Mathematics of Computational Imaging Systems:

Multilevel

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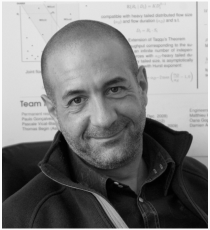
Joint work with:



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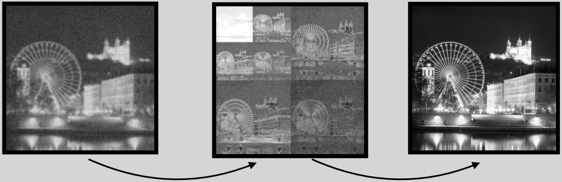
*Inria OCKHAM team, LIP, ENSL

Multiresolution/multilevel

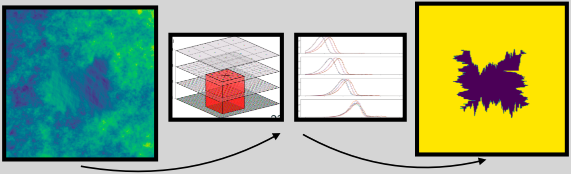
Multiresolution
to perform
image restoration

(~2000–2015)



Multiresolution
to perform
texture
segmentation

(~2014- now)



Multiresolution/multilevel

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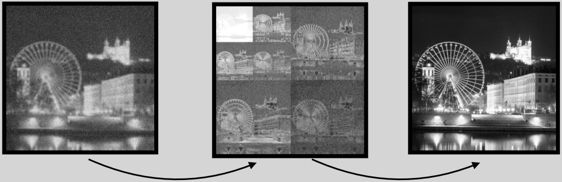

Multiresolution
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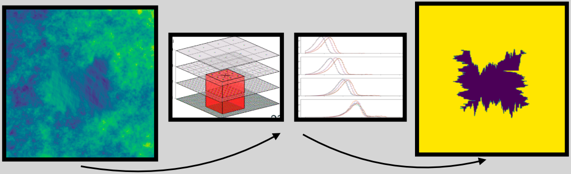
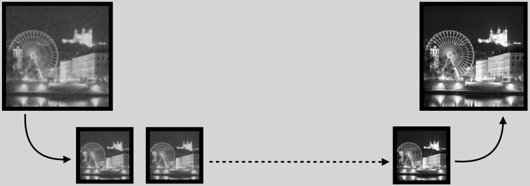
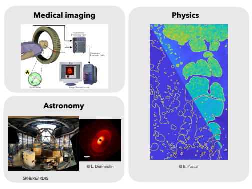


Multiresolution
to accelerate
algorithms

(~2016- now)



Inverse problems: variables and key equations



☛ **Variables**

- $\mathrm{z} \in \mathbb{R}^M$: data.
- $\overline{\mathrm{x}} \in \mathbb{R}^N$: unknown parameters.
- $\widehat{\mathrm{x}} \in \mathbb{R}^N$: estimated parameters.

☛ **Forward model**

$$\mathrm{z} = \mathcal{D}(\mathrm{A}\overline{\mathrm{x}})$$

Stochastic degradation **Linear** operator

☛ **Inverse problem**

$$\widehat{\mathrm{x}} = d_\Theta(\mathrm{z})$$

☛ **Goal**: Estimate $\widehat{\mathrm{x}}$ close to $\overline{\mathrm{x}}$ from z, A, noise statistic $\mathcal{D}$, and prior information on the class of image to recover.

Inversion $\widehat{\mathrm{x}} = d_\Theta(\mathrm{z})$

➡ [1922] **Maximum likelihood** (Fisher).

$$\widehat{\mathrm{x}} \in \underset{\mathrm{x}}{\mathrm{Argmin}}\ \frac{1}{2}\|\mathrm{Ax} - \mathrm{z}\|_2^2 = (\mathrm{A}^*\mathrm{A})^{-1}\mathrm{A}^*\mathrm{z}$$

➡ [1963] **Regularization** (Tikhonov, Huber)

$$\widehat{\mathrm{x}} \in \underset{\mathrm{x}}{\mathrm{Argmin}}\ \frac{1}{2}\|\mathrm{Ax} - \mathrm{z}\|_2^2 + \theta\|\mathrm{Lx}\|_2^2 \qquad \text{avec} \quad \theta > 0$$

➡ [2000] **Sparsity** (Donoho, Daubechies-Defrise-DeMol,...)

$$\widehat{\mathrm{x}} \in \underset{\mathrm{x}}{\mathrm{Argmin}}\ \frac{1}{2}\|\mathrm{Ax} - \mathrm{z}\|_2^2 + \theta\|\mathrm{Lx}\|_\star$$

➡ [2010] **"End to end" neural networks**

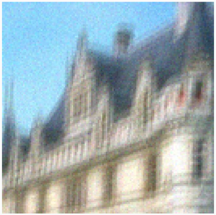
$$\widehat{\mathrm{x}} = \mathrm{NN}_\Theta(\mathrm{z})$$

➡ [2020] **Plug-and-Play**

$$0 \in \mathrm{A}^*(\mathrm{A}\widehat{\mathrm{x}} - \mathrm{z}) + \mathbf{B}(\widehat{\mathrm{x}})$$

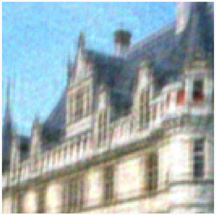
Summary of inverse problems in imaging

Original



Degraded

SNR = 13.4 dB

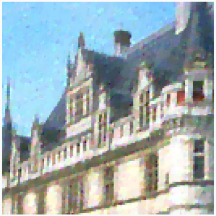


Tikhonov

SNR = 16.4 dB

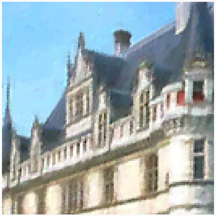
DTT

SNR = 16.6 dB



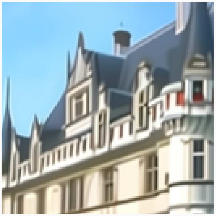
TV

SNR = 18.8 dB



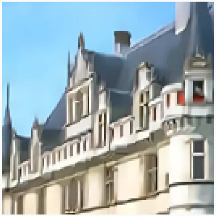
NLTV

SNR = 19.4 dB



PnP-DRUnet

SNR = 20.0 dB



PnP-ScCP

SNR = 20.2 dB

Focus in this presentation

➡ [1922] **Maximum likelihood** (Fisher).

