TD-TP: Proximity operator

1. Recall the definition of the space $\Gamma_0(\mathcal{H})$.
2. Recall the definition of the proximity operator.
3. Let $z \in \mathbb{R}^N$ and $f \in \Gamma_0(\mathcal{H})$. Let $g = f(\cdot - z)$, prove that
$$(\forall x \in \mathbb{R}^N) \quad \text{prox}_g(x) = z + \text{prox}_f(x - z).$$
4. Let $z \in \mathbb{R}^N$, $\alpha > 0$ and $f \in \Gamma_0(\mathcal{H})$. Let $g(x) = f(x) + \frac{\alpha}{2}\|x\|_2^2 + z^\top x$
$$(\forall x \in \mathbb{R}^N) \quad \text{prox}_g(x) = \text{prox}_{\frac{f}{\alpha+1}}(\frac{x - z}{\alpha + 1})$$
5. Derive the closed form expression of $\text{prox}_{\lambda|\cdot|}$.
6. Deduce from 5. the closed form expression of $\text{prox}_{\lambda\|\cdot\|_1}$. Why this operation is often encountered under the name soft-thresholding ?
7. Implement in Matlab the operation $\text{prox}_{\lambda\|\cdot\|_1}$.
8. If $L \in \mathbb{R}^{N\times N}$ such that $LL^* = \text{Id}$, recall the closed form expression for $\text{prox}_{\lambda\|L\cdot\|_1}$.
9. Let L denotes a wavelet transform. The matlab code to compute L and L^* is provided ('amr2D.m' and 'iamr2D.m').
   (a) In Matlab, download the original data from the file 'barbara256.mat' and the noisy data from the file 'barbara256_noisy.mat', denoted x and z respectively.
   (b) Compute the multiresolution analysis over 2 levels (rm = 2) with the wavelet 'sym3'. Add the command 'dwtmode('per') ;'. Display the wavelet coefficients.
   (c) Compute $\text{prox}_{\lambda\|L\cdot\|_1}$ with $\lambda = 10$ for all the details coefficients and $\lambda = 10^{-5}$ for the approximation coefficients.
   (d) Comment your results.