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Effective lower and upper bounds for the Fourier coefficients of powers of the modular invariant *j*

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Abstract. Using an elementary approach, we give precise effective lower and upper bounds for the Fourier coefficients of powers of the modular invariant j. Moreover, a straightforward adaptation of an old result of Rademacher yields a convergent series expansion of these Fourier coefficients and we show that this expansion allows to find a weaker version of these estimates in the general case and sharper ones in the case of j. Our results improve on previous ones by K. Mahler and O. Herrmann. In particular, we show that the Fourier coefficients of j are smaller than their asymptotically equivalent given by Petersson and Rademacher.

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1. Introduction

Let $\mathfrak{H} = \{z \in \mathbb{C};\, \Im(z) > 0\}$ denote Poincaré's half-plane. The modular invariant j can be defined by

$$j(\tau) = \frac{(1 + 240\sum_{n\geq 1}\sigma_3(n)e^{2i\pi n\tau})^3}{e^{2i\pi\tau}\prod_{n\geq 1}(1 - e^{2i\pi n\tau})^{24}}, \quad \text{for } \tau \in \mathfrak{H}, \tag{1}$$

where $\sigma_3(n) = \sum_{d|n} d^3$. It is a modular function[1] of weight 0. More precisely, j is holomorphic on $\mathfrak{H}$, meromorphic at infinity (with a simple pole of residue 1 at infinity) and it satisfies the following property of modular invariance

$$j\left(\frac{a\tau + b}{c\tau + d}\right) = j(\tau) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{2}$$

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[1] See [27] for basic definitions and results about modular functions.

From (1), we can define a function J holomorphic in the punctured unit disk $D' = \{z \in \mathbb{C} : 0 < |z| < 1\}$ such that $j(\tau) = J(e^{2i\pi\tau})$. We have

$$J(q) = \frac{\left(1 + 240\sum_{n\geq 1}\sigma_3(n)q^n\right)^3}{q\prod_{n\geq 1}(1-q^n)^{24}} \text{ for } q \in D'.$$

It follows that the coefficients of the Laurent expansion of J in 0 (which are the Fourier coefficients of j) are all nonnegative rational integers. For $m \in \mathbb{N} \setminus \{0\}$, we will denote j^m the function $\tau \in \mathfrak{H} \mapsto (j(\tau))^m$ and J^m the function $q \in D' \mapsto (J(q))^m$. We put $J^m(q) = \frac{1}{q^m} + \sum_{n\geq -m+1} c_m(n)q^n$ for all $m \geq 1$. When $m = 1$, we write $c(n)$ instead of $c_1(n)$. The computation of the coefficients $c(n)$ posed problems. The first seven were obtained in 1916 [4], the first twenty-four in 1939 [31] and the first hundred in 1953[2] [30].

In 1932, H. Petersson [19] proved $c(n) \sim_\infty \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}}$. In 1938, H. Rademacher [22] obtained by a different method the same result. Later, O. Herrmann [13] and K. Mahler [17], in order to give estimates about modular polynomials, established upper bounds for the $c_m(n)$. O. Herrmann obtained $c(n) \leq 6e^{4\pi\sqrt{n}}$, for $n \geq 1$, and K. Mahler showed that $c_m(n) \leq 1200e^{4\pi\sqrt{m(m+n)}}$, for $n, m \geq 1$.

These upper bounds did not seem optimal. Looking at tables of $c(n)$ that we computed, it appeared that the first $c(n)$ are smaller than the asymptotically equivalent given by Petersson and Rademacher. In this paper, we establish effective lower and upper bounds for the Fourier coefficients of j^m that give, in particular, the expected upper bound for $m = 1$: $c(n) \leq \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}}$ for all $n \geq 1$. We shall prove in Section 4 the following result.

Theorem 1.1. *For all $n, m \in \mathbb{N}$ such that $nm \geq 1000$ and $n \geq 4m\ln^2 m$, we have*

$$c_m(n) = \frac{1}{\sqrt{2}}\frac{m^{1/4}}{n^{3/4}}e^{4\pi\sqrt{nm}}\left(1 - \frac{3}{32\pi}\frac{1}{\sqrt{nm}} + \varepsilon_{n,m}\right) \quad \textit{with} \quad |\varepsilon_{n,m}| \leq \frac{0.055}{nm}.$$

When $m = 1$, the computation of the first thousand $c(n)$ shows that the theorem is true for all $n \geq 1$.

Remark 1.2. As the reader will see in the proof of Theorem 1.1, the conditions $nm \geq 1000$ and $n \geq 4m\ln^2 m$ are a compromise. By this, we mean that we can weaken one if we strengthen the other. Nevertheless, the reader should keep in mind that these conditions have to imply the necessary (but not sufficient: cf. Section 7) conditions $n \geq m$ and (6) that arise during the proofs.

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[2] Today, using a simple GP (the calculator of PARI [3]) program, we can get the first thousand in less than 26 seconds on a Celeron 466Mhz.

Let us mention that this elementary approach applies also to the Dedekind eta function and allows to find again some results, analogous to Theorem 1.1, about the partition function that are contained in [25, Chap. 14] (see also [18] [6]).

Reading [19] and [22], we saw that Petersson and Rademacher had obtained, not only an asymptotically equivalent but, in fact, a convergent series expansion for $c(n)$. Petersson used Poincaré series and Rademacher, inspired by his work on the partition function [20] [21], used Hardy and Ramanujan's circle method combined with a method of Kloostermann [16] extended by Estermann [9]. In this paper, we notice that an easy adaptation of Rademacher's work [22] yields a convergent series expansion for $c_m(n)$. This expansion allows to find a slightly weaker version of Theorem 1.1 but leads to the following improvement in the case $m = 1$:

Theorem 1.3. *For $k \geq 0$, let*

$$(1, k) = \frac{\prod_{j=0}^{k-1}(4 - (2j + 1)^2)}{4^k k!}.$$

For all $n \geq 1$, $p \geq 1$, we have

$$c(n) = \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}} \left(\sum_{k=0}^{p-1} \left(-\frac{1}{8\pi} \right)^k \frac{(1, k)}{n^{k/2}} + \frac{r_p(n)}{n^{p/2}} \right)$$

where

$$|r_p(n)| \leq \frac{1}{\sqrt{2}} \frac{|(1, p)|}{(4\pi)^p} + 62\sqrt{2}e^{-2\pi\sqrt{n}} n^{p/2}.$$

The outline of the paper is the following. In Section 2, using a theorem by Ingham, we give an asymptotically equivalent of $c_m(n)$ as n tends to infinity. Then, we prove in Section 3 some technical results necessary to the proof of Theorem 1.1 that we do in Section 4, which includes also estimates of the $c_m(n)$ for nonpositive n. We start Section 5 by giving, following [22], a convergent series expansion for any modular function of weight 0 holomorphic on $\mathfrak{H}$, from which we almost recover our main result. Then, we prove Theorem 1.3 and show that it implies, in particular, that $c(n) \leq \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}}$ for all $n \geq 1$, which improves on Herrmann's bounds [13]. We give in Section 6 general upper bounds for the Fourier coefficients $c_m(n)$, when $m \geq 2$, that improve on those given by Mahler in [17] and we gather in a corollary the upper bounds we obtained in the case $m \geq 2$. Finally, we examine numerically the conditions that appear in the proof of Theorem 1.1.

Our results improve on all the results known before. Let us mention that they have been used by A. Enge and F. Morain in [7] and are used in a work in progress by J.-B. Bost [5], in which he aims at replacing the classical estimates of Fourier coefficients in transcendence proofs with tools from Nevanlinna Theory. Our estimates allow to prove that some upper bounds obtained through Nevanlinna Theory are essentially optimal.

2. An asymptotically equivalent of $c_m(n)$ as n tends to infinity

In 1941, A. E. Ingham proved in [15] a tauberian theorem that implies the following result.

Theorem 2.1. [A. E. Ingham]. *Let* $f(z) = \sum_{k\geq 0} a_k z^k$ *a power series with real nonnegative coefficients and radius of convergence equal to* 1. *If there exist* $A > 0,\ \lambda,\ \alpha \in \mathbb{R}$ *such that*

$$f(x) \sim \lambda(\ln(1/x))^{\alpha} \exp\left(\frac{A}{\ln(1/x)}\right) \text{ as } x \to 1^-,$$

then

$$\sum_{k=0}^{n} a_k \sim \frac{\lambda}{2\sqrt{\pi}} \frac{A^{\alpha/2-1/4}}{n^{\alpha/2+1/4}} \exp(2\sqrt{An}) \text{ as } n \to +\infty.$$