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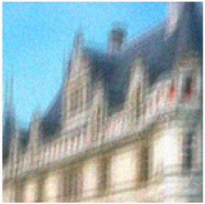
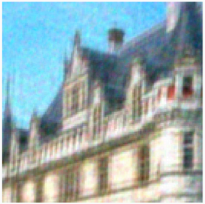
Original



Degraded
SNR = 13.4 dB



Tikhonov
SNR = 16.4 dB

DTT
SNR = 16.6 dB

TV
SNR = 18.8 dB

NLTV
SNR = 19.4 dB

PnP-DRUnet
SNR = 20.0 dB

PnP-ScCP
SNR = 20.2 dB[4]

Iterative scheme

➡ **Minimization problem** :

$$\widehat{\mathrm{x}} \in \underset{\mathrm{x}}{\mathrm{Argmin}}\; f(\mathrm{x}) + g(\mathrm{x})$$

with f and g either diff. with Lipschitz gradient or proximable.

➡ **Design of a recursive sequence of the form**

$$(\forall k \in \mathbb{N}) \qquad \mathrm{x}^{[k+1]} = \mathbf{T}(\mathrm{x}^{[k]}),$$

Gradient descent	$\mathbf{T} = \mathrm{Id} - \tau(\nabla f + \nabla g)$
Proximal point algorithm	$\mathbf{T} = \mathrm{prox}_{\tau(f+g)}$
Forward-Backward	$\mathbf{T} = \mathrm{prox}_{\tau g}(\mathrm{Id} - \tau \nabla f)$
Peaceman-Rachford	$\mathbf{T} = (2\,\mathrm{prox}_{\tau g} - \mathrm{Id}) \circ (2\,\mathrm{prox}_{\tau f} - \mathrm{Id})$
Douglas-Rachford	$\mathbf{T} = \mathrm{prox}_{\tau g}(2\,\mathrm{prox}_{\tau f} - \mathrm{Id}) + \mathrm{Id} - \mathrm{prox}_{\tau f}$

Iterative scheme

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$$\boxed{\widehat{\mathrm{x}} \in \underset{\mathrm{x}}{\mathrm{Argmin}}\ f(\mathrm{x}) + g(\mathrm{x})}$$

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Douglas-Rachford	$\mathbf{T} = \mathrm{prox}_{\tau g}(2\,\mathrm{prox}_{\tau f} - \mathrm{Id}) + \mathrm{Id} - \mathrm{prox}_{\tau f}$

Main goal : provide acceleration for high dimensional problems

High dimensional problems $\rightarrow$ high computation time.

Alternatives :

- **FISTA** [Beck & Teboulle, 2009] [Chambolle & Dossal, 2015],
- **Preconditionning** [Donatelli, 2019][Repetti et al., 2014],
- **Blocks methods** [Liu, 1996] [Chouzenoux et al., 2016] [Salzo, Villa 2022],
- **multiresolution strategy**

  ☛ Idea that comes from the PDE field [Nash, 2000].

  ☛ preliminary results for non-smooth optimization in [Parpas, 2017].

Common aim of these methods:

☛ improve the gradient/proximal gradient steps with well chosen rules.

Multilevel algorithm for smooth optimization

Some references:

- A. Javaherian and S. Holman, **A Multi-Grid Iterative Method for Photoacoustic Tomography**, IEEE Transactions on Medical Imaging, (2017)
- S. W. Fung and Z. Wendy, **Multigrid Optimization for Large-Scale Ptychographic Phase Retrieval**, SIAM Journal on Imaging Sciences, 13 (2020)
- J. Plier, F. Savarino, M. Kočvara, and S. Petra, **First-Order Geometric Multilevel Optimization for Discrete Tomography**, in Scale Space and Variational Methods in Computer Vision, A. Elmoataz, J. Fadili, Y. Quéau, J. Rabin, and L. Simon, eds., vol. 12679, Springer International Publishing, Cham, (2021)

→ Successful attempts of accelerating minimization in imaging.

→ Restricted to smooth optimization.

Multilevel algorithms

First order descent methods

Goal:

$$\min_{\mathrm{x}\in\mathbb{R}^N} F(\mathrm{x}) := f(\mathrm{x}) + g(\mathrm{x})$$

f and g proper, lower semi-continuous, and convex.

f is assumed differentiable with Lipschitz gradient.

g is not necessarily differentiable.

Build a sequence: $\mathrm{x}^{[k+1]} = \Phi(\mathrm{x}_k) = \mathrm{x}_k - D_k$

- If f and g are differentiable: Gradient descent

$$D_k = \tau_k(\nabla f(\mathrm{x}_k) + \nabla g(\mathrm{x}_k))$$

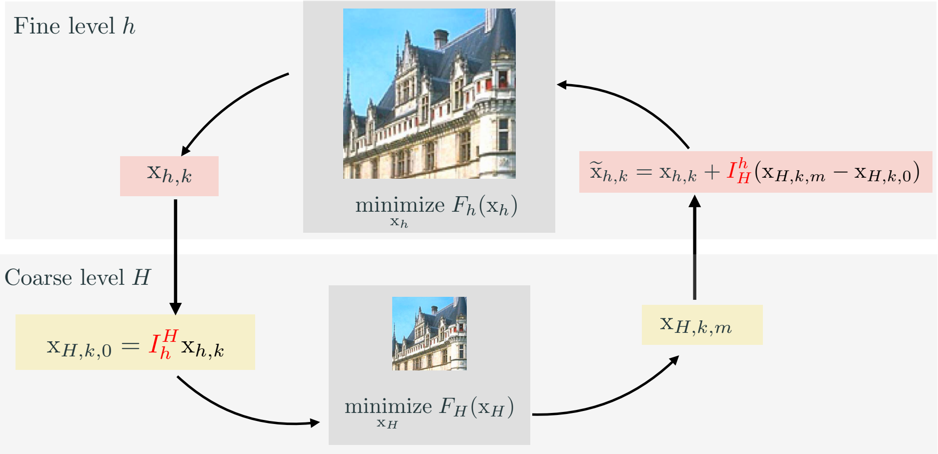
- If g is not differentiable: Proximal gradient descent

