. Let us prove it.

Remark. In fact the same holds with H and $\overline{H}$ replaced by arbitrary Banach spaces, but in this case one has to replace $C/\min(\sqrt{\alpha},\sqrt{\beta})$ by $C/\min(\alpha,\beta)$.

Proof of (c'). From Theorem 2.7,

$$\|f\otimes g\|_{B_1^{-\alpha-\beta}\left(H\hat{\otimes}\overline{H}\right)_+}\approx\|f\otimes g(0)\|+\left\|(1-|z|)^{\alpha+\beta}(f\otimes g)'\right\|_{L^1(\mathbb{D},dz;H\hat{\otimes}\overline{H})}.$$

Since $(f\otimes g)'=f'\otimes g+f\otimes g'$, (c') will clearly follow from the existence of a constant C depending on M only such that

$$\left\|(1-|z|)^{\alpha+\beta}f'\otimes g\right\|_{L^1(\mathbb{D},dz;H\hat{\otimes}\overline{H})}\leq\frac{C}{\sqrt{\alpha}}\|f\|_{B_2^{-\beta}(H)_+}\|g\|_{B_2^{-\alpha}\left(\overline{H}\right)_+}.$$

But by the Cauchy-Schwarz inequality, we get that

$$\begin{aligned}\left\|(1-|z|)^{\alpha+\beta} f' \otimes g\right\|_{L^1(\mathbb{D},dz;H\hat{\otimes}\overline{H})} \\ \leq \left\|(1-|z|)^{\beta+1/2} f'\right\|_{L^2(\mathbb{D},dz;H)} \left\|(1-|z|)^{\alpha-1/2} g\right\|_{L^2(\mathbb{D},dz;\overline{H})}\end{aligned}$$

For the first term, use again Theorem 2.7 to get

$$\left\|(1-|z|)^{\beta+1/2} f'\right\|_{L^2(\mathbb{D},dz;H)} \approx \|f\|_{B_2^{-\beta}(H)_+}, \tag{4}$$

whereas for the second term Theorem 2.6 implies

$$\left\|(1-|z|)^{\alpha-1/2} g\right\|_{L^2(\mathbb{D},dz;\overline{H})} \approx \frac{1}{\sqrt{\alpha}} \|g\|_{B_2^{-\alpha}(\overline{H})_+}.$$

□

3.3. **Right hand side of** (1) **for a general** p**.** Let us first reformulate the right-hand side of (1).

Denote by D the infinite diagonal matrix $D_{j,j} = 1/(1+j)$ and $D_{j,k} = 0$ if $j \neq k$. Let p, α and β as in Theorem 0.3. Define $\widetilde{\alpha} = \alpha + 1/2p$ and $\widetilde{\beta} = \beta + 1/2p$. Then for any φ

$$\Gamma_\varphi^{\alpha,\beta} = D^{1/2p} \Gamma_\varphi^{\widetilde{\alpha},\widetilde{\beta}} D^{1/2p},$$

and Theorem 2.4 implies that the map $I_{\widetilde{\alpha}+\widetilde{\beta}} : B_{p+}^{\widetilde{\alpha}+\widetilde{\beta}} \to B_{p+}^0$ is a regular isomorphism (with regular norms of the map and its inverse depending only on M if $|\alpha|, |\beta| \leq M$).

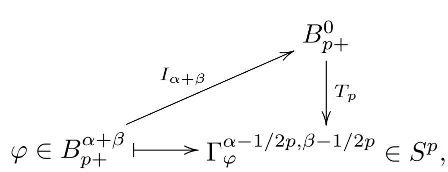
The main result of this section is

Lemma 3.1. *Let* $M > 0$*. Take* $0 < \alpha, \beta < M$ *and* $1 \leq p \leq \infty$*. The map*

$$\begin{aligned} T_p : B_{p+}^0 &\to S^p \text{ (or } B(\ell^2) \text{ if } p=\infty) \\ \varphi &\mapsto D^{1/2p} \left(\widehat{\varphi}(j+k) \frac{(1+j)^\alpha (1+k)^\beta}{(1+j+k)^{\alpha+\beta}} \right)_{j,k\geq 0} D^{1/2p} \end{aligned}$$

is regular, with regular norm less that $C/(\min(\alpha,\beta))^{1/2+1/2p}$ *for some constant* C *depending only on* M*.*