We apply this theorem to the function $(1-x)x^m J^m(x)$, $m \geq 1$, to obtain an asymptotically equivalent of $c_m(n)$ as $n \to +\infty$. For $m \in \mathbb{N} \setminus \{0\}$, we have $x^m J^m(x) = 1 + \sum_{n\geq 1} \widetilde{c}_m(n)x^n$ with $\widetilde{c}_m(n) = c_m(n-m)$, $n \geq 1$. Hence, $(1-x)x^m J^m(x) = 1 + \sum_{n\geq 1}(\widetilde{c}_m(n) - \widetilde{c}_m(n-1))x^n$. But,

$$(1-x)x^m J^m(x) = \frac{Q^{3m}(x)}{(1-x)^{24m-1}\prod_{n\geq 2}(1-x^n)^{24m}}$$

where $Q(x) = 1 + 240\sum_{n\geq 1}\sigma_3(n)x^n$. Therefore, $(1-x)x^m J^m(x)$ has real nonnegative coefficients. We aim at finding an asymptotically equivalent of $xJ(x)$ as x tends to 1^-. To do so, we use the modular properties of j. For $x \in]0, 1[$, let $t > 0$ such that $x = e^{-2\pi t}$: we have $J(x) = j(it)$. As x tends to 1^-, t tends to 0^+ and conversely. Thus, $J(x)$ which, from (2), is equal to $j\left(\frac{i}{t}\right) = e^{2\pi/t} + \sum_{n\geq 0} c(n)e^{-2\pi n/t}$ is asymptotic, as x tends to 1^-, to $e^{2\pi/t} = \exp\left(\frac{4\pi^2}{\ln(1/x)}\right)$. Hence, as x tends to 1^-, $(1-x)x^m J^m(x)$ is asymptotic to $(1-x)\exp\left(\frac{4\pi^2 m}{\ln(1/x)}\right)$ or yet to $\ln(1/x)\exp\left(\frac{4\pi^2 m}{\ln(1/x)}\right)$. Ingham's theorem applied with $\lambda = 1,\ \alpha = 1,\ A = 4\pi^2 m$, gives

$$1 + \sum_{p=1}^{n}(\widetilde{c}_m(p) - \widetilde{c}_m(p-1)) \underset{n\to+\infty}{\sim} \frac{1}{2\sqrt{\pi}} \frac{(4\pi^2 m)^{1/4}}{n^{3/4}} \exp(4\pi\sqrt{nm}).$$

Therefore, $\widetilde{c}_m(n) \underset{n\to+\infty}{\sim} \frac{1}{\sqrt{2}}\frac{m^{1/4}}{n^{3/4}}\exp(4\pi\sqrt{nm})$ and

$$c_m(n) = \widetilde{c}_m(n+m) \underset{n\to+\infty}{\sim} \frac{1}{\sqrt{2}}\frac{m^{1/4}}{(n+m)^{3/4}}\exp(4\pi\sqrt{(n+m)m}).$$

Finally, we obtain

$$c_m(n) \underset{n\to+\infty}{\sim} \frac{1}{\sqrt{2}}\frac{m^{1/4}}{n^{3/4}}\exp(4\pi\sqrt{nm}).$$

This result is also a corollary of Theorem 1.1 that we establish in the next two sections.

3. Preliminary results

3.1 An inequality

The Fourier expansion of j gives, since its coefficients are nonnegative, $|j(\tau)| \le j(i\Im(\tau))$ for all $\tau \in \mathfrak{H}$. Hence, let $x,\ y \in \mathbb{R}$, if $\tau = x+iy \in \mathfrak{H}$, $|j(\tau)| \le e^{2\pi y} + 744 + \sum_{n\ge 1} c(n)e^{-2\pi n y}$. The function $y \mapsto g(y) = \sum_{n\ge 1} c(n)e^{-2\pi n y}$ is a decreasing function of $y \in]0,+\infty[$. Consequently, for $y \ge 1$, $g(y) \le g(1) = j(i) - e^{2\pi} - 744$. Therefore, if $y \ge 1$, $j(iy) \le e^{2\pi y} + 1728 - e^{2\pi}$ and, using the modular relation (2), if $0 < y \le 1$, $j(iy) = j\left(\frac{i}{y}\right) \le e^{2\pi/y} + 1728 - e^{2\pi}$. So, we showed:

Lemma 3.1. *Let $x,\ y \in \mathbb{R}$, for $\tau = x+iy \in \mathfrak{H}$, we have*

$$|j(\tau)| \le j(iy) \le e^{2\pi\max(y,1/y)} + 1728 - e^{2\pi}.$$

Now, we prove two intermediate results.

3.2 Some technical lemmata

Lemma 3.2. *For $x > 0$, we have*

$$\int_0^x e^{-2\pi u^2}\mathrm{d}u = \frac{1}{2\sqrt{2}} - r_x \quad \textit{with} \quad 0 < r_x < \frac{1}{4\pi x}e^{-2\pi x^2}.$$

Proof. Since $\int_0^{+\infty} e^{-y^2}\mathrm{d}y = \sqrt{\pi}/2$, we have $\int_0^{+\infty} e^{-2\pi u^2}\mathrm{d}u = \frac{1}{2\sqrt{2}}$ and, for $x > 0$,

$$0 < r_x = \int_x^{+\infty} e^{-2\pi u^2}\mathrm{d}u < \int_x^{+\infty}\left(1+\frac{1}{4\pi u^2}\right)e^{-2\pi u^2}\mathrm{d}u = \frac{1}{4\pi x}e^{-2\pi x^2}.$$

□

Lemma 3.3. *For $\alpha > 5$, let*

$$A_\alpha = \int_0^{\alpha/2} e^{-2\pi u^2} \left(\left(\cos \frac{2\pi u^3}{\alpha\sqrt{1-\frac{u^2}{\alpha^2}}} \right) \left(1 - \frac{u^2}{\alpha^2}\right)^{-3/2} - 1 \right) \mathrm{d}u.$$

Then, we have

$$A_\alpha = -\frac{3}{64\pi\alpha^2\sqrt{2}} - \frac{15}{4096\pi^2\alpha^4\sqrt{2}} + B_\alpha \ \textit{with}\ |B_\alpha| \le \frac{0.355}{\alpha^6}.$$

Proof. For $u \in [0, \alpha/2]$, we write

$$\cos \frac{2\pi u^3}{\alpha\sqrt{1-u^2/\alpha^2}} = 1 - \frac{2\pi^2 u^6}{\alpha^2(1-u^2/\alpha^2)} + R_\alpha(u)$$

and

$$\left(1 - \frac{u^2}{\alpha^2}\right)^{-3/2} = 1 + \frac{3}{2}\frac{u^2}{\alpha^2} + S_\alpha(u)$$

with

$$\frac{1}{4!}\frac{(2\pi)^4 u^{12}}{\alpha^4(1-u^2/\alpha^2)^2} - \frac{1}{6!}\frac{(2\pi)^6 u^{18}}{\alpha^6(1-u^2/\alpha^2)^3} \le R_\alpha(u) \le \frac{1}{4!}\frac{(2\pi)^4 u^{12}}{\alpha^4(1-u^2/\alpha^2)^2}$$

and

$$\frac{15}{8}\frac{u^4}{\alpha^4} + \frac{35}{16}\frac{u^6}{\alpha^6} \le S_\alpha(u) \le \frac{15}{8}\frac{u^4}{\alpha^4} + 3.04\frac{u^6}{\alpha^6}.$$

Thus,

$$\begin{aligned} &\left(\cos \frac{2\pi u^3}{\alpha\sqrt{1-u^2/\alpha^2}}\right)\left(1 - \frac{u^2}{\alpha^2}\right)^{-3/2} - 1 \\ &\quad = \frac{3}{2}\frac{u^2}{\alpha^2} + S_\alpha(u) - \frac{2\pi^2 u^6}{\alpha^2(1-u^2/\alpha^2)^{5/2}} + \frac{R_\alpha(u)}{(1-u^2/\alpha^2)^{3/2}}. \end{aligned}$$

But,

$$\left(1 - \frac{u^2}{\alpha^2}\right)^{-5/2} = 1 + \frac{5}{2}\frac{u^2}{\alpha^2} + T_\alpha(u) \text{ with } \frac{35}{8}\frac{u^4}{\alpha^4} \le T_\alpha(u) \le 6.85\frac{u^4}{\alpha^4}$$

since $u \in [0, \alpha/2]$. Hence,

$$\begin{aligned} &\left(\cos \frac{2\pi u^3}{\alpha\sqrt{1-u^2/\alpha^2}}\right)\left(1 - \frac{u^2}{\alpha^2}\right)^{-3/2} - 1 \\ &\quad = \frac{3}{2}\frac{u^2}{\alpha^2} - \frac{2\pi^2 u^6}{\alpha^2} - \frac{5\pi^2 u^8}{\alpha^4} + U_\alpha(u) \end{aligned}$$

where

$$U_\alpha(u) = S_\alpha(u) - \frac{2\pi^2 u^6}{\alpha^2} T_\alpha(u) + \frac{R_\alpha(u)}{(1 - u^2/\alpha^2)^{3/2}}.$$

Then, from the inequalities above and

$$\left(1 - \frac{u^2}{\alpha^2}\right)^{-7/2} = 1 + V_\alpha(u) \text{ with } \frac{7}{2}\frac{u^2}{\alpha^2} \leq V_\alpha(u) \leq 7\frac{u^2}{\alpha^2},$$

$$\left(1 - \frac{u^2}{\alpha^2}\right)^{-9/2} = 1 + W_\alpha(u) \text{ with } \frac{9}{2}\frac{u^2}{\alpha^2} \leq W_\alpha(u) \leq 11\frac{u^2}{\alpha^2},$$

if $u \in [0, \alpha/2]$, we have

$$\begin{aligned} &\frac{15}{8}\frac{u^4}{\alpha^4} + \frac{35}{16}\frac{u^6}{\alpha^6} - 13.7\pi^2\frac{u^{10}}{\alpha^6} + \frac{(2\pi)^4 u^{12}}{4!\alpha^4}\left(1 + \frac{7}{2}\frac{u^2}{\alpha^2}\right) \\ &\quad - \frac{(2\pi)^6 u^{18}}{6!\alpha^6}\left(1 + 11\frac{u^2}{\alpha^2}\right) \\ &\quad \leq U_\alpha(u) \leq \frac{15}{8}\frac{u^4}{\alpha^4} + 3.04\frac{u^6}{\alpha^6} - \frac{35\pi^2}{4}\frac{u^{10}}{\alpha^6} + \frac{(2\pi)^4 u^{12}}{4!\alpha^4}\left(1 + 7\frac{u^2}{\alpha^2}\right). \end{aligned}$$