$$D_k = \mathrm{x}_k - \mathrm{prox}_{\tau_k g}\left(\mathrm{x}_k - \tau_k \nabla f(\mathrm{x}_k)\right)$$

Multilevel smooth optimization

Goal: Exploit hierarchy of approximations of the objective function.

Example: Two levels case with fine (h) and coarse (H) levels.



Design of I_H^h and I_h^H: Information transfer operators

Definition $I_h^H : \mathbb{R}^{N_h} \to \mathbb{R}^{N_H}$ (transfer from fine to coarse scales) and $I_H^h : \mathbb{R}^{N_H} \to \mathbb{R}^{N_h}$ (transfer from coarse to fine scales) are *coherent information transfer (CIT)* operators, if there exists $\nu > 0$ such that:

$$I_H^h = \nu (I_h^H)^T.$$

- particular case of squared grids reads:

$$I_h^H = \frac{1}{16} \underbrace{\begin{pmatrix} 2 & 1 & 0 & \dots & & & 0 \\ 0 & 1 & 2 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & & & & 0 \\ 0 & \dots & & 0 & 1 & 2 & 1 \end{pmatrix}}_{\sqrt{N_h}/2 \times \sqrt{N_h}} \otimes \underbrace{\begin{pmatrix} 2 & 1 & 0 & \dots & & & 0 \\ 0 & 1 & 2 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & & & & 0 \\ 0 & \dots & & 0 & 1 & 2 & 1 \end{pmatrix}}_{\sqrt{N_h}/2 \times \sqrt{N_h}} \in \mathbb{R}^{N_H \times N_h}$$

Design of F_H: First order coherence

Smoothed convex function *[Beck 2012, Definition 2.1]*

Let g be a convex, l.s.c., and proper function on $\mathbb{R}^N$.

For every $\gamma > 0$, g_γ is a smoothed convex function if there exist scalars η_1, η_2 satisfying $\eta_1 + \eta_2 > 0$ such that the following holds:

$$(\forall y \in \mathbb{R}^N) \qquad g(y) - \eta_1 \gamma \leqslant g_\gamma(y) \leqslant g(y) + \eta_2 \gamma.$$

Design of F_H: First order coherence

First order coherence *[Nash, 2000][Parpas et al. 1016, 2017]*
The first order coherence between the smoothed version of the objective function F_h at the fine level and the coarse level objective function F_H is verified in a neighbourhood of $y_h \in \mathbb{R}^{N_h}$ if the following equality holds:

$$\nabla F_H(I_h^H y_h) = I_h^H \nabla \left(f_h + g_{h,\gamma_h}\right)(y_h).$$

☛ **Impact**: Coherence up to order one in the neighbourhood of the current iterates $y_h = y_{h,k}$.

Design of F_H: First order coherence

Coarse model F_H for non-smooth functions

The coarse model F_H is defined for the point $y_h \in \mathbb{R}^{N_h}$ as:

$$F_H = f_H + g_{H,\gamma_H} + \langle v_H, \cdot \rangle, \tag{1}$$

where

$$v_H = I_h^H \left(\nabla f_h(y_h) + \nabla g_{h,\gamma_h}(y_h) \right) - \left(\nabla f_H(I_h^H y_h) + \nabla g_{H,\gamma_H}(I_h^H y_h) \right).$$

Lemma If F_H is given by definition (1), it necessarily verifies the first order coherence.

Proof.
Considering the gradient of the coarse model F_H and combining it with the definition of v_H, yields

$$\begin{aligned} \nabla F_H(I_h^H y_h) &= \nabla f_H(I_h^H y_h) + \nabla g_{H,\gamma_H}(I_h^H y_h) + v_H, \\ &= I_h^H \left(\nabla f_h(y_h) + \nabla g_{h,\gamma_h}(y_h) \right). \end{aligned}$$

Design of F_H: First order coherence

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$$F_H = f_H + g_{H,\gamma_H} + \langle v_H, \cdot \rangle,$$

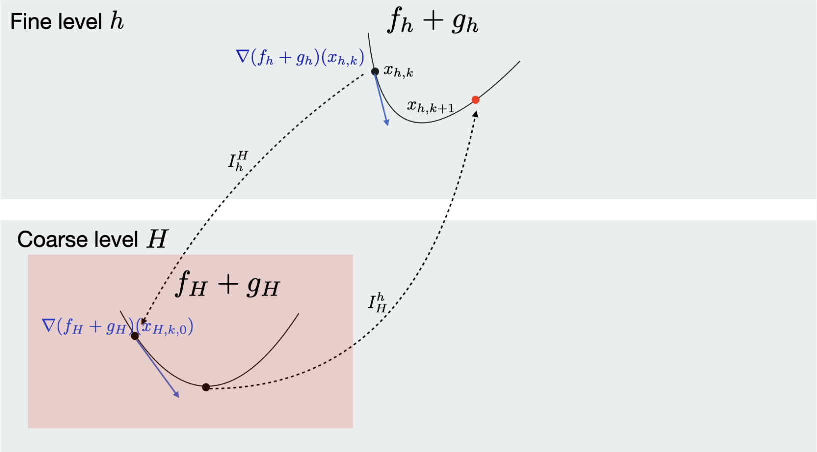
where

$$v_H = I_h^H \left(\nabla f_h(y_h) + \nabla g_{h,\gamma_h}(y_h) \right) - \left(\nabla f_H(I_h^H y_h) + \nabla g_{H,\gamma_H}(I_h^H y_h) \right).$$

