Mathematics of Computational Imaging Systems

Forward model

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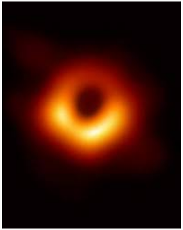
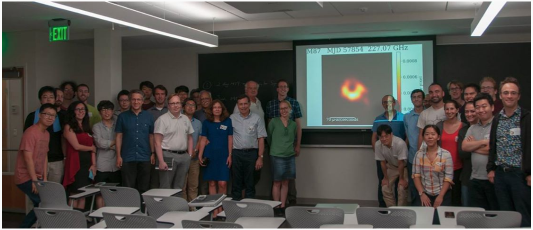
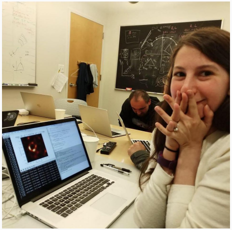
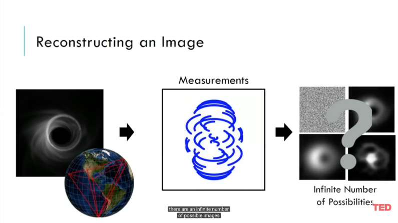
Computational Imaging Systems: serving other sciences



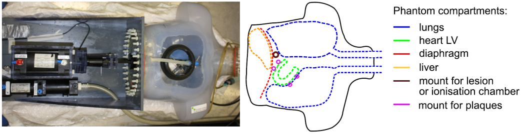


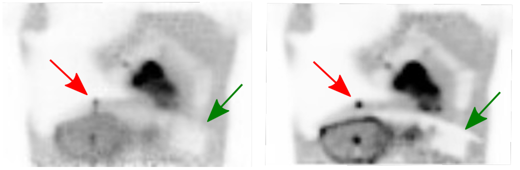




Black hole, Galaxy M87, Event Horizon Telescope (EHT)

Image analysis: serving other sciences





(source : F. Jolivet)

H2020 Nexis Project

Direct model

➡ Variables of interest

- $\mathrm{z} \in \mathbb{R}^M$: data/measurements.
- $\overline{\mathrm{x}} \in \mathbb{R}^N$: unknown image.
- $\widehat{\mathrm{x}} \in \mathbb{R}^N$: estimated image.

➡ Forward model:

$$\mathrm{z} = \mathcal{D}(\mathrm{A}\overline{\mathrm{x}})$$

Stochastic degradation ($\mathcal{D}$) — **Linear** operator (A)

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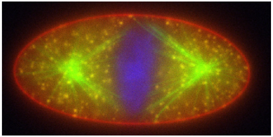
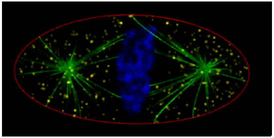


Degraded image z

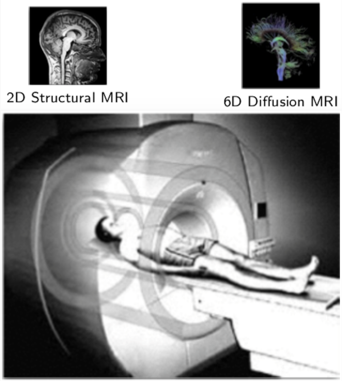
$\Leftarrow$



Original image $\overline{\mathrm{x}}$

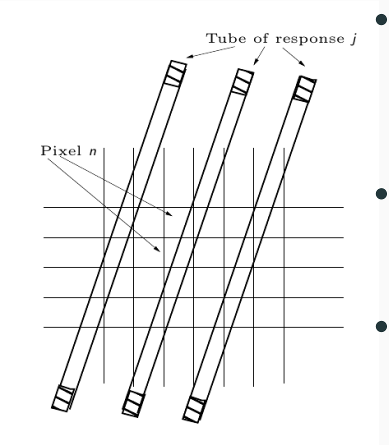
Direct model in medecine: MRI

$$z \simeq A\overline{x}$$



- $\overline{x} \in \mathbb{R}^N$: vectorized original (unknown) image.
- $A = \mathbf{MF}\colon \mathbb{R}^N \to \mathbb{C}^M$: measurement operator selecting (mask $\mathbf{M}\colon \mathbb{C}^N \to \mathbb{C}^M$) Fourier coefficients (2D Fourier transform $\mathbf{F}\colon \mathbb{R}^N \to \mathbb{C}^N$).
- z: vector containing the observed values (undersampled Fourier coefficients).