As explained above, this result is equivalent to the right-hand side inequality in (1). More precisely for $\alpha, \beta > 0$ and $1 \leq p \leq \infty$, we have the following factorization of $\varphi \mapsto \Gamma_\varphi^{\alpha-1/2p,\beta-1/2p}$:



where $I_{\alpha+\beta}$ is a regular isomorphism. Thus the above Lemma for this value of α, β and p is equivalent to the right-hand side inequality in (1) for the same p but with α and β replaced by $\alpha - 1/2p$, $\beta - 1/2p$. In the proof below, Pisier's Theorem 1.3 on interpolation of regular operators is used, but the reader unfamiliar with regular operators can as well directly use Stein's complex interpolation method with vector-valued Besov spaces and Schatten classes.

Proof of Lemma 3.1. We have already seen that the map T_p is regular when $p = 1$ or $p = \infty$. Therefore up to the change of density given by D, T_p is simultaneously completely bounded on B^0_{1+} and $B^0_{\infty+}$, which should imply that T_p is regular.

To check this more rigorously, we use Pisier's Theorem 1.3. Since the Besov space B^0_{p+} is a complemented subspace of $L^p(\mathbb{N}\times\mathbb{T})$ (where $\mathbb{N}\times\mathbb{T}$ is equipped with the product of the counting measure on $\mathbb{N}$ and the Lebesgue measure on $\mathbb{T}$), and since the projection map P is regular and is the same for every p, T_p naturally extends to a map $T_p \circ P : L^p(\mathbb{N}\times\mathbb{T}) \to S^p$ which is still completely bounded for $p = 1, \infty$.

We show that $T_p \circ P \in \left[CB(L^\infty, B(\ell^2)), CB(L^1, S^1)\right]_\theta$ (where the first L^∞ and L^1 spaces are $L^\infty(\mathbb{N}\times\mathbb{T})$ and $L^1(\mathbb{N}\times\mathbb{T})$). Since by the equivalence theorem for complex interpolation $[A_0, A_1]_\theta \subset [A_0, A_1]^\theta$ with constant 1 for any compatible Banach spaces A_0, A_1 (Theorem 4.3.1 in [1]), Theorem 1.3 will imply that $T_p \circ P$ is regular and hence its restriction to B^0_{p+}, T_p, too.

Consider the analytic map $f(z)$ with values in $CB(L^1, S^1)+CB(L^\infty, B(\ell^2))$ given by $f(z) = L_{D^{z/2}} R_{D^{z/2}} (T_\infty \circ P)$, where L and R stand for left and right multiplication maps (f takes in fact values in $CB(L^\infty, B(\ell^2))$). Then $f(1/p) = T_p \circ P$. The left and right multiplication by a unitary are complete isometries on both $B(\ell^2)$ and S^1. Therefore if $Re(z) = 0$, $\|f(z)\|_{CB(L^\infty,B(\ell^2))} = \|T_\infty \circ P\|_{CB(L^\infty,B(\ell^2))} \leq C/\sqrt{\min(\alpha,\beta)}$ and if $Re(z) = 1$, $\|f(z)\|_{CB(L^1,S^1)} = \|T_1 \circ P\|_{CB(L^1,S^1)} \leq C/\sqrt{\alpha\beta} \leq C/\min(\alpha,\beta)$. This proves that

$$\|T_p\|_{B_r(L^p,S^p)} \leq C/(\min(\alpha,\beta))^{1/2+1/2p}.$$

□

3.4. **Left-hand side of** (1)**.** In this section we assume that the right-hand side of (1) holds for $\alpha = \beta = 1$, that is to say the operator

$$\begin{array}{ccc} B^{1/p+2}_{p+} & \to & S^p \\ \varphi & \mapsto & \Gamma^{1,1}_\varphi \end{array}$$

is regular for every $1 \leq p \leq \infty$.