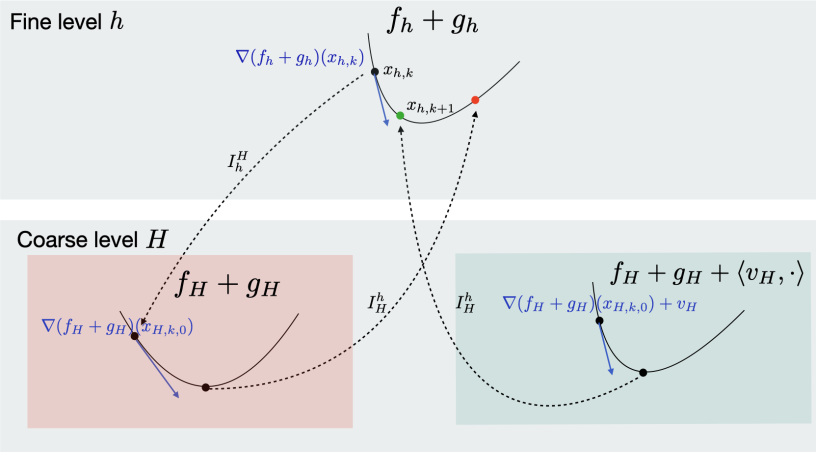
Remarks:

- ☛ Adding the linear term $\langle v_H, \cdot \rangle$ to $f_H + g_{H,\gamma_H}$ allows to impose the so-called *first order coherence*.
- ☛ if g_h and g_H are smooth by design, one can simply replace g_{H,γ_H} and g_{h,γ_h} by g_H and g_h.

Design of F_H: First order coherence



Design of F_H: First order coherence



IML FB: Multilevel algorithm for nonsmooth optimization

Set $x_{h,0}, y_{h,0} \in \mathbb{R}^N$, $t_{h,0} = 1$
while Stopping criterion is not met **do**
    if Descent condition **then**
        $s_{H,k,0} = I_h^H x_{h,k}$ Projection
        $s_{H,k,m} = \Phi_{H,m-1} \circ .. \circ \Phi_{H,0}(s_{H,k,0})$ Coarse minimization
        Set $\bar{\tau}_{h,k} > 0$,
        $\bar{x}_{h,k} = x_{h,k} + \bar{\tau}_{h,k} I_H^h \left(s_{H,k,m} - s_{H,k,0}\right)$ Coarse step update
    else
        $\bar{x}_{h,k} = x_{h,k}$
    end if
    $x_{h,k+1} = \Phi_{h,k}(\bar{x}_{h,k})$ Forward-Backward step
end while

Lemma *(Fine level decrease)*. Let assume that I_h^H and I_H^h are CIT operators and that F_H satisfies Definition (1) and $\Phi_{H,\bullet}$ allows a decrease of the coarse model. The iterations of IML FB ensure:

$$F_h(\mathrm{x}_h + \bar{\tau} I_H^h(s_{H,m} - s_{H,0})) \leqslant F_h(\mathrm{x}_h) + (\eta_1 + \eta_2)\gamma_h.$$

Proof.
This directly comes from the definition of a smoothed convex function:

$$\begin{aligned} & F_h(\mathrm{x}_h + \bar{\tau}_h I_H^h(s_{H,m} - s_{H,0})) \\ & \quad \leqslant (L_h + R_{h,\gamma_h})(y_h + \bar{\tau}_h I_H^h(s_{H,m} - s_{H,0})) + \eta_1\gamma_h \\ & \quad \leqslant (L_h + R_{h,\gamma_h})(\mathrm{x}_h) + \eta_1\gamma_h \\ & \quad \leqslant F_h(\mathrm{x}_h) + (\eta_1 + \eta_2)\gamma_h. \end{aligned}$$

$\square$

Lemma *(Fine level decrease)*. Let assume that I_h^H and I_H^h are CIT operators and that F_H satisfies Definition (1) and $\Phi_{H,\bullet}$ allows a decrease of the coarse model. The iterations of IML FB ensure:

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- Coarse level minimization step, leads to a decrease of F_h, up to a constant $(\eta_1 + \eta_2)\gamma_h$ that can be made arbitrarily small by driving γ_h to zero.
- Commonly found in the literature of multilevel algorithms.
- Not sufficient to guarantee the convergence of the generated sequence.

What has been done:

☛ Remarks on multilevel framework to non-smooth optimization:

+ Handles non-smooth g.
+ Smoothing to define the first order coherence.
− Requires explicit form of $\mathrm{prox}_g = \mathrm{prox}_{\varphi \circ \mathrm{L}}$.
− No convergence guarantee to a minimizer.

☛ Some references:

- V. Hovhannisyan, P. Parpas, and S. Zafeiriou, **MAGMA: Multilevel Accelerated Gradient Mirror Descent Algorithm for Large-Scale Convex Composite Minimization**, SIAM J. Imaging Sciences (2016)
- P. Parpas, **A Multilevel Proximal Gradient Algorithm for a Class of Composite Optimization Problems**, SIAM J. Scient. Comp., 39 (2017)
- G. Lauga, E. Riccietti, N. Pustelnik, and P. Goncalves Multilevel FISTA for Image Restoration, IEEE ICASSP, 2023.

IML FISTA

Motivations and contribution

☛ **Goal**:

- **inexact proximal** steps to handle state-of-the-art regularization: Total Variation (TV) and Non-Local Total Variation (NLTV).
- obtain **state-of-the-art convergence guarantees**.