Several integrations by parts (done with Maple) give

$$\begin{aligned} &K_1(\alpha) e^{-\frac{\pi\alpha^2}{2}} + K_2(\alpha)\int_0^{\alpha/2} e^{-2\pi u^2}\,\mathrm{d}u \\ &\qquad \leq A_\alpha \leq L_1(\alpha) e^{-\frac{\pi\alpha^2}{2}} + L_2(\alpha)\int_0^{\alpha/2} e^{-2\pi u^2}\,\mathrm{d}u \end{aligned}$$

with

$$\begin{aligned} K_1(\alpha) = {}& \frac{1}{1572864}\pi^5\alpha^{11} + \frac{277}{23592960}\pi^4\alpha^9 + \frac{973}{23592960}\pi^3\alpha^7 \\ & + \frac{1573}{1572864}\pi^2\alpha^5 + \frac{129471}{2621440}\pi\alpha^3 + \frac{3439207}{7864320}\alpha + \frac{8146573}{2621440}\frac{1}{\pi\alpha} \\ & + \frac{11070887}{524288}\frac{1}{\pi^2\alpha^3} + \frac{54546219}{524288}\frac{1}{\pi^3\alpha^5} + \frac{160044885}{524288}\frac{1}{\pi^4\alpha^7}, \\ K_2(\alpha) = {}& -\frac{3}{32}\frac{1}{\pi\alpha^2} - \frac{15}{2048}\frac{1}{\pi^2\alpha^4} - \frac{299481}{65536}\frac{1}{\pi^3\alpha^6} - \frac{160044885}{262144}\frac{1}{\pi^4\alpha^8}, \\ L_1(\alpha) = {}& -\frac{11}{49152}\pi^3\alpha^7 - \frac{45}{16384}\pi^2\alpha^5 + \frac{61}{49152}\pi\alpha^3 - \frac{897}{16384}\alpha \\ & - \frac{405303}{409600\pi\alpha} - \frac{424703}{81920\pi^2\alpha^3} - \frac{1275009}{81920\pi^3\alpha^5} \end{aligned}$$

and

$$L_2(\alpha) = -\frac{3}{32}\frac{1}{\pi\alpha^2} - \frac{15}{2048}\frac{1}{\pi^2\alpha^4} + \frac{1275009}{40960}\frac{1}{\pi^3\alpha^6}.$$

The function $L_2(\alpha)$ is negative when $\alpha > 5$. Hence, from Lemma 3.2, we have

$$\begin{aligned} A_\alpha &\leq e^{-\frac{\pi\alpha^2}{2}}\left(L_1(\alpha) - \frac{L_2(\alpha)}{2\pi\alpha}\right) + \frac{L_2(\alpha)}{2\sqrt{2}} \\ &\leq -\frac{1}{2\sqrt{2}}\left(\frac{3}{32}\frac{1}{\pi\alpha^2} + \frac{15}{2048}\frac{1}{\pi^2\alpha^4} - \frac{1275009}{40960}\frac{1}{\pi^3\alpha^6}\right) \\ &\quad + e^{-\frac{\pi\alpha^2}{2}}\left(\frac{61}{49152}\pi\alpha^3 + \frac{3}{64}\frac{1}{\pi^2\alpha^3} + \frac{15}{4096}\frac{1}{\pi^3\alpha^5}\right). \end{aligned}$$

Moreover, as $K_1(\alpha)$ is positive and $K_2(\alpha)$ is negative for all $\alpha > 0$, we get

$$\begin{aligned} A_\alpha &\geq -\frac{1}{2\sqrt{2}}\left(\frac{3}{32}\frac{1}{\pi\alpha^2} + \frac{15}{2048}\frac{1}{\pi^2\alpha^4} + \frac{299481}{65536}\frac{1}{\pi^3\alpha^6} + \frac{160044885}{262144}\frac{1}{\pi^4\alpha^8}\right) \\ &\geq -\frac{1}{2\sqrt{2}}\left(\frac{3}{32}\frac{1}{\pi\alpha^2} + \frac{15}{2048}\frac{1}{\pi^2\alpha^4} + \frac{1275009}{40960}\frac{1}{\pi^3\alpha^6}\right) \end{aligned}$$

if $\alpha > 5$. Thus, for $\alpha > 5$,

$$\begin{aligned} &\left|A_\alpha + \frac{1}{2\sqrt{2}}\left(\frac{3}{32}\frac{1}{\pi\alpha^2} + \frac{15}{2048}\frac{1}{\pi^2\alpha^4}\right)\right| \\ &\quad \leq \frac{1275009}{163840}\frac{\sqrt{2}}{\pi^3\alpha^6} + e^{-\frac{\pi\alpha^2}{2}}\left(\frac{61}{49152}\pi\alpha^3 + \frac{3}{64}\frac{1}{\pi^2\alpha^3} + \frac{15}{4096}\frac{1}{\pi^3\alpha^5}\right) \\ &\quad \leq \frac{1275009}{163840}\frac{\sqrt{2}}{\pi^3\alpha^6} + e^{-\frac{\pi\alpha^2}{2}}\left(\frac{61}{49152}\pi\alpha^3 + 4\cdot 10^{-5}\right) \leq \frac{0.355}{\alpha^6}, \end{aligned}$$

which is the estimate desired. □

4. Proof of the main result

Let $m \in \mathbb{N}^*$. We put $c_m(-m) = 1$. Cauchy's formula applied to $J^m(q) = \sum_{n\geq -m} c_m(n)q^n$, $|q| < 1$, gives: for $n \geq -m$, $c_m(n) = \frac{1}{2i\pi}\int_{\gamma_r}\frac{J^m(q)}{q^{n+1}}\mathrm{d}q$ where $\gamma_r : x \in [-1/2, 1/2] \mapsto re^{2i\pi x}$, $r \in]0, 1[$. Putting $r = e^{-2\pi y}$, $y > 0$, we get

$$\begin{aligned} c_m(n) &= e^{2\pi ny}\int_{-1/2}^{1/2} J^m\left(e^{2i\pi(x+iy)}\right)e^{-2i\pi nx}\mathrm{d}x \\ &= e^{2\pi ny}\int_{-1/2}^{1/2} j^m(x+iy)e^{-2i\pi nx}\mathrm{d}x. \end{aligned} \tag{3}$$

Moreover, we know that $c_m(n) \in \mathbb{N}$ and $\overline{j(x+iy)} = j(-x+iy)$. Hence, $c_m(n) = 2e^{2\pi ny}\,\Re\left(\int_0^{1/2} j^m(x+iy)e^{-2i\pi nx}\mathrm{d}x\right)$ for all $y > 0$.

As we shall see, the coefficients of the principal part of $J^m(q)$ at 0 are involved in the estimate of the $c_m(n)$ for $n \geq 1$. Thus, we need estimates of the $c_m(n)$ for n negative that we establish in the two propositions hereafter.

We start by a general (but not necessarily good) estimate of the coefficients $c_m(n)$.

Proposition 4.1. *For all $m \in \mathbb{N}^*$, $n \in \mathbb{Z}$, such that $n \geq -m$, we have*

$$c_m(n) \leq e^{2\pi n}1728^m.$$

Proof. If we choose y equal to 1, we get

$$c_m(n) = 2e^{2\pi n}\,\Re\left(\int_0^{1/2} j^m(x+i)e^{-2i\pi nx}\mathrm{d}x\right),$$

hence

$$c_m(n) \leq 2e^{2\pi n}\int_0^{1/2} j^m(i)\mathrm{d}x = e^{2\pi n}1728^m.$$

□

Now, we sharpen this result for some negative values of n.

Proposition 4.2. *For all $m \in \mathbb{N}^*$, $n \in \mathbb{Z}$, such that $-m+1 \leq n \leq -\frac{me^{2\pi}}{1728}$, we have*

$$c_m(n) \leq (1728 - e^{2\pi})^{m+n}\left(\frac{-n}{m+n}\right)^n\left(\frac{m}{m+n}\right)^m.$$

Proof. First, we notice that $m \geq 2$ necessarily. From Lemma 3.1 and (3), we have

$$c_m(n) \leq g_{m,n}(y) = \begin{cases} e^{2\pi ny}(e^{2\pi/y} + 1728 - e^{2\pi})^m & \text{if } y \in]0,1], \\ e^{2\pi ny}(e^{2\pi y} + 1728 - e^{2\pi})^m & \text{if } y \geq 1. \end{cases} \tag{4}$$

When $n \leq 0$, the function $g_{m,n}$ is decreasing on $]0,1]$. The derivative of $g_{m,n}$ on the interval $[1,+\infty[$ cancels at $y_{m,n} = \frac{1}{2\pi}\ln\left(\frac{-n}{m+n}(1728 - e^{2\pi})\right)$. The condition $y_{m,n} \geq 1$ is equivalent to $n \leq -\frac{me^{2\pi}}{1728}$. Hence, for $-m+1 \leq n \leq -\frac{me^{2\pi}}{1728}$,

$$c_m(n) \leq g_{m,n}(y_{m,n}) = (1728 - e^{2\pi})^{m+n}\left(\frac{-n}{m+n}\right)^n\left(\frac{m}{m+n}\right)^m.$$

For $-\frac{me^{2\pi}}{1728} < n < 0$, we notice that the best choice is $y = 1$ which gives $c_m(n) \leq e^{2\pi n}1728^m$. □

We start the proof of Theorem 1.1.

