Data processing and networks optimization

Part II: Optimization (Basics)

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Hilbert spaces

A (real) Hilbert space $\mathcal{H}$ is a complete real vector space endowed with an inner product $\langle \cdot \mid \cdot \rangle$. The associated norm is

$$(\forall x \in \mathcal{H}) \qquad \|x\| = \sqrt{\langle x \mid x \rangle}.$$

- Particular case: $\mathcal{H} = \mathbb{R}^N$ (Euclidean space with dimension N).

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$2^{\mathcal{H}}$ is the power set of $\mathcal{H}$, i.e. the family of all subsets of $\mathcal{H}$.

Hilbert spaces

Let $\mathcal{H}$ and $\mathcal{G}$ be two Hilbert spaces.
A linear operator $L\colon \mathcal{H} \to \mathcal{G}$ is bounded (or continuous) if

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$\mathcal{B}(\mathcal{H}, \mathcal{G})$: Banach space of bounded linear operators from $\mathcal{H}$ to $\mathcal{G}$.

Hilbert spaces

Let $\mathcal{H}$ and $\mathcal{G}$ be two Hilbert spaces.
Let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$. Its adjoint L^* is the operator in $\mathcal{B}(\mathcal{G}, \mathcal{H})$ defined as

$$(\forall (x, y) \in \mathcal{H} \times \mathcal{G}) \qquad \langle y \mid Lx \rangle_{\mathcal{G}} = \langle L^* y \mid x \rangle_{\mathcal{H}} .$$

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Example:

If $\quad L \colon \mathcal{H} \to \mathcal{H}^n \colon x \mapsto (x, \ldots, x)$

then $\quad L^* \colon \mathcal{H}^n \to \mathcal{H} \colon y = (y_1, \ldots, y_n) \mapsto \sum_{i=1}^{n} y_i$

Proof:

$$\langle Lx \mid y \rangle = \langle (x, \ldots, x) \mid (y_1, \ldots, y_n) \rangle = \sum_{i=1}^{n} \langle x \mid y_i \rangle = \left\langle x \mid \sum_{i=1}^{n} y_i \right\rangle$$

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$$(\forall (x, y) \in \mathcal{H} \times \mathcal{G}) \qquad \langle Lx \mid y \rangle = \langle x \mid L^* y \rangle \,.$$

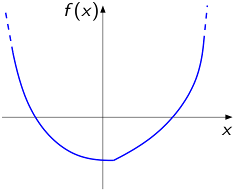
- We have $\|L^*\| = \|L\|$.
- If L is bijective (i.e. an isomorphism) then $L^{-1} \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ and $(L^{-1})^* = (L^*)^{-1}$.
- If $\mathcal{H} = \mathbb{R}^N$ and $\mathcal{G} = \mathbb{R}^M$ then $L^* = L^\top$.

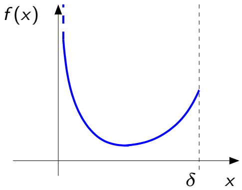
Functional analysis: definitions

Let $f : \mathcal{H} \to]-\infty, +\infty]$ where $\mathcal{H}$ is a Hilbert space.

- The domain of f is $\operatorname{dom} f = \{x \in \mathcal{H} \,|\, f(x) < +\infty\}$.
- The function f is proper if $\operatorname{dom} f \neq \varnothing$.

Domains of the functions ?





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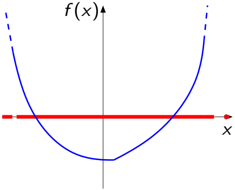
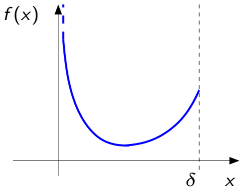


$\operatorname{dom} f = \mathbb{R}$
(proper)



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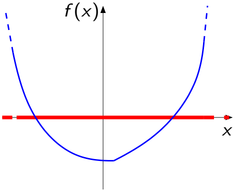

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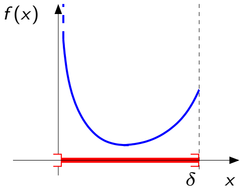