☛ **Proposed scheme**: **IML FISTA** a *convergent multilevel inexact and inertial proximal gradient algorithm*:

- prox_g is explicit.
- $\mathrm{prox}_{\varphi \circ \mathrm{L}}$ is not known under closed form.

Inexact FISTA for solving $\min_{\mathbf{x}} f(\mathbf{x}) + \varphi(\mathrm{L}\mathbf{x})$

Inexact FISTA [Aujol, Dossal, 2015]:

$$x_{k+1} \approx_{\epsilon_k} \operatorname{prox}_{\tau\varphi\circ\mathrm{L}}\left(y_k - \tau\nabla f(y_k) + e_k\right)$$
$$y_{k+1} = x_{k+1} + \alpha_k(x_{k+1} - x_k)$$

where α_k is chosen with $t_{k+1} = \left(\frac{k+a}{a}\right)^d$, $\alpha_k = \frac{t_k - 1}{t_{k+1}}$.

Contribution: update y_k through a multilevel step.

- How to construct such multilevel update ?
- How to guarantee convergence ?

Smoothing of F_h and F_H with the Moreau envelope

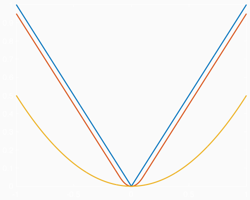
☛ **Moreau envelope of** g_H:

$$ {}^{\gamma}g_H = \inf_{y \in \mathcal{H}} g_H(y) + \frac{1}{2\gamma} \| \cdot - y \|^2 $$

☛ **Properties of the Moreau envelope**:

- $\nabla^{\gamma} g_H = \gamma^{-1}(\mathsf{Id} - \mathrm{prox}_{\gamma g_H})$
- $\nabla^{\gamma} g_H$ γ^{-1} - Lipschitz
- $\nabla \left({}^{\gamma}\varphi_H \circ \mathrm{L}_H \right)(\cdot) = \gamma_H^{-1} \mathrm{L}_H^{*} \left(\mathrm{L}_H \cdot - \mathrm{prox}_{\gamma_H \varphi_H}(\mathrm{L}_H \cdot) \right)$

☛ **Illustration**: Moreau envelope of l_1-norm for $\gamma = 0.1$ and $\gamma = 1$



First order coherence for g non-smooth

Coarse model F_H for non-smooth functions

$$F_H = f_H + ({}^{\gamma_H}\varphi_H \circ \mathrm{L}_H) + \langle v_H, \cdot \rangle$$

where

$$\begin{aligned} v_H = I_h^H & \left(\nabla f_h(y_h) + \nabla({}^{\gamma_h}\varphi_h \circ \mathrm{L}_h)(y_h)\right) \\ & - \left(\nabla f_H(I_h^H y_h) + \nabla({}^{\gamma_H}\varphi_H \circ \mathrm{L}_H)(I_h^H y_h)\right) \end{aligned}$$

Minimization scheme at coarse level:

$$\Phi_H := \nabla f_H + \nabla({}^{\gamma_H} g_H \circ \mathrm{L}_H)$$

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    $\bar{y}_{h,k} = y_{h,k} + \bar{\tau}_{h,k} I_H^h \left(s_{H,k,m} - s_{H,k,0}\right)$
   Coarse step update with size $\bar{\tau}_{h,k}$
  else
    $\bar{y}_{h,k} = y_{h,k}$
  end if
  $x_{h,k+1} = \Phi_i^{\epsilon_{h,k}}(\bar{y}_{h,k})$ Forward-backward step
  $t_{h,k+1} = \left(\frac{k+a}{a}\right)^d$, $\alpha_{h,k} = \frac{t_{h,k}-1}{t_{h,k+1}}$
  $y_{h,k+1} = x_{h,k+1} + \alpha_{h,k}(x_{h,k+1} - x_{h,k})$. Inertial step
end while

Convergence of IML FISTA

Multilevel steps interpreted as gradient errors

☛ FISTA steps allow errors on the computation of the backward and on the forward steps:

$$x_{h,k+1} \simeq_{\epsilon_{h,k}} \operatorname{prox}_{\tau_h \varphi_h \circ \mathrm{L}_h} \left(y_{h,k} - \tau_h \nabla f_h \left(y_{h,k} \right) + e_{h,k} \right)$$
$$y_{h,k+1} = x_{h,k+1} + \alpha_{h,k}(x_{h,k+1} - x_{h,k})$$

☛ Rewriting coarse corrections:

$$e_{h,k} = \tau_h \Big(\nabla f_h(y_{h,k}) - \nabla f_h(\bar{y}_{h,k}) + \frac{\bar{\tau}_{h,k}}{\tau_h} I_H^h (s_{H,k,m} - s_{H,k,0}) \Big)$$

☛ Multilevel steps = bounded errors on the gradient

Convergence analysis

Lemma *(Coarse corrections are finite)*
Let β_h and β_H be the Lipschitz constants of f_h and f_H, respectively. Assume that we compute at most p coarse corrections.
Let $\tau_h, \tau_H \in (0, +\infty)$ be the step sizes taken at fine and coarse levels, respectively.
Assume that $\tau_H < \beta_H^{-1}$ and that $\tau_h < \beta_h^{-1}$ and denote $\bar{\tau}_h = \sup_k \bar{\tau}_{h,k}$.
Then the sequence $(e_{h,k})_{k\in\mathbb{N}}$ in $\mathbb{R}^{N_h}$ generated by IML FISTA is defined as:

$$e_{h,k} = \tau_h \left(\nabla f_h(y_{h,k}) - \nabla f_h(\bar{y}_{h,k}) + (\tau_h)^{-1} \bar{\tau}_{h,k} I_H^h (s_{H,k,m} - s_{H,k,0}) \right),$$

if a coarse correction has been computed, and $e_{h,k} = 0$ otherwise. This sequence is such that $\sum_{k\in\mathbb{N}} k\|e_{h,k}\| < +\infty$.