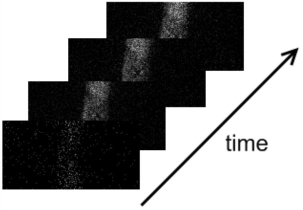
Direct model in medecine: Tomography



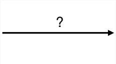
$$z \simeq A\overline{x}$$

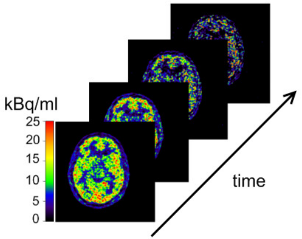
- $\overline{x} = (\overline{x}_n)_{1 \leq n \leq N} \in \mathbb{R}^N$: vector consisting of the (unknown) values of the original image of size $N = N_1 \times N_2$.
- $A = (A_{m,n})_{1 \leq m \leq M, 1 \leq n \leq N}$: probability to detect an event in the tube/line of response.
- $z = (z_m)_{1 \leq m \leq M} \in \mathbb{R}^M$: vector containing the observed values (sinogram).

Application examples in medecine: Tomography



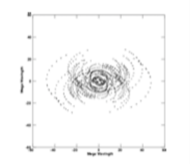
Degraded images





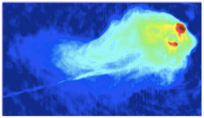
Reconstructed images

Direct model in Astronomy: Radio-interferometry



Fourier sampling

Very Large Array, New Mexico

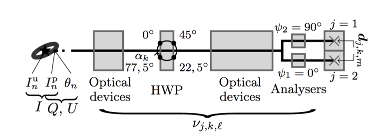


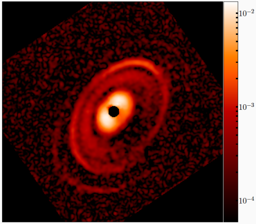
Cygnus A

$$\mathrm{z} \simeq \mathrm{A}\overline{\mathrm{x}}$$

- $\overline{\mathrm{x}} \in \mathbb{R}^N$: vectorized original (unknown) 2D image.
- $\mathrm{A} = \mathbf{GF}\colon \mathbb{R}^N \to \mathbb{C}^M$: measurement operator selecting Fourier coefficients.
  - $\mathbf{F}\colon \mathbb{R}^N \to \mathbb{C}^{\overline{N}}$: 2D Fourier transform (with zero-padding),
  - $\mathbf{G} \in \mathbb{C}^{M \times \overline{N}}$: (de)-gridding matrix modelling non-uniform (undersampled) Fourier transform, and direction (in)dependent effect (calibration artefacts).
- z: vector containing the observed values (undersampled Fourier coefficients)

Application examples in astronomy: High-contrast imagery





RXJ 1615 (Avenhaus et al. 2018)

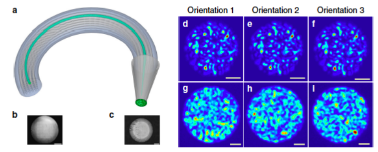
$$\mathrm{z}_{j,k} \simeq \underbrace{\sum_{m=1}^{3} \nu_{j,k,m}\, \mathrm{T}_{j,k}\mathrm{H}\, \mathrm{S}_m}_{\mathrm{A}_{j,k}}$$

- $\mathrm{H}\colon \mathbb{R}^N \to \mathbb{R}^N$: invariant blur.
- $\mathrm{T}_{j,k}\colon \mathbb{R}^N \to \mathbb{R}^{M_j}$: geometric transform of the j-th polariser during the k-th acquisition.
- Stokes versus Jones formalisms:

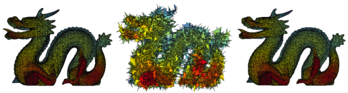
$$\mathrm{I}_{j,k}^{\mathrm{det}} = \frac{1}{2}\mathrm{I_u} + \mathrm{I_p}\cos^2(\theta - 2\alpha_k - \psi_j)$$

$$\Leftrightarrow \quad \mathrm{I}_{j,k}^{\mathrm{det}} = \nu_{j,k,1}\,\mathsf{I} + \nu_{j,k,2}\,\mathsf{Q} + \nu_{j,k,3}\,\mathsf{U}$$

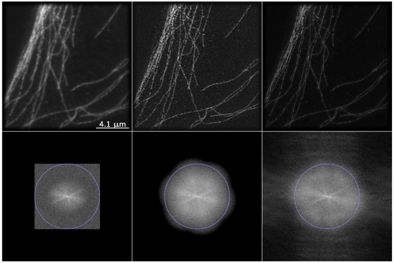
Many others!



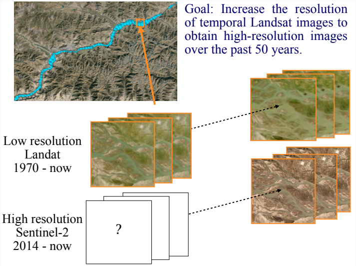
Photon imaging



3D Mesh denoising



Structured Illumination Microscopy



Remote sensing