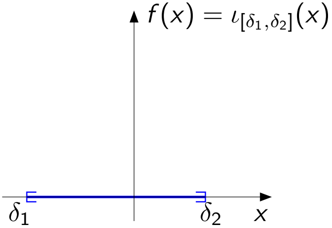
$\operatorname{dom} f =]0, \delta]$
(proper)

Functional analysis: definitions

Let $C \subset \mathcal{H}$.
The **indicator function of C** is

$$(\forall x \in \mathcal{H}) \qquad \iota_C(x) = \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{otherwise.} \end{cases}$$

Example : $C = [\delta_1, \delta_2]$



Convergence in Hilbert spaces

Let $\mathcal{H}$ be a Hilbert space.
Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ and $\widehat{x} \in \mathcal{H}$.

- $(x_n)_{n\in\mathbb{N}}$ **converges strongly** to $\widehat{x}$ if

$$\lim_{n\to+\infty} \|x_n - \widehat{x}\| = 0.$$

  It is denoted by $x_n \to \widehat{x}$.
- $(x_n)_{n\in\mathbb{N}}$ **converges weakly** to $\widehat{x}$ if

$$(\forall y \in \mathcal{H}) \qquad \lim_{n\to+\infty} \langle y \mid x_n - \widehat{x}\rangle = 0.$$

  It is denoted by $x_n \rightharpoonup \widehat{x}$.

Remark: $x_n \to x \Rightarrow x_n \rightharpoonup x$.
In a finite dimensional Hilbert space, strong and weak convergences are equivalent.

Convergence in Hilbert spaces

Let S be a subset of a Hilbert space $\mathcal{H}$.

- S is **bounded** if it is included in a ball.
- S is **closed** if the limit of every converging sequence of elements of S belongs to S.
- S is **compact** if, from every sequence $(x_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$, one can extract a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which converges to a point of S.

- If S is compact, then it is closed and bounded.
- The converse property holds, when $\mathcal{H}$ is finite dimensional.

Limits inf and sup

Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of elements in $[-\infty,+\infty]$.
Its infimum limit is $\liminf \xi_n = \lim_{n\to+\infty} \inf \left\{\xi_k \;\middle|\; k \geq n\right\} \in [-\infty,+\infty]$ and its supremum limit is $\limsup \xi_n = \lim_{n\to+\infty} \sup \left\{\xi_k \;\middle|\; k \geq n\right\} \in [-\infty,+\infty]$.

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- $\limsup \xi_n = -\liminf(-\xi_n)$
- $\lim_{n\to+\infty} \xi_n = \overline{\xi} \in [-\infty,+\infty]$ if and only if $\liminf \xi_n = \limsup \xi_n = \overline{\xi}$.

Epigraph

Let $f : \mathcal{H} \to \,]-\infty, +\infty]$. The **epigraph** of f is

$$\operatorname{epi} f = \left\{ (x, \zeta) \in \operatorname{dom} f \times \mathbb{R} \;\middle|\; f(x) \leq \zeta \right\}$$

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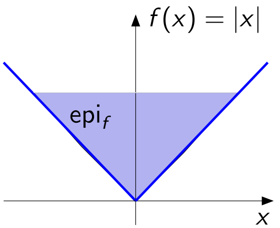

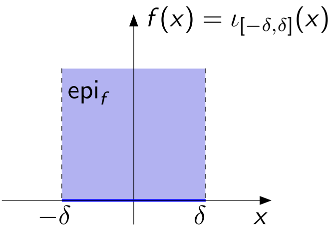

Lower semi-continuity

Let $f : \mathcal{H} \to]-\infty, +\infty]$.
f is a lower semi-continuous (l.s.c.) function at $x \in \mathcal{H}$ if, for every sequence $(x_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$,

$$x_n \to x \quad \Rightarrow \quad \liminf f(x_n) \geq f(x).$$

Lower semi-continuity

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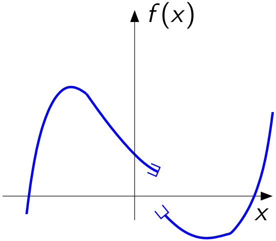
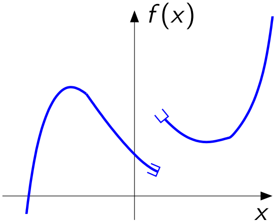
- l.s.c. functions ?