Inexact proximal step

The ϵ-subdifferential of g at $z \in \text{dom } g$ is defined as:

$$\partial_\epsilon g(z) = \{y \in \mathbb{R}^N \mid g(x) \geqslant g(z) + \langle x - z, y \rangle - \epsilon, \forall x \in \mathbb{R}^N\}.$$

Type 0 approximation [Combettes, Wajs, 2005]

$z \in \mathbb{R}^N$ is a type 0 approximation of $\text{prox}_{\gamma g}(y)$ with precision ϵ, and we write $z \simeq_0 \text{prox}_{\gamma g}(y)$, if and only if $\|z - \text{prox}_{\gamma g}(y)\| \leqslant \sqrt{2\gamma\epsilon}$.

Type 1 approximation [Villa et al., 2013]

$z \in \mathbb{R}^N$ is a type 1 approximation of $\text{prox}_{\gamma g}(y)$ ith precision ϵ, and we write $z \simeq_1 \text{prox}_{\gamma g}(y)$, if and only if $0 \in \partial_\epsilon \left(g(z) + \frac{1}{2\gamma}\|z - y\|^2 \right)$.

Type 2 approximation [Villa et al., 2013]

$z \in \mathbb{R}^N$ is a type 2 approximation of $\text{prox}_{\gamma g}(y)$ with precision ϵ, and we write $z \simeq_2 \text{prox}_{\gamma g}(y)$, if and only if $\frac{y-z}{\gamma} \in \partial_\epsilon g(z)$.

Inexact proximity operator step

- At each iteration of fine level minimization we need to compute

$$\mathrm{prox}_{\gamma\varphi_h \circ \mathrm{L}_h}(x) = x - \mathrm{L}_h^* \widehat{u}$$

with:

$$\widehat{u} \in \underset{u \in \mathbb{R}^K}{\operatorname{argmin}} \frac{1}{2}\|\mathrm{L}_h^* u - x\|^2 + \gamma\varphi_h^*(u)$$

which can be solved iteratively with accuracy ϵ so that:

$$x - \mathrm{L}_h^* \widehat{u}_\epsilon \simeq_\epsilon \mathrm{prox}_{\gamma\varphi_h \circ \mathrm{L}_h}(x)$$

- Equivalent to:

$$\frac{\mathrm{L}_h^* \widehat{u}_\epsilon}{\gamma} \in \partial_\epsilon \left(\varphi_h \circ \mathrm{L}_h\right) \left(x - \mathrm{L}_h^* \widehat{u}_\epsilon\right)$$

Inexact proximity operator step

☛ At each iteration of fine level minimization we need to compute

$$\operatorname{prox}_{\gamma\varphi_h \circ \mathrm{L}_h}(x) = x - \mathrm{L}_h^* \widehat{u}$$

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which can be solved iteratively with accuracy ϵ so that:

$$x - \mathrm{L}_h^* \widehat{u}_\epsilon \simeq_\epsilon \operatorname{prox}_{\gamma\varphi_h \circ \mathrm{L}_h}(x)$$

☛ Equivalent to:

$$\frac{\mathrm{L}_h^* \widehat{u}_\epsilon}{\gamma} \in \partial_\epsilon \left(\varphi_h \circ \mathrm{L}_h\right)\left(x - \mathrm{L}_h^* \widehat{u}_\epsilon\right)$$

$\Rightarrow$ **Type 2 approximation**

Convergence analysis

Theorem

Considering $\forall k \in \mathbb{N}^*,\ \alpha_{h,k} = 0$ and the sequence $(\epsilon_{h,k})_{k\in\mathbb{N}}$ is such that $\sum_{k\in\mathbb{N}} \sqrt{\|\epsilon_{h,k}\|} < +\infty$. Set $x_{h,0} \in \mathbb{R}^{N_h}$ and choosing approximation of Type 0, the sequence $(x_{h,k})_{k\in\mathbb{N}}$ generated by IML FISTA converges to a minimizer of F_h.

Convergence analysis

Theorem

Let $\forall k \in \mathbb{N}^*,\ t_{h,k+1} = \left(\frac{k+a}{a}\right)^d$, with (a, d) satisfying the conditions in [Aujol, Dossal, 2015 – Definition 3.1], and that the assumptions of Lemma 29 hold. Moreover, if we assume that:

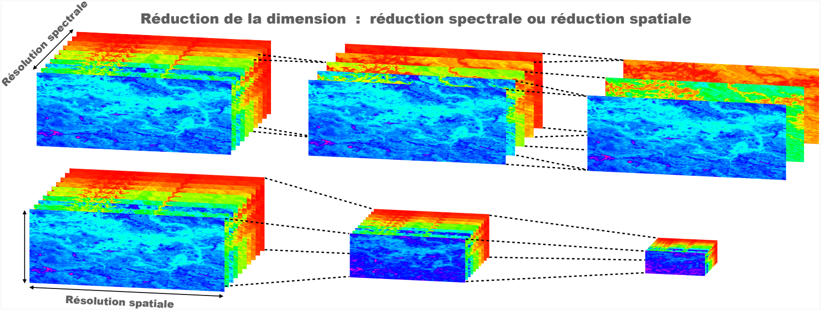
- $\sum_{k=1}^{+\infty} k^d \sqrt{\epsilon_{h,k}} < +\infty$ in the case of Type 1 approximation,
- $\sum_{k=1}^{+\infty} k^{2d} \epsilon_{h,k} < +\infty$ in the case of Type 2 approximation.