All along this section, we assume that $n \geq m$.

Inequalities (4) lead us to choose $y \in]0, 1]$ so that $e^{2\pi n y} e^{2\pi m/y}$ is minimal. This is done for $y = \sqrt{\frac{m}{n}}$. Hence, we study $\Re(K)$ with

$$K = \int_0^{1/2} j^m\left(x + i\sqrt{\frac{m}{n}}\right) e^{-2i\pi n x}\mathrm{d}x.$$

Thanks to (2), we have

$$j\left(x + i\sqrt{\frac{m}{n}}\right) = j\left(\frac{-x + i\sqrt{m/n}}{x^2 + m/n}\right). \tag{5}$$

To take this fundamental property into account, we have to compare 1, $\sqrt{\frac{m}{n}}$ and $\frac{\sqrt{m/n}}{x^2+m/n}$. Let $a_{m,n}$ denote $\left(\frac{m}{n}\right)^{1/4}\left(1 - \sqrt{\frac{m}{n}}\right)^{1/2}$, the inequality $\frac{\sqrt{m/n}}{x^2+m/n} \leq 1$ is equivalent to $x \geq a_{m,n}$. We notice that $a_{m,n} \in [0, 1/2]$, for $a_{m,n} \leq 1/2$ is equivalent to $\sqrt{\frac{m}{n}}\left(1 - \sqrt{\frac{m}{n}}\right) \leq \frac{1}{4}$ i.e. $\left(\sqrt{\frac{m}{n}} - \frac{1}{2}\right)^2 \geq 0$. Hence, we write $K = L + M$ with

$$L = \int_0^{a_{m,n}} j^m\left(x + i\sqrt{\frac{m}{n}}\right) e^{-2i\pi n x}\mathrm{d}x,$$

$$M = \int_{a_{m,n}}^{1/2} j^m\left(x + i\sqrt{\frac{m}{n}}\right) e^{-2i\pi n x}\mathrm{d}x.$$

4.1 An upper bound for $|M|$

From (5), we have $M = \int_{a_{m,n}}^{1/2} j^m\left(\frac{-x+i\sqrt{m/n}}{x^2+m/n}\right) e^{-2i\pi n x}\mathrm{d}x$. We have just seen that, for $x \geq a_{m,n}$, $\frac{\sqrt{m/n}}{x^2+m/n} \leq 1$. Then,

$$\begin{aligned} |M| &\leq \int_{a_{m,n}}^{1/2} j^m\left(i\frac{\sqrt{m/n}}{x^2 + m/n}\right)\mathrm{d}x \\ &\leq \int_{a_{m,n}}^{1/2} (e^{2\pi(x^2+m/n)\sqrt{n/m}} + 1728 - e^{2\pi})^m \mathrm{d}x \end{aligned}$$

from Lemma 3.1. This is equivalent to

$$|M| \leq e^{2\pi m\sqrt{m/n}} \int_{a_{m,n}}^{1/2} e^{2\pi x^2\sqrt{nm}}(1 + (1728 - e^{2\pi})e^{-2\pi(x^2+m/n)\sqrt{n/m}})^m \mathrm{d}x,$$

which gives

$$
\begin{aligned}
|M| &\leq e^{2\pi m\sqrt{m/n}} \int_{a_{m,n}}^{1/2} e^{2\pi x^2\sqrt{nm}} (1 + (1728 - e^{2\pi}) e^{-2\pi(a_{m,n}^2+m/n)\sqrt{n/m}})^m \mathrm{d}x \\
&= e^{2\pi m\sqrt{m/n}} \left(\frac{1728}{e^{2\pi}}\right)^m \int_{a_{m,n}}^{1/2} e^{2\pi x^2\sqrt{nm}} \mathrm{d}x.
\end{aligned}
$$

As $\int_{a_{m,n}}^{1/2} e^{2\pi x^2\sqrt{nm}} \mathrm{d}x \leq \int_{a_{m,n}}^{1/2} e^{\pi x\sqrt{nm}} \mathrm{d}x$, we finally get

$$
|M| \leq e^{2\pi m\sqrt{m/n}} \left(\frac{1728}{e^{2\pi}}\right)^m \frac{1}{\pi\sqrt{nm}} e^{\pi\sqrt{nm}/2}.
$$

4.2 An estimate of L

From (5), we have $L = \int_0^{a_{m,n}} j^m \left(\frac{-x+i\sqrt{m/n}}{x^2+m/n}\right) e^{-2i\pi nx} \mathrm{d}x$ i.e.

$$
\begin{aligned}
L = {} & \underbrace{\int_0^{a_{m,n}} e^{-2i\pi m\left(\frac{-x+i\sqrt{m/n}}{x^2+m/n}\right)} e^{-2i\pi nx} \mathrm{d}x}_{L_1} \\
& + \underbrace{\sum_{p>-m} c_m(p) \int_0^{a_{m,n}} e^{2i\pi p\left(\frac{-x+i\sqrt{m/n}}{x^2+m/n}\right)} e^{-2i\pi nx} \mathrm{d}x}_{L_2}.
\end{aligned}
$$

First, we compute an upper bound for $|L_2|$. Let $p_0 = -\left\lfloor \frac{me^{2\pi}}{1728} \right\rfloor - 1$ where $\lfloor x \rfloor$ denotes the floor of x. We have $L_2 = \int_0^{a_{m,n}} e^{-2i\pi nx} \left(\sum_{p=-m+1}^{p_0} + \sum_{p=p_0+1}^{-1} + \sum_{p\geq 0} \right) = L_{21} + L_{22} + L_{23}$ where $L_{21} = 0$ if $m = 1$ and $L_{22} = 0$ if $m = 1, 2, 3$.

We have, for all $m \geq 2$,

$$
|L_{21}| \leq \sum_{p=-m+1}^{p_0} c_m(p) \int_0^{a_{m,n}} e^{-\frac{2\pi p\sqrt{m/n}}{x^2+m/n}} \mathrm{d}x \leq \sum_{p=-m+1}^{p_0} c_m(p) a_{m,n} e^{-2\pi p\sqrt{n/m}}.
$$

Hence, if λ_p denotes $-\frac{p}{m} \in \left[\frac{e^{2\pi}}{1728}, 1 - \frac{1}{m}\right]$, Proposition 4.2 gives

$$
\begin{aligned}
|L_{21}| \leq a_{m,n} \sum_{p=-m+1}^{p_0} h^m(\lambda_p) \text{ with } & \ln(h(\lambda)) \\
&= (1-\lambda)\ln(1728 - e^{2\pi}) - \lambda\ln\lambda - (1-\lambda)\ln(1-\lambda) + 2\pi\lambda\sqrt{n/m}.
\end{aligned}
$$

The derivative $h'(\lambda)$ cancels if and only if $\lambda = ((1728 - e^{2\pi})e^{-2\pi\sqrt{n/m}} + 1)^{-1}$. Let us recall that $\lambda_p \in \left[\frac{e^{2\pi}}{1728}, 1 - \frac{1}{m}\right]$. We have $((1728 - e^{2\pi})e^{-2\pi\sqrt{n/m}} + 1)^{-1} \geq 1 - \frac{1}{m}$ if and only if

$$n \geq m \left(\frac{1}{2\pi} \ln((1728 - e^{2\pi})(m-1)) \right)^2. \tag{6}$$

In this case, for all $\lambda \in \left[\frac{e^{2\pi}}{1728}, 1 - \frac{1}{m}\right]$, $m \geq 2$, we have $h(\lambda) \leq h\left(1 - \frac{1}{m}\right)$ which implies $h^m(\lambda_p) \leq (1728 - e^{2\pi})\, m\, e^{1+2\pi(\sqrt{mn} - \sqrt{n/m})}$.

Finally, we obtain

$$|L_{21}| \leq \frac{(1728 - e^{2\pi})^2}{1728} \frac{m^{9/4}}{n^{1/4}} e^{1+2\pi(\sqrt{mn} - \sqrt{n/m})}$$

for all $m, n \in \mathbb{N}$ satisfying $m \geq 2$ and condition (6) that we assume until the upper bound for $|L_2|$ is achieved.