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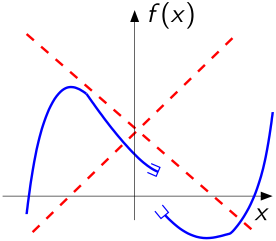

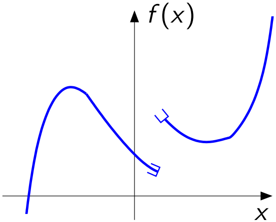

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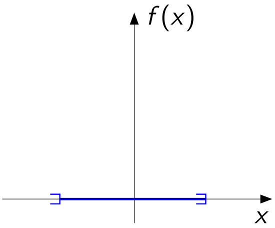

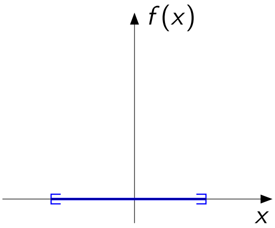

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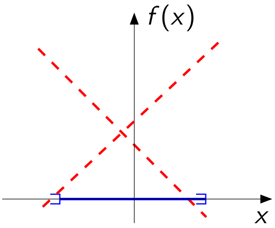

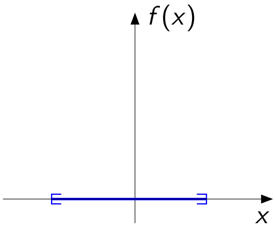

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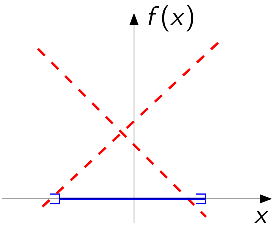
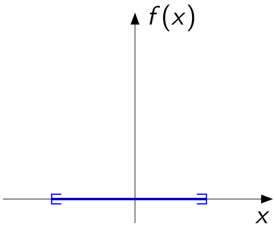




Lower semi-continuity

- Every continuous function on $\mathcal{H}$ is l.s.c.
- Every finite sum of l.s.c. functions is l.s.c.
- Let $(f_i)_{i \in I}$ be a family of l.s.c functions.
  $\sup_{i \in I} f_i$ is l.s.c.

Minimizers

Let S be a nonempty set of a Hilbert space $\mathcal{H}$.
Let $f : S \to]-\infty, +\infty]$ be a proper function and let $\widehat{x} \in S$.

- $\widehat{x}$ is a **local minimizer** of f if there exists an open neigborhood O of $\widehat{x}$ such that

$$(\forall x \in O \cap S) \qquad f(\widehat{x}) \leq f(x).$$

- $\widehat{x}$ is a **(global) minimizer** of f if

$$(\forall x \in S) \qquad f(\widehat{x}) \leq f(x).$$

Minimizers

Let S be a nonempty set of a Hilbert space $\mathcal{H}$.
Let $f : S \to]-\infty, +\infty]$ be a proper function and let $\widehat{x} \in S$.

- $\widehat{x}$ is a strict local minimizer of f if there exists an open neigborhood O of $\widehat{x}$ such that

$$(\forall x \in (O \cap S) \setminus \{\widehat{x}\}) \qquad f(\widehat{x}) < f(x).$$