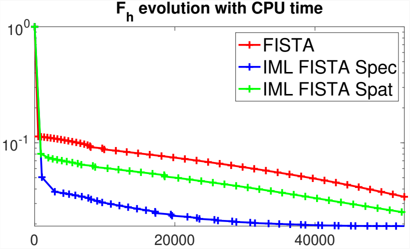
Let $(x_{h,k})_{k\in\mathbb{N}}$ the sequence generated by IML FISTA, then

- The sequence $\left(k^{2d}\left(F_h(x_{h,k}) - F_h(x^*)\right)\right)_{k\in\mathbb{N}}$ belongs to $\ell_\infty(\mathbb{N})$.
- The sequence $(x_{h,k})_{k\in\mathbb{N}}$ converges to a minimizer of F_h.

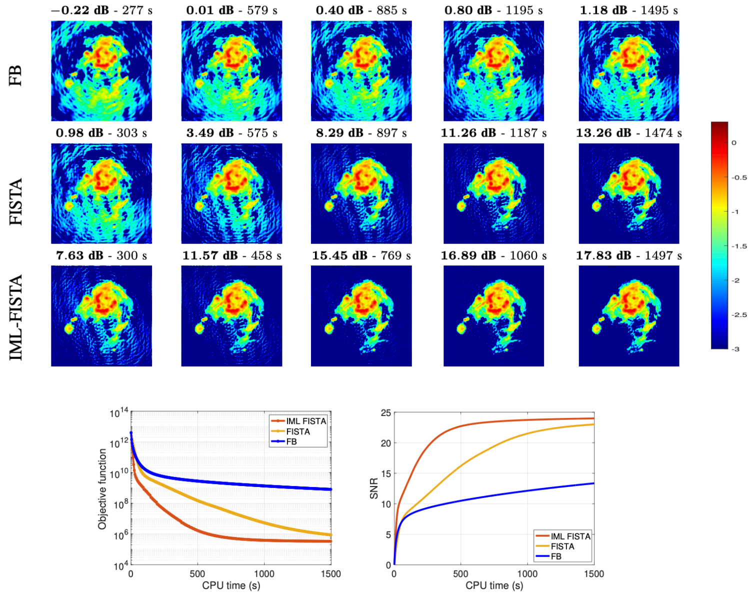
Numerical experiments

Numerical experiments on hyperspectral images





Numerical experiments in radio-interferometric imaging



Partial conclusions

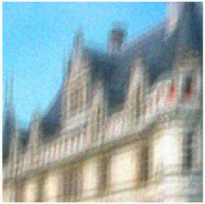
- Unifying and extended convergence guarantees for IML FB.
- Convergent IML FISTA.
- IML FISTA much faster than FISTA for large scale problems.

Future works:

- Deeper analysis of the design of I_h^H and I_H^h.
- Improve the rule to go from fine to a coarser step.
- What about multilevel PnP and unfolded networks ?

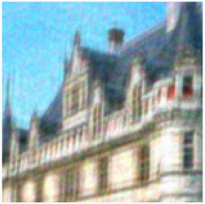
Perspective: Towards deep learning

Original



Degraded

SNR = 13.4 dB



Tikhonov

SNR = 16.4 dB

DTT

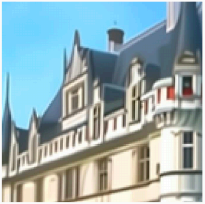
SNR = 16.6 dB

TV

SNR = 18.8 dB

NLTV

SNR = 19.4 dB



PnP-DRUnet

SNR = 20.0 dB

PnP-ScCP

SNR = 20.2 dB[4]

Perspective: Towards deep learning

➡ [1922] **Maximum likelihood** (Fisher).

$$\widehat{\mathrm{x}} \in \underset{\mathrm{x}}{\mathrm{Argmin}} \, \frac{1}{2}\|\mathrm{Ax}-\mathrm{z}\|_2^2 = (\mathrm{A}^*\mathrm{A})^{-1}\mathrm{A}^*\mathrm{z}$$

➡ [1963] **Regularization** (Tikhonov, Huber)

$$\widehat{\mathrm{x}} \in \underset{\mathrm{x}}{\mathrm{Argmin}} \, \frac{1}{2}\|\mathrm{Ax}-\mathrm{z}\|_2^2 + \theta\|\mathrm{Lx}\|_2^2 \qquad \text{avec} \quad \theta > 0$$

➡ [2000] **Sparsity** (Donoho, Daubechies-Defrise-DeMol,...)

$$\widehat{\mathrm{x}} \in \underset{\mathrm{x}}{\mathrm{Argmin}} \, \frac{1}{2}\|\mathrm{Ax}-\mathrm{z}\|_2^2 + \theta\|\mathrm{Lx}\|_\star$$

➡ [2010] **"End to end" neural networks**

$$\widehat{\mathrm{x}} = \mathrm{NN}_\Theta(\mathrm{z})$$

➡ [2020] **Plug-and-Play**

$$0 \in \mathrm{A}^*(\mathrm{A}\widehat{\mathrm{x}}-\mathrm{z}) + \mathbf{B}(\widehat{\mathrm{x}})$$

3- Make the algorithm robust and faster with multilevel strategy

Multilevel Plug-and-play

- **FB-PnP**:

$$\mathrm{x}^{[k+1]} = \boxed{\mathrm{d}_\Theta}\left(\mathrm{x}^{[k]} - \gamma \mathrm{A}^\top (\mathrm{A}\mathrm{x}^{[k]} - \mathrm{z}) \right)$$