Fix now $1 \leq p \leq \infty$ and $\alpha, \beta > -1/2p$. We prove that the map $\Gamma^{\alpha,\beta}_\varphi \mapsto \varphi$ is regular from the subspace of S^p (or $B(\ell^2)$ if $p = \infty$) formed of all the matrices of the form $\Gamma^{\alpha,\beta}_\varphi$ to $B^{1/p+\alpha+\beta}_{p+}$.

For $\psi \in B_{p'+}^{1/p'+2}$ define the matrix

$$\begin{aligned}\widetilde{\Gamma}_\psi^{1,1} &= \left(\frac{D_j^{\alpha+1}}{(1+j)^\alpha} \frac{D_k^{\beta+1}}{(1+k)^\beta} \widehat{\psi}(j+k) \right)_{j,k\geq 0} \\ &= diag\left(\frac{D_j^{\alpha+1}}{(1+j)^{\alpha+1}} \right) \cdot \Gamma_\psi^{1,1} \cdot diag\left(\frac{D_k^{\beta+1}}{(1+k)^{\beta+1}} \right).\end{aligned}$$

First note that since $\sup_{-1/2\leq\alpha\leq M} \sup_{j\geq 0} D_j^{\alpha+1}/(1+j)^{\alpha+1} < \infty$ the assumption with p' implies that the operator $T : \psi \mapsto \widetilde{\Gamma}_\psi^{1,1}$ is also regular from $B_{p'+}^{1/p'+2}$ to $S^{p'}$ with regular norm bounded by some constant depending only on M.

Recall that by Theorem 2.3, $B_{p+}^{-1/p'-2} \simeq (B_{p'+}^{1/p'+2})^*$ if $p>1$ (and $B_{p'+}^{1/p'+2} \simeq (B_{p+}^{-1/p'-2})^*$ if $p<\infty$). Since $B_{p+}^{1/p-3}$ is complemented in $\ell_p^{1/p-3}(\mathbb{N}; L^p)$ with a regular complementation map, Theorem 1.4 implies that the dual map $T^* : S^p \to B_{p+}^{-1/p'-2} = B_{p+}^{1/p-3}$ is also regular.

It is now enough to compute explicitly the restriction of T^* to the set of matrices of the form $\Gamma_\varphi^{\alpha,\beta}$ to conclude. Indeed for any analytic $\varphi : \mathbb{T} \to \mathbb{C}$ such that $\Gamma_\varphi^{\alpha,\beta} \in S^p$ (or $B(\ell^2)$), and any $\psi \in B_{p'+}^{1/p'+2}$ we have

$$\begin{aligned}\left\langle T^* \Gamma_\varphi^{\alpha,\beta}, \psi \right\rangle &= \left\langle \Gamma_\varphi^{\alpha,\beta}, T\psi \right\rangle \\ &= \sum_{j,k\geq 0} D_j^{\alpha+1} D_k^{\beta+1} \widehat{\varphi}(j+k) \widehat{\psi}(j+k) \\ &= \sum_{n\geq 0} D_n^{\alpha+\beta+3} \widehat{\varphi}(n) \widehat{\psi}(n) \\ &= \langle \widetilde{I}_{\alpha+\beta+3}\varphi, \psi \rangle.\end{aligned}$$