- $\widehat{x}$ is a strict (global) minimizer of f if

$$(\forall x \in S \setminus \{\widehat{x}\}) \qquad f(\widehat{x}) < f(x).$$

Existence of a minimizer

Weierstrass theorem

Let S be a nonempty compact set of a Hilbert space $\mathcal{H}$.
Let $f : S \to \,]-\infty, +\infty]$ be a proper l.s.c function such that $\operatorname{dom} f \cap S \neq \varnothing$.
Then, there exists $\widehat{x} \in S$ such that

$$f(\widehat{x}) = \inf_{x \in S} f(x).$$

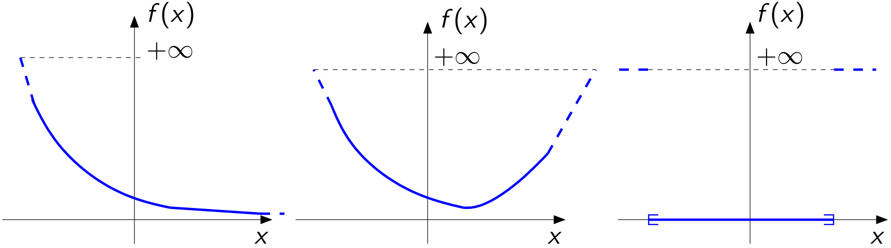
Existence of a minimizer

Let $\mathcal{H}$ be a Hilbert space. Let $f : \mathcal{H} \to \,]-\infty, +\infty]$.
f is coercive if $\lim_{\|x\| \to +\infty} f(x) = +\infty$.

Existence of a minimizer

Let $\mathcal{H}$ be a Hilbert space. Let $f : \mathcal{H} \to \,]-\infty, +\infty]$.
f is **coercive** if $\lim_{\|x\|\to+\infty} f(x) = +\infty$.

Coercive functions ?



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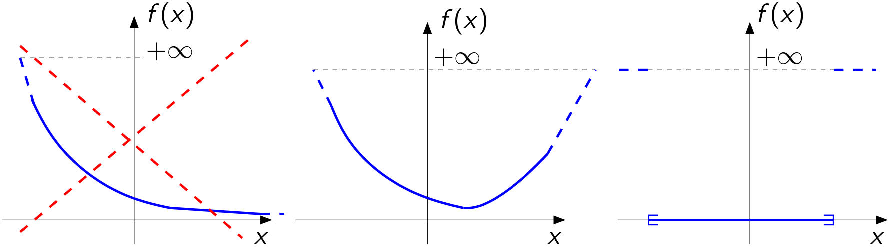


Existence of a minimizer

Theorem

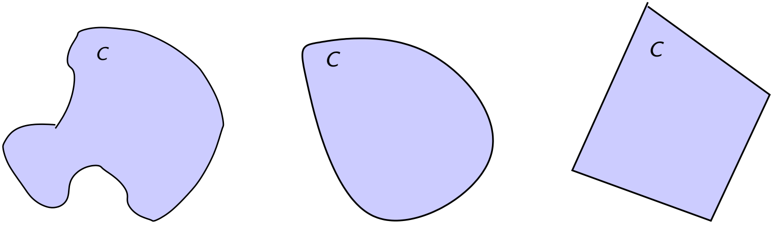
Let $\mathcal{H}$ be a finite dimensional Hilbert space.
Let $f : \mathcal{H} \to \,]-\infty, +\infty]$ be a proper l.s.c. coercive function.
Then, the set of minimizers of f is a nonempty compact set.

Convex set

$C \subset \mathcal{H}$ is a convex set if

$$(\forall (x,y) \in C^2)(\forall \alpha \in]0,1[) \qquad \alpha x + (1-\alpha) y \in C$$

Convex sets ?



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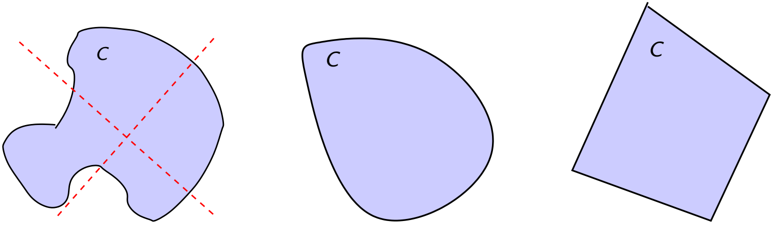