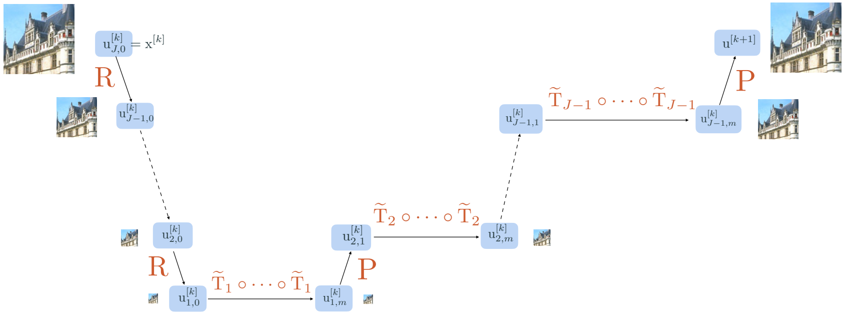
- **Multilevel FB-PnP**:

$$\mathrm{u}^{[k]} = \mathrm{ML}(\mathrm{x}^{[k]})$$

$$\mathrm{x}^{[k+1]} = \boxed{\mathrm{d}_\Theta}\left(\mathrm{u}^{[k]} - \gamma \mathrm{A}^\top (\mathrm{A}\mathrm{u}^{[k]} - \mathrm{z}) \right)$$

- We denote: $\mathbf{T}(\mathrm{u}^{[k]}) = \mathrm{d}_\Theta(\mathrm{u}^{[k]} - \gamma \mathrm{A}^\top (\mathrm{A}\mathrm{u}^{[k]} - \mathrm{z})$
- Multilevel framework when $\mathrm{d}_\Theta = \mathrm{prox}_f$: [Lauga, Riccietti, Pustelnik, Goncalves, 2024]

ML-step



Main ingredients

- R: restriction operator
- P: prolongation operator
- $\widetilde{T}_j \circ \ldots \circ \widetilde{T}_j$: coarser updates to insure first order coherence

Numerical experiments

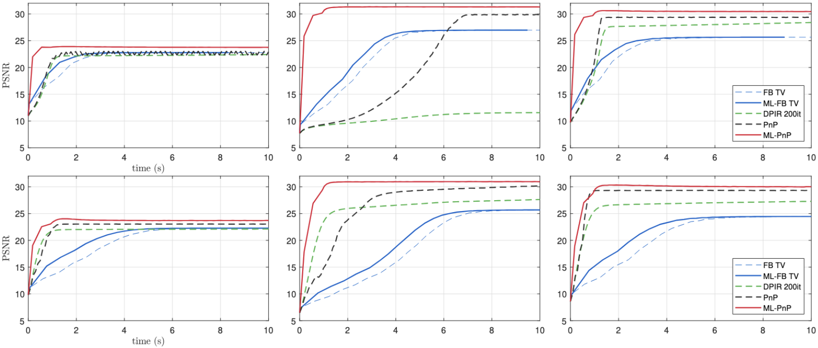
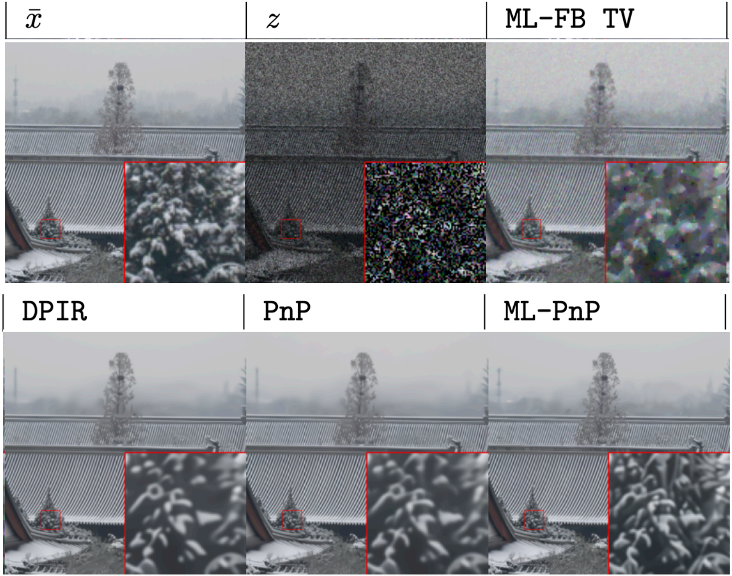
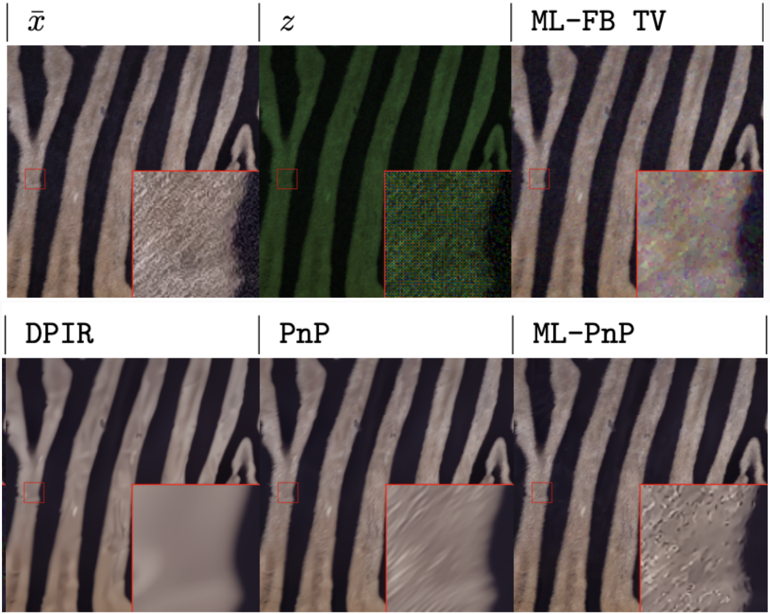


Figure 3: Reconstruction performance of multilevel algorithms and state of the art measured in PSNR with respect to time. First row presents results for the inpainting problem with 50% missing pixels, second row presents results for the demosaïcing problem. Each column is respectively associated with the images in the rows of Fig. 4 (inpainting) and Fig. 5 (demosaicing).

Numerical experiments



Numerical experiments



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- G. Lauga, A. Repetti, E. Riccietti, N. Pustelnik, P. Gonçalves, Y.Wiaux, A multilevel framework for accelerating uSARA in radio-interferometric imaging, EUSIPCO, Lyon, 2024.
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