Then, we turn to L_{22}. For all $m \geq 4$,

$$|L_{22}| \leq \sum_{p=p_0+1}^{-1} \int_0^{a_{m,n}} c_m(p) e^{-\frac{2\pi p \sqrt{m/n}}{x^2 + m/n}} \, \mathrm{d}x \leq \sum_{p=p_0+1}^{-1} a_{m,n} e^{2\pi p} 1728^m e^{-2\pi p \sqrt{n/m}}$$

from Proposition 4.1. This implies

$$|L_{22}| \leq \left(\frac{m}{n}\right)^{1/4} 1728^m e^{2\pi(\sqrt{n/m} - 1)} \frac{e^{2\pi(\sqrt{n/m}-1) m e^{2\pi}/1728} - 1}{e^{2\pi(\sqrt{n/m}-1)} - 1}.$$

When $m \geq 4$, from (6), we have $e^{2\pi(\sqrt{n/m}-1)} \geq 6$ which yields

$$|L_{22}| \leq \left(\frac{m}{n}\right)^{1/4} 1728^m \frac{6}{5} e^{2\pi \frac{e^{2\pi}}{1728}\sqrt{nm}} e^{-2\pi m \frac{e^{2\pi}}{1728}} \leq \frac{6}{5} \left(\frac{m}{n}\right)^{1/4} e^{5.51m} e^{2\pi \frac{e^{2\pi}}{1728}\sqrt{nm}}.$$

Lastly, we give an upper bound for $|L_{23}|$. We have

$$|L_{23}| \leq \int_0^{a_{m,n}} \sum_{p \geq 0} c_m(p) e^{-\frac{2\pi p \sqrt{m/n}}{x^2 + m/n}} \, \mathrm{d}x.$$

As $\frac{\sqrt{m/n}}{x^2 + m/n} \geq 1$ for all $x \in [0, a_{m,n}]$, we get

$$|L_{23}| \leq \int_0^{a_{m,n}} \left(\sum_{p \geq 0} c_m(p) e^{-2\pi p} \right) \mathrm{d}x = a_{m,n} j^m(i) \leq \left(\frac{m}{n}\right)^{1/4} 1728^m.$$

We gather our results to get an upper bound for $|L_2|$.

For $m = 1$, we have $|L_2| = |L_{23}| \leq \frac{1}{n^{1/4}} 1728$.

For $m = 2,\ 3$, we obtain

$$|L_2| \le |L_{21}| + |L_{23}| \le \frac{m^{1/4}}{n^{1/4}} 1728^m + \frac{(1728 - e^{2\pi})^2}{1728} \frac{m^{9/4}}{n^{1/4}} e^{1+2\pi(\sqrt{mn}-\sqrt{n/m})}$$

For $m \ge 4$, we have

$$\begin{aligned} |L_2| &\le |L_{21}| + |L_{22}| + |L_{23}| \\ &\le \frac{m^{1/4}}{n^{1/4}} 1728^m + \frac{(1728 - e^{2\pi})^2}{1728} \frac{m^{9/4}}{n^{1/4}} e^{1+2\pi(\sqrt{mn}-\sqrt{n/m})} \\ &\quad + \frac{6}{5} \frac{m^{1/4}}{n^{1/4}} e^{5.51m + 2\pi \frac{e^{2\pi}}{1728}\sqrt{nm}}. \end{aligned}$$

The conditions $m \ge 2$ and (6) imply $n \ge 1.27m$, from which follows

$$\left(\frac{m}{n}\right)^{1/4} 1728^m \le \left(\frac{m}{n}\right)^{1/4} e^{5.51m + 2\pi \frac{e^{2\pi}}{1728}\sqrt{nm}}.$$

Thus, for $m \ge 4$, we obtain,

$$\begin{aligned} |L_2| &\le \frac{(1728 - e^{2\pi})^2}{1728} \frac{m^{9/4}}{n^{1/4}} e^{1+2\pi(\sqrt{mn}-\sqrt{n/m})} + \frac{11}{5} \frac{m^{1/4}}{n^{1/4}} e^{5.51m + 2\pi \frac{e^{2\pi}}{1728}\sqrt{nm}} \\ &\le 2238 \frac{m^{9/4}}{n^{1/4}} e^{2\pi(\sqrt{mn}-\sqrt{n/m})} + \frac{11}{5} \frac{m^{1/4}}{n^{1/4}} e^{5.51m + 0.62\pi\sqrt{nm}}. \end{aligned}$$

One can check immediately that this inequality remains true when $m \in \{1, 2, 3\}$.

Remark 4.3. The condition (6) is a consequence (when $m \ge 2$) of the condition $n \ge 4m \ln^2 m$.

Now we estimate L_1. We make the change of variable $u = \frac{n^{3/4} m^{1/4} x}{(nx^2+m)^{1/2}}$. This gives

$$\begin{aligned} L_1 &= \frac{m^{1/4}}{n^{3/4}} e^{2\pi\sqrt{mn}} \int_0^{\sqrt{n} a_{m,n}} e^{-2i\pi \frac{u^3}{(mn)^{1/4}(1-u^2/\sqrt{mn})^{1/2}}} e^{-2\pi u^2} \frac{\mathrm{d}u}{(1 - u^2/\sqrt{mn})^{3/2}} \\ &= \frac{m^{1/4}}{n^{3/4}} e^{2\pi\sqrt{mn}} \Bigg(\underbrace{\int_0^{\frac{(mn)}{2}^{1/4}}}_{L_{11}} + \underbrace{\int_{\frac{(mn)}{2}^{1/4}}^{\sqrt{n} a_{m,n}}}_{L_{12}} \Bigg). \end{aligned}$$

First,

$$\begin{aligned} |L_{12}| &\le \int_{\frac{(mn)}{2}^{1/4}}^{\sqrt{n} a_{m,n}} e^{-2\pi u^2} \frac{\mathrm{d}u}{(1 - u^2/\sqrt{mn})^{3/2}} \le \left(\frac{n}{m}\right)^{3/4} \int_{\frac{(mn)}{2}^{1/4}}^{\sqrt{n} a_{m,n}} e^{-2\pi u^2} \mathrm{d}u \\ &\le \left(\frac{n}{m}\right)^{3/4} \int_{\frac{(mn)}{2}^{1/4}}^{+\infty} e^{-2\pi u^2} \mathrm{d}u \le \frac{1}{2\pi} \frac{\sqrt{n}}{m} e^{-\pi\sqrt{mn}/2} \end{aligned}$$

from Lemma 3.2. Then,

$$\operatorname{Re} L_{11} = \int_0^{\frac{(mn)}{2}^{1/4}} e^{-2\pi u^2} \left(\left(\cos \frac{2\pi u^3}{(mn)^{1/4}\sqrt{1-\frac{u^2}{\sqrt{mn}}}} \right) \left(1-\frac{u^2}{\sqrt{mn}}\right)^{-3/2} -1 \right) \mathrm{d}u$$

$$+ \int_0^{\frac{(mn)}{2}^{1/4}} e^{-2\pi u^2} \mathrm{d}u.$$

Lemmata 3.3 and 3.2 give $\operatorname{Re} L_{11} = \left(1 - \frac{3}{32\pi\sqrt{mn}} - \frac{15}{2048\pi^2 mn}\right)\frac{1}{2\sqrt{2}} + C_{(mn)^{1/4}}$ with $C_{(mn)^{1/4}} = B_{(mn)^{1/4}} - \int_{\frac{(mn)}{2}^{1/4}}^{+\infty} e^{-2\pi u^2} \mathrm{d}u$ and

$$|C_{(mn)^{1/4}}| \leq \frac{e^{-\pi\sqrt{mn}/2}}{2\pi (mn)^{1/4}} + \frac{0.355}{(mn)^{3/2}}. \tag{7}$$

4.3 End of the proof of Theorem 1.1

For all $m, n \in \mathbb{N} \setminus \{0\}$, we have

$$\begin{aligned} c_m(n) &= 2e^{2\pi\sqrt{nm}}(\operatorname{Re} L_1 + \operatorname{Re} L_2 + \operatorname{Re} M) \\ &= \frac{1}{\sqrt{2}} \frac{m^{1/4}}{n^{3/4}} e^{4\pi\sqrt{nm}} \left(1 - \frac{3}{32\pi} \frac{1}{\sqrt{nm}} + \frac{1}{nm} E_{n,m}\right) \end{aligned}$$

with

$$\begin{aligned} E_{n,m} = {}& 2\sqrt{2} n^{7/4} m^{3/4} e^{-2\pi\sqrt{nm}} (\operatorname{Re} M + \operatorname{Re} L_2) \\ & - \frac{15}{2048\pi^2} + 2\sqrt{2} nm C_{(nm)^{1/4}} + 2\sqrt{2} nm \operatorname{Re} L_{12}. \end{aligned}$$

We shall prove in the sequel that $|E_{n,m}| \leq 0.055$ if $nm \geq 1000$ and $n \geq 4m \ln^2 m$.

First, $|2\sqrt{2} nm \operatorname{Re} L_{12}| \leq \frac{\sqrt{2}}{\pi} n^{3/2} e^{-\pi\sqrt{nm}/2} \leq \frac{\sqrt{2}}{\pi} (nm)^{3/2} e^{-\pi\sqrt{nm}/2} \leq 10^{-17}$ if $nm \geq 1000$.

If $nm \geq 1000$, we have from (7)

$$|2\sqrt{2} nm C_{(nm)^{1/4}}| \leq 2\sqrt{2} \left(\frac{(mn)^{3/4} e^{-\pi\sqrt{mn}/2}}{2\pi} + \frac{0.355}{(mn)^{1/2}} \right) \leq 0.03176.$$

Then,

$$\begin{aligned} |2\sqrt{2} n^{7/4} m^{3/4} e^{-2\pi\sqrt{nm}} \operatorname{Re} L_2| \leq {}& 4476\sqrt{2} m^3 n^{3/2} e^{-2\pi\sqrt{n/m}} \\ & + \frac{22\sqrt{2}}{5} n^{3/2} m e^{5.51m - 1.38\pi\sqrt{nm}}. \end{aligned}$$

The function $f_m : x \mapsto x^{3/2}e^{-2\pi\sqrt{x/m}}$ is decreasing for $x \ge \max(1, 4m\ln^2 m)$.