Minimizers over convex sets

Let $\mathcal{H}$ be a Hilbert space and let $f\colon \mathcal{H} \to \left]-\infty, +\infty\right]$ be a proper function. f is Gâteaux differentiable at $x \in \operatorname{dom} f$ if there exists $\nabla f(x) \in \mathcal{H}$ such that

$$(\forall y \in \mathcal{H}) \qquad \langle \nabla f(x) \mid y \rangle = \lim_{\substack{\alpha \to 0 \\ \alpha \neq 0}} \frac{f(x + \alpha y) - f(x)}{\alpha}.$$

Theorem

Let C be a nonempty convex subset of a Hilbert space $\mathcal{H}$. Let $f\colon C \to \left]-\infty, +\infty\right]$ be Gâteaux differentiable at $\widehat{x} \in C$. If $\widehat{x}$ is a local minimizer of f, then

$$(\forall y \in C) \quad \langle \nabla f(\widehat{x}) \mid y - \widehat{x} \rangle \geq 0.$$

If C is a vector space or $\widehat{x} \in \operatorname{int}(C)$, then the condition reduces to

$$\nabla f(\widehat{x}) = 0.$$

Convex function: definitions

$f : \mathcal{H} \to \,]-\infty, +\infty]$ is a convex function if

$$\big(\forall (x,y) \in \mathcal{H}^2\big)(\forall \alpha \in]0,1[) \quad f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$$

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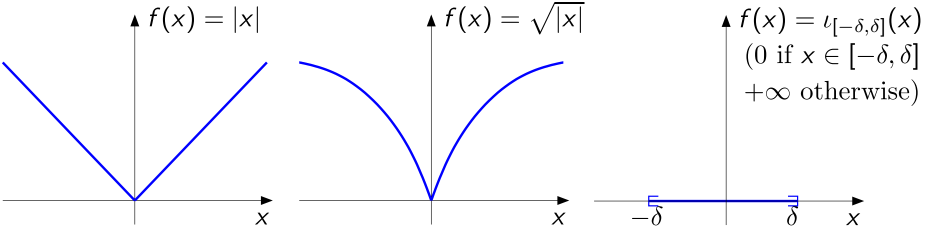

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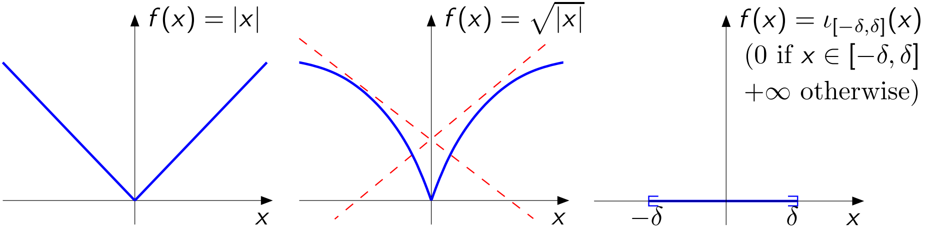

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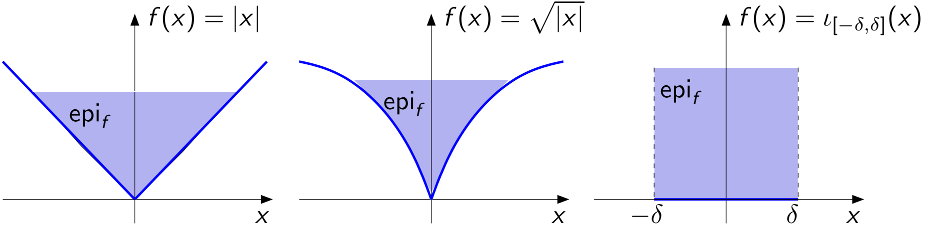

