Approximation Theory and Proof Assistants: Certified Computations

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Chapter 2. Orthogonal Polynomials - Chebyshev series

Section 2.1. Orthogonal Polynomials

Let $(a, b) \subset \mathbb{R}$ be an open interval, and let w be a weight function, that is to say $w : (a, b) \to (0, \infty)$ is a continuous function. We assume

$$\forall n \in \mathbb{N}, \quad \int_a^b |x|^n w(x) \mathrm{d}x < \infty.$$

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$$\forall n \in \mathbb{N}, \quad \int_a^b |x|^n w(x) \mathrm{d}x < \infty.$$

This is the case, for instance, if (a, b) is bounded and

$$\int_a^b w(x) \mathrm{d}x < \infty.$$

Let

$$\mathcal{E}(w) = \left\{ f \in \mathcal{C}((a,b)) : \|f\|_2 := \left(\int_a^b f(x)^2 w(x) \mathrm{d}x \right)^{1/2} < \infty \right\}.$$

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Observe that $\mathbb{R}[x] \subset \mathcal{E}(w)$. The space $\mathcal{E}(w)$ is equipped with an inner product

$$\langle f, g \rangle = \int_a^b f(x) g(x) w(x) \mathrm{d}x;$$

and $\| \cdot \|_2$ is the norm associated to this inner product.

Section 2.1. Orthogonal Polynomials

Definition 1

A family of orthogonal polynomials associated with w is a sequence $(p_n) \in \mathbb{R}[x]^{\mathbb{N}}$ where $\deg p_k = k$ for all k, and

$$i \neq j \quad \Rightarrow \quad \langle p_i, p_j \rangle = 0.$$

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For any weight w, there exists a family of orthogonal polynomials associated with w. If additionally we request that the p_k are all monic, this family is unique.

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Gram-Schmidt orthogonalization process

Theorem 3

The polynomials $(p_n)_{n\in\mathbb{N}}$ *satisfy the recurrence relation*

$$p_n(x) = (x - \alpha_n)p_{n-1}(x) - \beta_n p_{n-2}(x) \qquad (n \geqslant 2)$$

with

$$\alpha_n = \frac{\langle x p_{n-1}, p_{n-1} \rangle}{\|p_{n-1}\|_2^2}, \qquad \beta_n = \frac{\|p_{n-1}\|_2^2}{\|p_{n-2}\|_2^2}.$$

Example 4

$(-1, 1)$	$w(x) = (1 - x^2)^{-1/2}$	Chebyshev polynomials of the first kind (up to normalization)
$(-1, 1)$	$w(x) = 1$	Legendre polynomials
$(0, +\infty)$	$w(x) = e^{-x}$	Laguerre polynomials
$(-\infty, \infty)$	$w(x) = e^{-x^2}$	Hermite polynomials

Exercise

Prove that the first statement of Example 4 is correct.

Section 2.1. Orthogonal Polynomials

Theorem 5

For any weight w and for all n, the polynomial p_n has n distinct zeros in (a, b).

Section 2.1. Orthogonal Polynomials

Theorem 6

Let $f \in \mathcal{E}(w)$, $n \in \mathbb{N}$. There exists a unique best $L_2(w)$ polynomial approximation in $\mathbb{R}_n[x]$to f, denoted $p_{2,n}$:

$$\|f - p_{2,n}\|_2 = \min_{p \in \mathbb{R}_n[x]} \|f - p\|_2.$$

It is characterized by

$$\forall p \in \mathbb{R}_n[x], \quad \langle f - p_{2,n}, p \rangle = 0.$$

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Exercise

Prove the previous theorem.

Theorem 7

If (a, b) is bounded, then for all $f \in \mathcal{E}(w)$, we have $p_{2,n} \xrightarrow{\|\cdot\|_2} f$ as $n \to \infty$.