For $m \le 7$, as $n \ge 1000/7 \ge \max(1, 28\ln^2 7)$,

$$\begin{aligned} 4476\sqrt{2}n^{3/2}m^3e^{-2\pi\sqrt{n/m}} &\le 4476\sqrt{2}m^3 f_m\left(\frac{1000}{7}\right) \\ &\le 4476\sqrt{2}\cdot 7^3 f_7\left(\frac{1000}{7}\right) \le 0.0018. \end{aligned}$$

For $m = 8$, we have $n \ge 32\ln^2 8$ i.e. $n \ge 139$ which implies

$$\begin{aligned} 4476\sqrt{2}n^{3/2}m^3e^{-2\pi\sqrt{n/m}} &= 4476\sqrt{2}n^{3/2}8^3e^{-2\pi\sqrt{n/8}} \\ &\le 2291712\sqrt{2}f_8(139) \le 0.02244. \end{aligned}$$

For $m \ge 9$, as $n \ge 4m\ln^2 m$, we have

$$\begin{aligned} 4476\sqrt{2}n^{3/2}m^3e^{-2\pi\sqrt{n/m}} &\le 4476\sqrt{2}m^3 f_m(4m\ln^2 m) \\ &= 35808\sqrt{2}m^{9/2-4\pi}\ln^3 m. \end{aligned}$$

The function $x \mapsto x^{9/2-4\pi}\ln^3 x$ is decreasing for $x \ge \exp\left(\frac{3}{4\pi-9/2}\right)$. Hence, if $m \ge 9 \ge \left(\exp\left(\frac{3}{4\pi-9/2}\right)\right)$, we get

$$4476\sqrt{2}n^{3/2}m^3e^{-2\pi\sqrt{n/m}} \le 0.011.$$

Now, we give an upper bound for $\frac{22\sqrt{2}}{5}n^{3/2}me^{5.51m-1.38\pi\sqrt{nm}}$. We notice that

- if $m \le 10$ then $1000 \le nm \le 10n$ hence $n \ge 10m$;
- if $m \ge 10$ then $n \ge 4m\ln^2 m \ge 4m\ln^2 10 \ge 20m$.

It follows

$$\begin{aligned} \frac{22\sqrt{2}}{5}n^{3/2}me^{5.51m-1.38\pi\sqrt{nm}} &\le \frac{22\sqrt{2}}{5}(nm)^{3/2}e^{5.51\sqrt{10}\sqrt{nm}-1.38\pi\sqrt{nm}} \\ &\le \frac{22\sqrt{2}}{5}(nm)^{3/2}e^{2.5\sqrt{nm}} \\ &\le \frac{22\sqrt{2}}{5}(1000)^{3/2}e^{-2.5\sqrt{1000}} \le 10^{-29}. \end{aligned}$$

Finally,

$$\begin{aligned} |2\sqrt{2}n^{7/4}m^{3/4}e^{-2\pi\sqrt{nm}}\mathrm{Re}\,M| &\le \frac{2\sqrt{2}}{\pi}n^{5/4}m^{1/4}e^{2\pi m\sqrt{m/n}}\left(\frac{1728}{e^{2\pi}}\right)^m e^{-3\pi\sqrt{nm}/2} \\ &\le \frac{2\sqrt{2}}{\pi}(nm)^{5/4}(1728)^m e^{-3\pi\sqrt{nm}/2} \end{aligned}$$

since $n \geq m$. As $n \geq 10m$, we have

$$\begin{aligned}|2\sqrt{2}n^{7/4}m^{3/4}e^{-2\pi\sqrt{nm}}\operatorname{Re} M| &\leq \frac{2\sqrt{2}}{\pi}(nm)^{5/4}e^{\sqrt{0.1}\ln 1728\sqrt{nm}-3\pi\sqrt{nm}/2}\\ &\leq \frac{2\sqrt{2}}{\pi}(nm)^{5/4}e^{-2.3\sqrt{nm}}\\ &\leq \frac{2\sqrt{2}}{\pi}(1000)^{5/4}e^{-2.3\sqrt{1000}} \leq 10^{-27}.\end{aligned}$$

These upper bounds collected give

$$|E_{n,m}| \leq \frac{15}{2048\pi^2} + 10^{-17} + 0.03176 + 0.02244 + 10^{-29} + 10^{-27} \leq 0.055$$

if $nm \geq 1000$ and $n \geq 4m\ln^2 m$, which concludes the proof.

5. Some corollaries of Rademacher's series expansion of the coefficients $c(n)$

Let

$$S(a, b; c) = \sum_{\substack{1\leq u,v\leq c\\ uv=1 \bmod c}} e^{2i\pi\left(\frac{au+bv}{c}\right)} \text{ where } a,\ b,\ c \in \mathbb{Z} \text{ with } c > 0$$

be the Kloosterman sum. Let

$$I_1(z) = \sum_{\nu=0}^{+\infty} \frac{(z/2)^{2\nu+1}}{\nu!(\nu+1)!}$$

be the Bessel function of first order with purely imaginary argument.

In [22], Rademacher gives a series expansion of the Fourier coefficients of j. A straightforward adaptation of the proof yields the following result.

Theorem 5.1. *Let f be a modular function of weight 0, holomorphic on $\mathfrak{H}$, having the Fourier expansion*

$$f(\tau) = \sum_{n\geq -m_f} a(n)q^n, \text{ with } q = e^{2i\pi\tau},$$

for all $\tau \in \mathfrak{H}$. Then, we have, for $n \geq 1$,

$$a(n) = \frac{2\pi}{\sqrt{n}}\sum_{\ell=1}^{m_f} a(-\ell)\sqrt{\ell}\sum_{k=1}^{+\infty}\frac{S(n, -\ell; k)}{k} I_1\left(\frac{4\pi\sqrt{n\ell}}{k}\right).$$

Hence, in the particular case of j^m, we have

Theorem 5.2. *For $m,\ n \geq 1$, we have*

$$c_m(n) = \frac{2\pi}{\sqrt{n}} \sum_{\ell=1}^{m} c_m(-\ell)\sqrt{\ell} \sum_{k=1}^{+\infty} \frac{S(n,-\ell;k)}{k} I_1\left(\frac{4\pi\sqrt{n\ell}}{k}\right).$$

In this section, we first derive two estimates of the $c_m(n)$ from this theorem. Then, we focus on the peculiar case of the $c(n)$.

5.1 An estimate of the Fourier coefficients of j^m with an arbitrarily small error term

For $n \geq 1,\ \ell \geq 1,\ N \geq 1$, we set $R_{\ell,N}(n) = \sum_{k=N+1}^{+\infty} \frac{S(n,-\ell;k)}{k} I_1\big(\frac{4\pi\sqrt{n\ell}}{k}\big)$.

We first prove the following result.

Lemma 5.3. *For all positive integers $\ell,\ n,\ k$, we have*

$$|S(n,-\ell;k)| \leq 9(n,\ell,k)^{1/2} k^{3/4}.$$

Proof. From [29], we know that

$$|S(n,-l;k)| \leq (n,l,k)^{1/2} k^{1/2} \tau(k) \tag{8}$$

where (n,l,k) is the g.c.d. of n, l and k and $\tau(k)$ denotes the number of positive divisors of k. Thus, it remains to prove that $\tau(k) \leq 9k^{1/4}$, which we do following [12, §18.1]. Let $\mathcal{P}$ denote the set of prime numbers and, for all $p \in \mathcal{P}$, let v_p denote the p-adic valuation. We have $k = \prod_{p\in\mathcal{P}} p^{v_p(k)}$ As $\tau(k) = \prod_{p\in\mathcal{P}}(v_p(k)+1)$, we have to prove $\frac{\tau(k)}{k^{1/4}} = \prod_{p\in\mathcal{P}}\big(\frac{v_p(k)+1}{p^{v_p(k)/4}}\big) \leq 9$. If $p \geq 16$, we have $p^{1/4} \geq 2$ and then $\frac{v_p(k)+1}{p^{v_p(k)/4}} \leq \frac{v_p(k)+1}{2^{v_p(k)}} \leq 1$. Hence,

$$\frac{\tau(k)}{k^{1/4}} \leq \prod_{\substack{p\in\mathcal{P},\\ p\leq 16}} \frac{v_p(k)+1}{p^{v_p(k)/4}}.$$

Let u_p be the function $x \mapsto \frac{x+1}{p^{x/4}}$ for $p \in \mathcal{P}$. When $p \leq 16$, the respective maxima of the u_p over $\mathbb{N}$ are $u_2(5),\ u_3(3),\ u_5(2),\ u_7(1),\ u_{11}(1)$ and $u_{13}(1)$, which implies $\frac{\tau(k)}{k^{1/4}} \leq 9$. □

Then, following [25, §121], we get, for $\ell \geq 1,\ N \geq 1$,

$$\begin{aligned} |R_{\ell,N}(n)| &\leq 9(n,\ell)^{1/2} \sum_{k=N+1}^{+\infty} k^{-1/4} \left| I_1\left(\frac{4\pi\sqrt{n\ell}}{k}\right)\right| \\ &= 9(n,\ell)^{1/2} \sum_{k=N+1}^{+\infty} k^{-1/4} \sum_{\nu=0}^{+\infty} \frac{(2\pi\sqrt{n\ell}/k)^{2\nu+1}}{\nu!(\nu+1)!} \end{aligned}$$

$$< 9(n,\ell)^{1/2}\sum_{\nu=0}^{+\infty}\frac{(2\pi\sqrt{n\ell})^{2\nu+1}}{\nu!(\nu+1)!}\int_N^{+\infty}\frac{\mathrm{d}x}{x^{2\nu+5/4}}$$

$$= 9(n,\ell)^{1/2}\sum_{\nu=0}^{+\infty}\frac{(2\pi\sqrt{n\ell})^{2\nu+1}}{\nu!(\nu+1)!(2\nu+1/4)}\frac{1}{N^{2\nu+1/4}}$$

$$\leq 36(n,\ell)^{1/2}N^{3/4}I_1\left(\frac{4\pi\sqrt{n\ell}}{N}\right).$$

Thus, for $m,\ n \geq 1$, $(N_1,\dots,N_m) \in (\mathbb{N}\setminus\{0\})^m$, we have

$$c_m(n) = \frac{2\pi}{\sqrt{n}}\sum_{\ell=1}^{m}c_m(-\ell)\sqrt{\ell}\left(\sum_{k=1}^{N_\ell}\frac{S(n,-\ell;k)}{k}I_1\left(\frac{4\pi\sqrt{n\ell}}{k}\right)+R_{\ell,N_\ell}(n)\right) \tag{9}$$

with $|R_{\ell,N_\ell}(n)| \leq 36(n,\ell)^{1/2}N_\ell^{3/4}I_1\big(\frac{4\pi\sqrt{n\ell}}{N_\ell}\big)$.