We used that for all $\alpha, \beta \in \mathbb{R}$, and all $n \in \mathbb{N}$

$$\sum_{j+k=n} D_j^\alpha D_k^\beta = D_n^{\alpha+\beta+1},$$

which follows from the equality $\sum_{n\geq 0} D_n^\alpha x^n = (1+x)^{-\alpha-1}$ for $|x|<1$. Indeed, the Cauchy product $\sum_{j+k=n} D_j^\alpha D_k^\beta$ is the coefficient of x^n in the power series expansion of the product $(1+x)^{-\alpha-1} \cdot (1+x)^{-\beta-1}$, and this product is equal to $(1+x)^{-\alpha-\beta-1-1} = \sum_{n\geq 0} D_n^{\alpha+\beta+1} x^n$.

Thus we have that $T^* \Gamma_\varphi^{\alpha,\beta} = \widetilde{I}_{\alpha+\beta+3}\varphi$. By Theorem 2.4, $\left(\widetilde{I}_{\alpha+\beta+3}\right)^{-1}$ is regular as a map from $B_{p+}^{1/p-3}$ to $B_{p+}^{1/p+\alpha+\beta}$. Hence the map $\Gamma_\varphi^{\alpha,\beta} \mapsto \varphi$ is regular from the subspace of S^p formed of all the matrices of the form $\Gamma_\varphi^{\alpha,\beta}$ to $B_{p+}^{1/p+\alpha+\beta}$. This concludes the proof (it is immediate from the proof that the regular norm of this map only depends on M).

3.5. **Optimality of the constants.** In this last part we first show that the inequality

$$C^{-1}\left\|\varphi\right\|_{B_p^{1/p}(E)_+} \leq \left\|\Gamma_\varphi\right\|_{S^p[E]} \leq C\sqrt{p}\left\|\varphi\right\|_{B_p^{1/p}(E)_+} \tag{5}$$

in Theorem 0.1 is optimal even when $E=\mathbb{C}$ (up to constants not depending on p). This observation is due to Éric Ricard who kindly allowed to reproduce his proof here.

The fact that the left-hand side of (5) is optimal is obvious: indeed if $\varphi(z)=1$ then Γ_φ is a rank one orthogonal projection and hence $\left\|\Gamma_\varphi\right\|_{S^p} = 1 = \left\|\varphi\right\|_{B^{1/p}_{p+}}$ for any p.

For the right-hand side inequality consider the positive integer n such that $n \leq p < n+1$. Let $a_1,\dots,a_n \in \mathbb{C}$ and consider the function $\varphi_a = \sum_{k=0}^n a_k z^{2^k}$. We clearly have

$$\left\|\varphi_a\right\|_{B^{1/p}_{p+}} = (\sum_{k=0}^n 2^k |a_k|^p)^{1/p} \leq 2^{n+1/p} \max_k |a_k| \leq 4 \max_k |a_k|,$$

and the following lemma therefore implies that the ratio $\left\|\varphi_a\right\|_{B^{1/p}_{p+}} / \left\|\Gamma_{\varphi_a}\right\|_{S^p}$ can be as small as $12/\sqrt{n}$, which shows the optimality of the right-hand side of (5).

Lemma 3.2. *For any $1 \leq p \leq \infty$ and any (finite) sequence $a=(a_k)_{k\geq 0}$ we have*

$$\left\|\Gamma_{\varphi_a}\right\|_{S^p} \geq \frac{1}{3}\|a\|_{\ell^2}.$$

Proof. Since $\|\cdot\|_{S^p} \geq \|\cdot\|_{B(\ell^2)}$ for any $1 \leq p \leq \infty$, and since by Nehari's Theorem

$$\|\Gamma_{\varphi_a}\|_{B(\ell^2)} = \|\varphi_a\|_{H^{1*}},$$

the statement follows from the inequality $\|\varphi_a\|_{H^{1*}} \geq \|a\|_{\ell^2}/3$, which is the dual inequality of the classical Paley inequality

$$\left(\sum_{k\geq 0} |\widehat{f}(2^k)|^2\right)^{1/2} \leq 3\|f\|_{H^1}$$

which holds for any $f \in H^1(\mathbb{T})$. □

We now state the result mentioned in the introduction, that shows that the statement of Theorem 0.1 is also optimal in the sense of (2) :