Section 2.2. A little bit of quadrature: Gauss methods

Let w be a weight function over (a, b), and let $f \in \mathcal{C}([a, b])$. We briefly study methods which approximate the integral

$$\int_a^b f(x)w(x)\mathrm{d}x$$

with a sum of the form

$$\sum_{k=0}^{n} w_k f(x_k), \qquad w_k \in \mathbb{R}, \quad x_k \in [a, b] \text{ pairwise distinct.}$$

Section 2.2. A little bit of quadrature: Gauss methods

First of all, if $\ell_k(x) = \prod_{\substack{j=0, \\ j \neq k}}^{n} \frac{x - x_j}{x_k - x_j}$, observe that if

$$p(x) = \sum_{k=0}^{n} f(x_k)\ell_k(x) \in \mathbb{R}_n[x]$$

interpolates f at the points $x_0, \ldots, x_n$, then our approximation for the integral is equal to $\int_a^b p(x)w(x)\mathrm{d}x = \sum_{k=0}^{n} w_k f(x_k)$ with

$$w_k = \int_a^b \ell_k(x)w(x)\mathrm{d}x \text{ for } k = 0, \ldots, n.$$

Section 2.2. A little bit of quadrature: Gauss methods

Theorem 8

There exists a unique choice of the points x_k and the weights w_k such that, whenever $f \in \mathbb{R}_{2n+1}[x]$,

$$\int_a^b f(x)w(x)\mathrm{d}x = \sum_{k=0}^{n} w_k f(x_k).$$

These points x_k belong to (a, b) and are the roots of the $(n+1)$-th orthogonal polynomial associated to w.

Section 2.2. A little bit of quadrature: Clenshaw-Curtis quadrature

Remark

The Chebyshev polynomials of the first kind satisfy

$$\int_{-1}^{1} T_k(x)\mathrm{d}x = \begin{cases} \frac{2}{1-k^2}, & k \in 2\mathbb{N}, \\ 0, & k \notin 2\mathbb{N}. \end{cases}$$

If $p = \sum_{k=0}^{n} c_k T_k$*, we deduce that the integral with weight* $w = 1$ *is given by*

$$\int_{-1}^{1} p(x)\mathrm{d}x = \sum_{\substack{0 \leqslant k \leqslant n \\ k \in 2\mathbb{N}}} \frac{2c_k}{1-k^2}.$$

Section 2.3. Lebesgue constants

For simplicity, we assume $[a, b] = [-1, 1]$.

Definition 9

We say that a linear mapping $L : \mathcal{C}([-1,1]) \to \mathbb{R}_n[x]$ is a projection onto $\mathbb{R}_n[x]$ if $Lp = p$ for all $p \in \mathbb{R}_n[x]$. The operator norm

$$\Lambda = \sup_{f \in \mathcal{C}([-1,1])} \frac{\|Lf\|_\infty}{\|f\|_\infty}$$

is called the Lebesgue constant for the projection.

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Proposition

Let Λ be the Lebesgue constant for the linear projection L of $\mathcal{C}([-1, 1])$ onto $\mathbb{R}_n[x]$. Let $f \in \mathcal{C}([-1, 1])$ and let $p = Lf$. Let p^* denote the minimax approximation to f. Then, we have

$$\|f - p\|_\infty \leqslant (1 + \Lambda)\|f - p^*\|_\infty.$$

2.3.1. Lebesgue constants for polynomial interpolation

Let $x_0, \ldots, x_n$ be pairwise distinct points in $[-1, 1]$. Consider the Lagrange interpolation operator

$$L_n : \mathcal{C}([-1,1]) \to \mathbb{R}_n[x], \qquad L_n f(x) = \sum_{k=0}^{n} f(x_k)\ell_k(x).$$

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Theorem 10

The Lebesgue constant of degree-n Lagrange interpolation at $x_0, \dots, x_n$ is equal to

$$\max_{x\in[-1,1]} \sum_{k=0}^{n} |\ell_k(x)|.$$

2.3.1. Lebesgue constants for polynomial interpolation

Theorem 11

The Lebesgue constant Λ_n satisfies

$$\frac{2}{\pi}\left(\log(n+1)+\gamma+\log\frac{4}{\pi}\right)\leqslant\Lambda_n,\ \textit{where}\ \frac{2}{\pi}\left(\gamma+\log\frac{4}{\pi}\right)=0.52125\ldots$$