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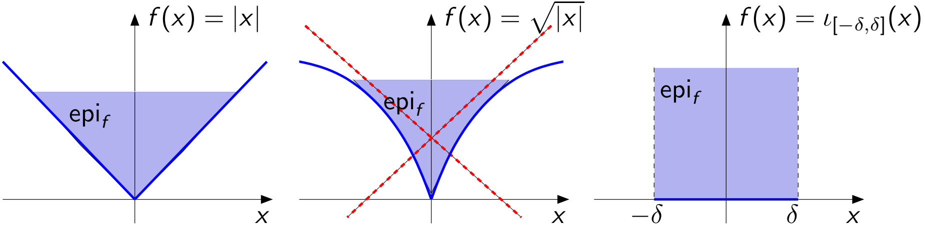

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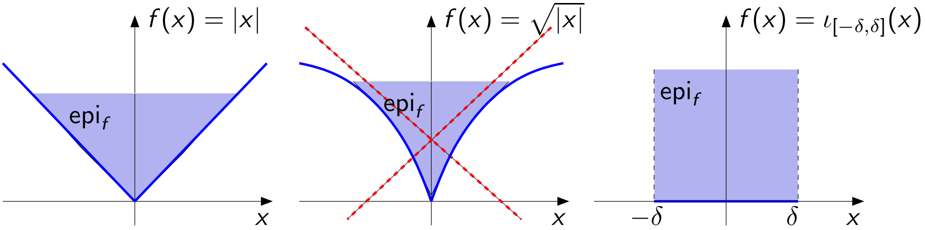

- If $f : \mathcal{H} \to \,]-\infty, +\infty]$ is convex, then $\operatorname{dom} f$ is convex.
- $f : \mathcal{H} \to [-\infty, +\infty[$ is concave if $-f$ is convex.

Convex functions: properties

- Every finite sum of convex functions is convex.
- Let $(f_i)_{i\in I}$ be a family of convex functions. $\sup_{i\in I} f_i$ is convex.
- $\Gamma_0(\mathcal{H})$: class of convex, l.s.c., and proper functions from $\mathcal{H}$ to $]-\infty,+\infty]$.
- $\iota_C \in \Gamma_0(\mathcal{H}) \Leftrightarrow C$ is a nonempty closed convex set.
  Proof: $\mathrm{epi}_{\iota_C} = C \times [0,+\infty[$.

Strictly convex functions

Let $\mathcal{H}$ be a Hilbert space. Let $f\colon \mathcal{H} \to \,]-\infty,+\infty]$.
f is strictly convex if

$$(\forall x \in \operatorname{dom} f)(\forall y \in \operatorname{dom} f)(\forall \alpha \in]0,1[)$$
$$x \neq y \quad \Rightarrow \quad f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha) f(y).$$

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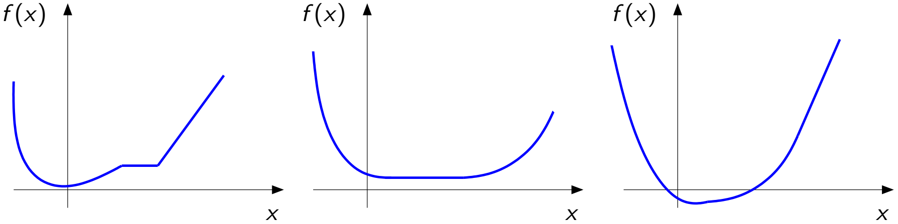
Strictly convex functions ?



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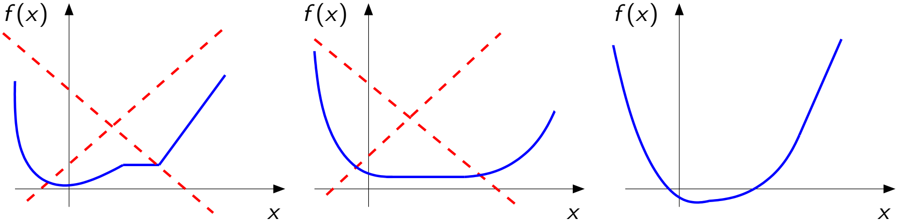
Strictly convex functions ?



Characterization of twice differentiable convex functions

Let $\mathcal{H}$ be a Hilbert space.
Let $f\colon \mathcal{H} \to]-\infty, +\infty]$ be a twice (Fréchet) differentiable function on its domain. Assume that $\operatorname{dom} f$ is a convex set.