Proposition 3.3. *Let $T : Hank_p \to X$ be a regular isomorphism between $Hank_p$ and a subspace X of a commutative L^p space. Then*

$$\|T\|_{reg}\|T^{-1}\|_{reg} \geq c\sqrt{p}.$$

Proof. Fix $1 \leq p < \infty$. It is enough to show that we can find two operator space structures E_1 and E_2 on ℓ^2 and an element $x \in Hank_p \otimes \ell^2$ such that

$$\|x\|_{S^p[E_2]} \geq c\sqrt{p} \text{ and } \|x\|_{S^p[E_1]} \leq 1. \tag{6}$$

Indeed, if $T : Hank_p \to X \subset L^p(\Omega, \mu)$ is as above, we have that

$$\begin{aligned} \|x\|_{S^p[E_2]} &\leq \|T^{-1}\|_{reg}\|T \otimes id(x)\|_{L^p(\Omega;E_2)} \\ &= \|T^{-1}\|_{reg}\|T \otimes id(x)\|_{L^p(\Omega;E_1)} \\ &\leq \|T^{-1}\|_{reg}\|T\|_{reg}\|x\|_{S^p[E_2]}. \end{aligned}$$

If (6) holds we exactly get $c\sqrt{p} \leq \|T\|_{reg}\|T^{-1}\|_{reg}$.

We claim that if N is the integer such that $N \leq p < N+1$, (6) holds for $E_1 = OH$, $E_2 = R$ and $x = (\sqrt{\lambda_{i+j}}e_{i+j})_{i,j\geq 0}$ where $(e_n)_{n\geq 0}$ is an orthonormal family in ℓ^2 and $(\lambda_n)_{n\geq 0}$ is a sequence of nonnegative real numbers with the following properties: (i) $\lambda_n = 0$ for all $n \geq 2^N$, (ii) $\sum_n \lambda_n \geq c^2 p$ and (iii) there exists a function $\varphi \in L^\infty(\mathbb{T})$ such that $\|\varphi\|_\infty \leq 1/4$ and $\lambda_n = \widehat{\varphi}(n)$ for all $n \geq 0$. Such a sequence can be obtained from the sequence $\big(1/(n+1)\big)_{n\geq 0}$ using a smooth truncation (since $1/(n+1)$ is the n-th Fourier coefficient of the bounded function defined by $e^{i\theta} \mapsto -i\theta e^{-i\theta}$ for $\theta \in [0, 2\pi]$).

Indeed since $x \in M_{2^N}(E)$, Hölder's inequality and [7, Theorem 1.5] imply that

$$\|x\|_{S^\infty[E_i]} \leq \|x\|_{S^p[E_i]} \leq (2^N)^{1/p}\|x\|_{S^\infty[E_i]} \leq 2\|x\|_{S^\infty[E_i]}. \tag{7}$$

Moreover remember that from the definition of the row Hilbert space R and the operator Hilbert space OH ([8, Chapter 7]), for matrices x_n,

$$\|\sum_n x_n \otimes e_n\|^2_{S^\infty[R]} = \|\sum_n x_n x_n^*\|$$

and

$$\begin{aligned} \|\sum_n x_n \otimes e_n\|^2_{S^\infty[OH]} &= \|\sum_n x_n \otimes \overline{x_n}\| \\ &= \sup_{a,b\in S^2, \|a\|_2\|b\|_2<1} Tr\left(\sum_n a x_n b^* x_n^*\right). \end{aligned}$$

Here we have $x = \sum_n x_n \otimes e_n$ with $(x_n)_{i,j} = \sqrt{\lambda_n}1_{i+j=n}$.

