TP1: Smooth optimization

Let $\overline{x}$ be the vectorized representation of an image available in the file 'cameraman.mat' and let $z = H\overline{x} + \epsilon$ be the vectorized representation of an image degraded with a blur operator and an additive white Gaussian noise. The associated file is 'cameraman_degraded.mat'. Both images are displayed in Figure 1.

Original Degraded

FIGURE 1 – Image to be reconstructed before and after degradation.

1. Display with MATLAB the original and degraded images. Specify the size of both images.

2. The PSF in the frequency domain can be downloaded from 'blur.mat'. Display the histogram of the noise. Compute the mean and the variance.

3. We aim to obtain a restored image $\widehat{x}$, from the degraded image z. For real application, $\overline{x}$ is not available but for the purpose of this exercise we provide it in order to evaluate restoration error. First, we are going to implement Tikhonov-based reconstruction :

$$\widehat{x} \in \underset{x}{\operatorname{Argmin}} \underbrace{\|Hx - z\|_2^2 + \lambda \|\Gamma x\|_2^2}_{f(x)} \tag{1}$$

with $\lambda > 0$ and Γ models Laplacien operator whose associated PSF in the frequency domain can be obtained in 'laplacien.mat'.

   (a) Discuss the impact of λ.

   (b) Prove that
$$\widehat{x} = (H^*H + \lambda \Gamma^*\Gamma)^{-1} H^* z. \tag{2}$$

   (c) Implement the closed form estimation of $\widehat{x}$ using Fourier transform in order to manage with the inversion efficiently.

(d) Compute the mean square error between $\widehat{x}$ and $\underline{x}$ for different values of λ. What do you observe ?

(e) Implement the estimation of $\widehat{x}$ using a gradient descent in order to minimize f. At each iteration k, display the value $f(x^{[k]})$.

4. Now we assume that $z = DH\overline{x} + \epsilon$ where D models a decimation operator. The decimation is encoded in the file 'decimation.mat'. Download the new observation from the file 'cameraman_cs.mat'.

   (a) What is the size of this new observation ? What is the size of the non-zero value of D. What can you deduce ?

   (b) Display the decimated image by replacing the missing pixels with NaN.

   (c) The decimation can be downloaded from 'decimation.mat'. We aim to solve

$$\widehat{x} \in \underset{x}{\operatorname{Argmin}} \underbrace{\|DHx - z\|_2^2 + \lambda\|\Gamma x\|_2^2}_{g(x)}$$

   Is it still possible to handle with the inversion in the Fourier domain ? Implement the gradient descent. At each iteration k, display the value $f(x^{[k]})$. Display the evolution of $g(x^{[k]})$ w.r.t k

   (d) Implement Armijo rule to select the descent step-size. Plot $g(x^{[k]})$ w.r.t k for different choice of σ, β and s (cf. course).