Additionally,

- *for Chebyshev nodes (of the first and the second kinds), we have the bound*

$$\Lambda_n\leqslant\frac{2}{\pi}\log(n+1)+1\ \textit{and}\ \Lambda_n\sim\frac{2}{\pi}\log n\ \textit{as}\ n\to+\infty;$$

- *for equispaced points,*

$$\Lambda_n>\frac{2^{n-2}}{n^2}\ \textit{and}\ \Lambda_n\sim\frac{2^{n+1}}{en\log n}\ \textit{as}\ n\to+\infty.$$

2.3.1. Lebesgue constants for polynomial interpolation

Remark

We deduce from this theorem that Chebyshev interpolants (i.e. interpolation polynomials at Chebyshev nodes) are "near-best" approximations:

- $\Lambda_{15} = 2.76\ldots$*: one loses at most 2 bits if one uses a Chebyshev interpolant instead of the minimax polynomial;*
- $\Lambda_{30} = 3.18\ldots$*: one loses at most 2 bits if one uses a Chebyshev interpolant instead of the minimax polynomial;*
- $\Lambda_{100} = 3.93\ldots$*: one loses at most 2 bits if one uses a Chebyshev interpolant instead of the minimax polynomial;*
- $\Lambda_{100000} = 8.32\ldots$*: one loses at most 4 bits if one uses a Chebyshev interpolant instead of the minimax polynomial.*

2.3.2. Lebesgue constants for L_2 best approximation

When the L_2 space under consideration is $L_2\left([-1,1], \frac{1}{\sqrt{1-x^2}}\right)$, the best polynomial approximation $p_{2,n}$ is called the truncated Chebyshev series of order n.

Theorem 12

The Lebesgue constant for the $L_2\left([-1,1], \frac{1}{\sqrt{1-x^2}}\right)$ projection onto $\mathbb{R}_n[x]$ is

$$\Lambda_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\sin((n+1/2)t)}{\sin(t/2)}\right| \mathrm{d}t.$$

We have

$$\Lambda_n \leqslant \frac{4}{\pi^2}\log(n+1)+3 \ \textit{and}\ \Lambda_n \sim \frac{4}{\pi^2}\log n \ \textit{as}\ n \to +\infty.$$

2.3.2. Lebesgue constants for L_2 best approximation

Remark

We deduce from this theorem that truncated Chebyshev series are "near-best" approximations:

- $\Lambda_{15} = 4.12\ldots$*: one loses at most 3 bits if one uses the truncated Chebyshev series instead of the minimax polynomial;*
- $\Lambda_{30} = 4.39\ldots$*: one loses at most 3 bits if one uses the truncated Chebyshev series instead of the minimax polynomial;*
- $\Lambda_{100} = 4.87\ldots$*: one loses at most 3 bits if one uses the truncated Chebyshev series instead of the minimax polynomial;*
- $\Lambda_{100000} = 7.66\ldots$*: one loses at most 3 bits if one uses the truncated Chebyshev series instead of the minimax polynomial.*

2.3.3. Corollary: A first statement on the convergence of Chebyshev interpolants and truncated Chebyshev series

Let $f \in \mathcal{C}([a,b])$. The modulus of continuity of f is the function ω defined as

$$\text{for all } \delta > 0,\ \omega(\delta) = \sup_{\substack{|x-y|<\delta,\\ x,y\in[a,b]}} |f(x) - f(y)|.$$

2.3.3. Corollary: A first statement on the convergence of Chebyshev interpolants and truncated Chebyshev series

Let $f \in \mathcal{C}([a,b])$. The modulus of continuity of f is the function ω defined as

$$\text{for all } \delta > 0,\ \omega(\delta) = \sup_{\substack{|x-y| < \delta, \\ x,y \in [a,b]}} |f(x) - f(y)|.$$