Now, we prove that a slightly weaker version of Theorem 1.1 is a consequence of this estimate.

5.2 (Almost) Theorem 1.1 as a corollary of estimate (9)

For $m,\ n \geq 1$, we fix $N_1 = \dots = N_{m-1} = 1$ and $N_m = 2$. This gives

$$c_m(n) = 2\pi\sqrt{\frac{m}{n}}\left(I_1(4\pi\sqrt{nm}) + \frac{(-1)^{n-m}}{2}I_1(2\pi\sqrt{nm}) + R_{m,2}(n)\right)$$
$$+\frac{2\pi}{\sqrt{n}}\sum_{\ell=1}^{m-1}c_m(-\ell)\sqrt{\ell}(I_1(4\pi\sqrt{n\ell})+R_{\ell,1}(n))$$
$$= \frac{2\pi}{\sqrt{n}}(\sqrt{m}I_1(4\pi\sqrt{nm})+R(n))$$

with

$$|R(n)| \leq 62\sqrt{m}\,(n,m)^{1/2}I_1(2\pi\sqrt{nm})$$
$$+37\sum_{\ell=1}^{m-1}c_m(-\ell)\sqrt{\ell}(n,\ell)^{1/2}I_1(4\pi\sqrt{n\ell}).$$

For all $z \in \mathbb{C}$, we have $I_1(z) = -\frac{i}{2}(H_1^{(1)}(iz) + H_1^{(2)}(iz))$ where $H_1^{(1)}$ and $H_1^{(2)}$ are respectively the first and second Hankel functions[3] [28, §3.6]. As

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[3] also called Bessel functions of the third kind.

$H_1^{(1)}(ix)$ is real for all $x > 0$ (cf. [8, §7.2.2] or [28, §7.2, p. 196]), we know that $I_1(x) = \Im(H_1^{(2)}(ix))/2$ for $x > 0$. Following [28, Chap. 7], we establish an asymptotic expansion of $I_1(x)$ (where $x > 0$) with an effective remainder term.

For all $x > 0$, we have (see [28, p. 168])

$$H_1^{(2)}(ix) = i\sqrt{\frac{2}{\pi x}}\frac{e^x}{\Gamma\left(\frac{3}{2}\right)}\int_0^{\infty \exp i\beta} e^{-u}u^{1/2}\left(1-\frac{u}{2x}\right)^{1/2}\mathrm{d}u \tag{10}$$

where $0 < \beta < \pi/2$. For all $p \in \mathbb{N}\setminus\{0\}$,

$$\left(1-\frac{u}{2x}\right)^{1/2} = \sum_{k=0}^{p-1}\frac{\left(-\frac{1}{2}\right)_k}{k!}\left(\frac{u}{2x}\right)^k + \frac{\left(-\frac{1}{2}\right)_p}{(p-1)!}\left(\frac{u}{2x}\right)^p \times \int_0^1 (1-t)^{p-1}\left(1-\frac{ut}{2x}\right)^{1/2-p}\mathrm{d}t \tag{11}$$

where $(\alpha)_n$ is the Pochhammer symbol defined by $(\alpha)_0 = 1$ and $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$. We choose (for simplicity) $\beta = \pi/4$. For all $t \in [0, 1]$ and u on the half-line $[0, \infty e^{i\pi/4})$, we have $\left|1-\frac{ut}{2x}\right| \geq \sin(\pi/4) = \frac{\sqrt{2}}{2}$ hence $\left|\left(1-\frac{ut}{2x}\right)^{1/2-p}\right| \leq 2^{\frac{2p-1}{4}}$. Then, in (10), we replace $\left(1-\frac{u}{2x}\right)^{1/2}$ by its expansion (11) and integrate term-by-term. This gives

$$H_1^{(2)}(ix) = i\sqrt{\frac{2}{\pi x}}\,e^x\left(\sum_{k=0}^{p-1}\frac{\left(-\frac{1}{2}\right)_k\left(\frac{3}{2}\right)_k}{k!(2x)^k} + \frac{r_{H_1^{(2)},p}}{x^p}\right)$$

with

$$|r_{H_1^{(2)},p}| \leq \frac{2^{\frac{2p-1}{4}}}{(p-1)!}\frac{\left|\left(-\frac{1}{2}\right)_p\right|}{\Gamma\left(\frac{3}{2}\right)2^p}\left|\int_0^1(1-t)^{p-1}\mathrm{d}t\int_0^{\infty\exp\left(i\frac{\pi}{4}\right)} e^{-u}u^{p+1/2}\mathrm{d}u\right|$$

$$\leq \frac{2^{\frac{2p-1}{4}}}{p!}\frac{\left|\left(-\frac{1}{2}\right)_p\right|}{\Gamma\left(\frac{3}{2}\right)2^p}2^{\frac{2p+3}{4}}\Gamma\left(p+\frac{3}{2}\right) = \frac{\sqrt{2}}{p!}\left|\left(-\frac{1}{2}\right)_p\left(\frac{3}{2}\right)_p\right|.$$

If we write, after Hankel [11, p. 494],

$$(1,k) = (-1)^k\frac{\left(-\frac{1}{2}\right)_k\left(\frac{3}{2}\right)_k}{k!} = \frac{\Gamma\left(k+\frac{3}{2}\right)}{k!\Gamma\left(-k+\frac{3}{2}\right)} = \frac{\prod_{j=0}^{k-1}(4-(2j+1)^2)}{4^k k!}, \tag{12}$$

we obtained, for all $x > 0$ and $p \in \mathbb{N}\setminus\{0\}$,

$$I_1(x) = \frac{e^x}{\sqrt{2\pi x}}\left(\sum_{k=0}^{p-1}\frac{(-1)^k(1,k)}{(2x)^k} + \frac{r_{I_1,p}}{x^p}\right) \tag{13}$$

with $|r_{I_1,p}| \le \frac{1}{\sqrt{2}}|(1,p)|$. Notice that this formula does not give a convergent series expansion. This was done by Hadamard [10] whose work implies, for all $x > 0$,

$$I_1(x) = \frac{e^x}{\sqrt{2\pi x}} \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_k \gamma\left(k+\frac{3}{2}, 2x\right)}{\Gamma\left(\frac{3}{2}\right) k!(2x)^k}$$

where γ denotes Legendre's incomplete Gamma-function defined by $\gamma(a,u) = \int_0^u t^{a-1}e^{-t}\mathrm{d}t$. As $\left(-\frac{1}{2}\right)_k \le 0$ for all $k \ge 1$, we get, in particular, for all $x > 0$,

$$I_1(x) \le \frac{e^x}{\sqrt{2\pi x}}. \tag{14}$$

Until the end of the section, we assume that $n,\ m \in \mathbb{N}$**,** $nm \ge 1000$ **and** $n \ge 4m\ln^2 m$**.**

Let $p_1 = \left\lfloor \frac{me^{2\pi}}{1728} \right\rfloor$. We have $p_1 \ge 1$ if and only if $m \ge 4$.

For all $m \ge 2$, we have, from inequality (14),

$$\begin{aligned}
S_1 &:= \frac{2\pi}{\sqrt{n}} \sum_{\ell=p_1+1}^{m-1} 37c_m(-\ell)\sqrt{\ell}(n,\ell)^{1/2} I_1(4\pi\sqrt{n\ell}) \\
&\le \sum_{\ell=p_1+1}^{m-1} \frac{37\sqrt{2}}{2} c_m(-\ell) \left(\frac{\ell}{n}\right)^{3/4} e^{4\pi\sqrt{n\ell}} \\
&= \frac{\sqrt{2}}{2} \frac{m^{1/4}}{n^{3/4}} \frac{e^{4\pi\sqrt{nm}}}{nm} \sum_{\ell=p_1+1}^{m-1} 37nc_m(-\ell)(m\ell)^{3/4} e^{4\pi\sqrt{n}(\sqrt{\ell}-\sqrt{m})} \\
&\le \frac{\sqrt{2}}{2} \frac{m^{1/4}}{n^{3/4}} \frac{e^{4\pi\sqrt{nm}}}{nm} \underbrace{37nm^{3/2} e^{-2\pi\sqrt{nm}} \sum_{\ell=p_1+1}^{m-1} c_m(-\ell) e^{2\pi\ell\sqrt{n/m}}}_{S_1'}.
\end{aligned}$$

If we proceed as in Subsection 4.2, we obtain, under the assumptions $m \ge 2$ and (6),

$$S_1' \le 37nm^{7/2} \frac{(1728 - e^{2\pi})^2}{1728} e^{1-2\pi\sqrt{n/m}} \le 82771nm^{7/2} e^{-2\pi\sqrt{n/m}}.$$

The function $g_m : x \mapsto xe^{-2\pi\sqrt{x/m}}$ is decreasing for $x \ge \max(1, 4m\ln^2 m)$.