We thus have $\|x\|^2_{S^\infty[R]} = \sum_{n\geq 0}\lambda_n$, and the left-hand side of (7) together with the assumption (ii) proves the first inequality in (6).

For the second inequality in (6), we prove that $\|x\|^2_{S^\infty[OH]} \leq \|\varphi\|_{L^\infty}$ (which is enough by the right-hand side of (7) and assumption (iii)). Fix a, b in the unit ball of S^2. If we denote by f and g the functions in the unit ball of $L^2(\mathbb{T}^2)$ defined by $f(z, z') = \sum_{i,j\geq 0} \overline{a_{i,j}} z^i z'^j$ and $g(z, z') = \sum_{i,j\geq 0} b_{i,j} z^i z'^j$, we have that

$$Tr\left(\sum_n a x_n b^* x_n^*\right) = \int_{\mathbb{T}^2} \varphi(zz')\overline{fg(z,z')}dzdz',$$

which implies that $\|x\|_{S^\infty[OH]} \leq \|\varphi\|_{L^\infty}$ since $\|\overline{fg(z,z')}\|_{L^1(\mathbb{T}^2)} \leq 1$. This concludes the proof. $\square$

3.6. **The projection.** As in the introduction, P_{Hank} will denote the natural projection from the space of infinite $\mathbb{N}\times\mathbb{N}$ matrices onto the space of Hankel matrices. The boundedness properties of P_{Hank} stated in Theorem 0.2 are formal consequences of Theorem 0.1.

Proof of Theorem 0.2. Let $1 < p, p' < \infty$, with $1/p + 1/p' = 1$. Since for the identification $(S^p)^* = S^{p'}$, ${P_{Hank}}^* = P_{Hank}$, we can restrict ourselves to the case when $1 < p \leq 2$. We thus have to show that

$$\|P_{Hank}\|_{S^p\to S^p} \approx \|P_{Hank}\|_{B_r(S^p,S^p)} \approx \sqrt{p'} \tag{8}$$

up to constants not depending on p.

This follows from Theorem 0.1. More precisely let $T : \psi \mapsto \Gamma_\psi$ defined from $B^{1/p'}_{p'+}$ to $S^{p'}$. Then by Theorem 0.1, we have that

$$\|T\|_{B^{1/p'}_{p'+}\to S^{p'}} \approx \|T\|_{B_r(B^{1/p'}_{p'+},S^{p'})} \approx \sqrt{p'}.$$

As in part 3.4 this implies (for the natural dualities) that

$$\|T^*\|_{S^p\to B^{-1/p'}_{p+}} \approx \|T^*\|_{B_r(S^p,B^{-1/p'}_{p+})} \approx \sqrt{p'}.$$

But $T^*(a_{j,k})_{j,k\geq 0} = \sum_{j,k\geq 0} a_{j,k} z^{j+k}$. Thus we have the following factorization of P_{Hank}:

$$\begin{array}{ccc} S^p & \xrightarrow{P_{Hank}} & S^p \\ \big\downarrow{\scriptstyle T^*} & & \big\uparrow{\scriptstyle T} \\ B^{-1/p'}_{p+} & \xrightarrow{I_{-1}} & B^{1/p}_{p+} \end{array} .$$

This concludes the proof since I_{-1} (resp. T) is a regular isomorphism between $B^{-1/p'}_{p+}$ and $B^{1/p}_{p+}$ (resp. between $B^{1/p}_{p+}$ and the subspace of Hankel matrices in S^p), and the regular norms of these isomorphisms as well as their inverses can be dominated uniformly in p (recall that $1 < p \leq 2$). $\square$

Acknowledgement. The author would like to thank his adviser Gilles Pisier for suggesting the problem. He also thanks Quanhua Xu for pointing out the result from Lemma 2.6, and Éric Ricard for allowing to reproduce his argument in part 3.5, and for other useful discussions regarding the optimality of the constants.

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