Proposition

If f is a continuous function over $[0,1]$, ω its modulus of continuity, then we have

$$\|f - B_n(f,\cdot)\|_\infty = \tfrac{9}{4}\omega\left(n^{-\frac{1}{2}}\right).$$

2.3.3. Corollary: A first statement on the convergence of Chebyshev interpolants and truncated Chebyshev series

Theorem 13

If f is Lipschitz continuous over $[a, b]$, then

1. *the sequence of interpolation polynomials at the Chebyshev nodes uniformly converges to f.*
2. *The truncated Chebyshev series of f uniformly converges to f.*

Section 2.4.2. Convergence

Remark

The Chebyshev expansion of f is the Fourier expansion of $f(\cos t)$, so that many results on the convergence of Chebyshev expansions can be deduced from corresponding results in the well-developed theory of Fourier series.

Theorem 14

Let f be continuous on $[-1, 1]$. Denote by (a_k) its sequence of Chebyshev coefficients, by (f_n) its sequence of truncated Chebyshev expansions and by $(p_n)_{n \in \mathbb{N}}$ the sequence of interpolation polynomials of f at the Chebyshev nodes. Then

1. *The coefficients a_k tend to 0 when $k \to \infty$.*

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1. *The coefficients a_k tend to 0 when $k \to \infty$.*
2. *If f is Lipschitz continuous on $[-1,1]$, then (f_n) converges absolutely and uniformly to f and (p_n) converges uniformly to f.*

Section 2.4.2. Convergence

Theorem 14

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1. *The coefficients a_k tend to 0 when $k \to \infty$.*
2. *If f is Lipschitz continuous on $[-1,1]$, then (f_n) converges absolutely and uniformly to f and (p_n) converges uniformly to f.*
3. *If f is $\mathcal{C}^m$ and $f^{(m)}$ is Lipschitz continuous, then $a_k = O(1/k^{m+1})$, $\|f - f_n\|_\infty = O(n^{-m})$ and $\|f - p_n\|_\infty = O(n^{-m})$.*

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1. *The coefficients a_k tend to 0 when $k \to \infty$.*
2. *If f is Lipschitz continuous on $[-1, 1]$, then (f_n) converges absolutely and uniformly to f and (p_n) converges uniformly to f.*
3. *If f is $\mathcal{C}^m$ and $f^{(m)}$ is Lipschitz continuous, then $a_k = O(1/k^{m+1})$, $\|f - f_n\|_\infty = O(n^{-m})$ and $\|f - p_n\|_\infty = O(n^{-m})$.*
4. *If f is analytic inside the ellipse $\left|z + \sqrt{z^2 - 1}\right| \leqslant r$ with $r > 1$, then $a_k = O(r^{-k})$, $\|f - f_n\|_\infty = O(r^{-n})$ and $\|f - p_n\|_\infty = O(r^{-n})$.*

Section 2.4.2. Convergence

Theorem 15

Let f be continuous on $[-1,1]$. Denote by (f_n) its sequence of truncated Chebyshev expansions and by $(p_n)_{n\in\mathbb{N}}$ the sequence of interpolation polynomials of f at the Chebyshev nodes. Then

5. *Let P_n^* denote the minimax polynomial of degree at most n of f. If $f \in \mathcal{C}^{n+1}([-1,1])$, there exists $\xi_1, \xi_2, \xi_3 \in (-1,1)$ such that*

$$\|f - P_n^*\|_\infty = \frac{|f^{(n+1)}(\xi_1)|}{2^n(n+1)!};$$

$$\|f - f_n\|_\infty = \frac{|f^{(n+1)}(\xi_2)|}{2^n(n+1)!};$$

$$\|f - p_n\|_\infty = \frac{|f^{(n+1)}(\xi_3)|}{2^n(n+1)!}.$$