- f is convex if and only if, for every $x \in \operatorname{dom} f$,

$$(\forall z \in \mathcal{H}) \qquad \langle z \mid \nabla^2 f(x) z \rangle \geq 0.$$

- If, for every $x \in \operatorname{dom} f$,

$$(\forall z \in \mathcal{H} \setminus \{0\}) \qquad \langle z \mid \nabla^2 f(x) z \rangle > 0,$$

  then f is strictly convex.

Minimizers of a convex function

Theorem

Let $\mathcal{H}$ be a Hilbert space. Let $f\colon \mathcal{H} \to \,]-\infty, +\infty]$ be a proper convex function such that $\mu = \inf f > -\infty$.

- $\big\{x \in \mathcal{H} \;\big|\; f(x) = \mu\big\}$ is convex.
- Every local minimizer of f is a global minimizer.
- If f is strictly convex, then there exists at most one minimizer.

Existence and uniqueness of a minimizer

Theorem

Let $\mathcal{H}$ be a Hilbert space and C a closed convex subset of $\mathcal{H}$. Let $f \in \Gamma_0(\mathcal{H})$ such that $\operatorname{dom} f \cap C \neq \varnothing$.

If f is coercive or C is bounded, then there exists $\widehat{x} \in C$ such that

$$f(\widehat{x}) = \inf_{x \in C} f(x).$$

If, moreover, f is strictly convex, this minimizer $\widehat{x}$ is unique.

Exercise 1

Provide an example of a function $f : \mathbb{R} \to \mathbb{R}$ and a nonempty set $C \subset \mathbb{R}$ such that

- f is nonconvex
- C is convex
- $f + \iota_C$ is convex.

Exercise 2

1. Let $f\colon \mathcal{H} \to \left]-\infty, +\infty\right]$ be a convex function.
   Prove that for every $\zeta \in \mathbb{R}$, the lower level set

$$\operatorname{lev}_{\leq \zeta} f = \big\{ x \in \mathcal{H} \;\big|\; f(x) \leq \zeta \big\}$$

   is convex.

2. Show that the converse is false by providing an example of a nonconvex function the lower level sets of which are all convex.

Exercise 3

Let $A \in \mathbb{R}^{M\times N}$ and $z \in \mathbb{R}^M$. Let $f : \mathbb{R}^N \to \mathbb{R} : x \mapsto \|Ax - z\|$ and let $g : \mathbb{R}^N \to \mathbb{R} : x \mapsto \|Ax - z\|^2$.

1. Prove that f and g are convex.
2. Give a necessary and sufficient condition on A for g to be strictly convex.
3. Can f be strictly convex ?
4. Find the minimizers of g.
5. What are the minimizers of f ?

Exercise 4

Let $y \in \mathbb{R}$. Show that

$$f : \mathbb{R} \to \mathbb{R} : x \mapsto \log(1 + \exp(-yx))$$

is convex. When is it strictly convex ?

Exercise 5

Let $\mathcal{H}$ be a Hilbert space and let $f\colon \mathcal{H} \to\ ]-\infty, +\infty]$ be a convex function. Let g be the perspective function of f defined as

$$(\forall (x,t) \in \mathcal{H} \times \mathbb{R}) \qquad g(x,t) = \begin{cases} t\, f(x/t) & \text{if } t > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

1. How is the epigraph of g related to the epigraph of f ?
2. Deduce that g is a convex function.
3. As a consequence of this result, show that the Kullback-Leibler divergence defined as

$$\big(\forall x = (x^{(i)})_{1 \le i \le N} \in \mathbb{R}^N\big)\big(\forall y = (y^{(i)})_{1 \le i \le N} \in \mathbb{R}^N\big)$$
$$h(x,y) = \begin{cases} \sum_{i=1}^N x^{(i)} \ln(x^{(i)}/y^{(i)}) & \text{if } (x,y) \in (\,]0,+\infty[^N)^2 \\ +\infty & \text{otherwise,} \end{cases}$$

is convex.