For $m \le 7$, as $n \ge 1000/7 \ge \max(1, 28\ln^2 7)$,

$$\begin{aligned}
82771nm^{7/2}e^{-2\pi\sqrt{n/m}} &\le 82771m^{7/2} g_m\left(\frac{1000}{7}\right) \\
&\le 82771 \cdot 7^{7/2} g_7\left(\frac{1000}{7}\right) \le 0.0051.
\end{aligned}$$

For $m = 8$, we have $n \geq 32 \ln^2 8$ i.e. $n \geq 139$ which implies

$$\begin{aligned} 82771 n m^{7/2} e^{-2\pi\sqrt{n/m}} &= 82771 \cdot 8^{7/2} n e^{-2\pi\sqrt{n/8}} \\ &\leq 84757504\sqrt{2} g_8(139) \leq 0.0704. \end{aligned}$$

For $m \geq 9$, as $n \geq 4m \ln^2 m$, we have

$$82771 n m^{7/2} e^{-2\pi\sqrt{n/m}} \leq 82771 m^{7/2} g_m(4m \ln^2 m) = 331084 m^{9/2-4\pi} \ln^2 m.$$

The function $x \mapsto x^{9/2-4\pi} \ln^2 x$ is decreasing for $x \geq \exp\left(\frac{2}{4\pi-9/2}\right)$. Hence, if $m \geq 9 \geq \left(\exp\left(\frac{2}{4\pi-9/2}\right)\right)$, we get

$$82771 n m^{7/2} e^{-2\pi\sqrt{n/m}} \leq 331084 \cdot 9^{9/2-4\pi} \ln^2 9 \leq 0.0321$$

Finally, we obtained $S_1' \leq 0.0704$ under the assumptions $nm \geq 1000$ and $n \geq 4m \ln^2 m$.

For all $m \geq 4$,

$$\begin{aligned} S_2 &:= \frac{2\pi}{\sqrt{n}} \sum_{\ell=1}^{p_1} 37 c_m(-\ell)\sqrt{\ell}(n,\ell)^{1/2} I_1(4\pi\sqrt{n\ell}) \\ &\leq \sum_{\ell=1}^{p_1} \frac{37\sqrt{2}}{2} \frac{1728^m}{e^{2\pi\ell}} \left(\frac{\ell}{n}\right)^{3/4} e^{4\pi\sqrt{n\ell}} \\ &= \frac{\sqrt{2}}{2} \frac{m^{1/4}}{n^{3/4}} \frac{e^{4\pi\sqrt{nm}}}{nm} \underbrace{\sum_{\ell=1}^{p_1} 37n \frac{1728^m}{e^{2\pi\ell}} (m\ell)^{3/4} e^{4\pi\sqrt{n}(\sqrt{\ell}-\sqrt{m})}}_{S_2'} \end{aligned}$$

from Proposition 4.1 and (14). As $e^{4\pi\sqrt{n}(\sqrt{\ell}-\sqrt{m})} = e^{\frac{4\pi\sqrt{n}(\ell-m)}{\sqrt{\ell}+\sqrt{m}}} \leq e^{\frac{4\pi\sqrt{n}(\ell-m)}{2\sqrt{m}}}$, it follows that

$$S_2' \leq 37 n m^{3/2} 1728^m e^{-2\pi\sqrt{nm}} \sum_{\ell=1}^{p_1} e^{2\pi\ell(\sqrt{n/m}-1)}.$$

Here again, under the assumptions $m \geq 4$ and (6), Subsection 4.2 gives (cf. the estimate of $|L_{22}|$)

$$S_2' \leq \frac{222}{5} n m^{3/2} e^{5.51m} e^{2\pi\left(\frac{e^{2\pi}}{1728}-1\right)\sqrt{nm}} \leq \frac{222}{5} \frac{(nm)^{3/2}}{n^{1/2}} e^{5.51m-1.38\pi\sqrt{nm}}.$$

As $m \geq 4$, the condition $n \geq 4m \ln^2 m$ implies $\sqrt{n} \geq 4\sqrt{m} \ln 2 \geq 8 \ln 2$. Then, we get

$$S_2' \leq \frac{111}{20 \ln 2} (nm)^{3/2} e^{(5.51/4 \ln 2 - 1.38\pi)\sqrt{nm}} \leq 2 \cdot 10^{-27}$$

if $nm \geq 1000$.

For $m \geq 1$, we have

$$S_3 := 124\pi \sqrt{\frac{m}{n}}\,(n,m)^{1/2} I_1(2\pi\sqrt{nm})$$

$$\leq 62\left(\frac{m}{n}\right)^{3/4} e^{2\pi\sqrt{nm}} = \frac{\sqrt{2}}{2}\frac{m^{1/4}}{n^{3/4}}\frac{e^{4\pi\sqrt{nm}}}{nm}\underbrace{62\sqrt{2}nm^{3/2}e^{-2\pi\sqrt{nm}}}_{S_3'}.$$

But $S_3' \leq 62\sqrt{2}(nm)^{3/2}e^{-2\pi\sqrt{nm}} \leq 2\cdot 10^{-80}$ if $nm \geq 1000$.

As $(1,1) = \frac{3}{4}$ and $(1,2) = -\frac{15}{32}$, the expression (13) gives

$$I_1(x) = \frac{e^x}{\sqrt{2\pi x}}\left(1 - \frac{3}{8x} + \frac{r_{I_1,2}}{x^2}\right) \tag{15}$$

with $|r_{I_1,2}| \leq \frac{15\sqrt{2}}{64}$. It follows that

$$c_m(n) = \frac{1}{\sqrt{2}}\frac{m^{1/4}}{n^{3/4}}e^{4\pi\sqrt{nm}}\left(1 - \frac{3}{32\pi}\frac{1}{\sqrt{nm}} + \frac{1}{nm}\rho_{n,m}\right)$$

with $|\rho_{n,m}| \leq \frac{15\sqrt{2}}{1024\pi^2} + S_1' + S_2' + S_3' \leq 0.0725$ for all $m, n \in \mathbb{N}$ such that $nm \geq 1000$ and $n \geq 4m\ln^2 m$.

The result obtained here is slightly weaker than the one proved in the previous section. But in the case $m = 1$, Rademacher's approach leads to a better result than the one provided by our approach, as we are going to see in the next subsection.

5.3 The case of j

According to Subsection 5.2, we have

$$c(n) = \frac{2\pi}{\sqrt{n}} I_1(4\pi\sqrt{n}) + R'(n)$$

with

$$|R'(n)| \leq \frac{124\pi}{\sqrt{n}} I_1(2\pi\sqrt{n}).$$

From (13) and (14), we get, for $n \geq 1$, $p \geq 1$,

$$c(n) = \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}}\left(\sum_{k=0}^{p-1}\frac{(-1)^k(1,k)}{(8\pi\sqrt{n})^k} + \frac{r_p(n)}{n^{p/2}}\right) \tag{16}$$

where $(1, k)$ is given by (12) and

$$|r_p(n)| = \left| \frac{r_{I_1,p}}{(4\pi\sqrt{n})^p} + 62\sqrt{2}e^{-2\pi\sqrt{n}} \right| \leq \frac{\sqrt{2}}{2}\frac{|(1, p)|}{(4\pi)^p} + 62\sqrt{2}e^{-2\pi\sqrt{n}}n^{p/2}.$$

Note that the maximum over $\mathbb{R}^+$ of the function $x \mapsto 62\sqrt{2}e^{-x}(2x)^p$ is reached at $x = p$.

For example, if $p = 3$, we have

$$|r_3(n)| = \frac{\sqrt{2}}{2}\frac{105}{128(4\pi)^3} + 62\sqrt{2}e^{-2\pi\sqrt{n}}(\sqrt{n})^3 \leq 0.0003$$

if $n \geq 10$. Hence, for all $n \geq 10$, we have

$$c(n) = \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}} \left(1 - \frac{3}{32\pi\sqrt{n}} - \frac{15}{2048\pi^2 n} + \varepsilon_n \right) \text{ with } |\varepsilon_n| \leq \frac{3 \cdot 10^{-4}}{n^{3/2}}. \tag{17}$$

The computation of the first ten $c(n)$ (cf. Section 7) shows that this remains true for $n \geq 3$.

A careful analysis of Section 4 would give a similar result. But, when dealing with parameter $p \geq 4$, we need a more precise estimate than Lemma 3.3 to establish from our approach a result similar to the one obtained from (16).

To end this section, we want to check that $c(n) \leq \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}}$ for all $n \geq 1$ as claimed in the introduction. Relation (17) implies $c(n) \leq \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}}$ for all $n \geq 10$. The computation of the first ten $c(n)$ shows that this is indeed satisfied for all $n \geq 1$. Note that this result can also be obtained from Theorem 1.1. This improves on the result $c(n) \leq 6e^{4\pi\sqrt{n}}$, for $n \geq 1$, given in [13].

6. Upper bounds for the coefficients $c_m(n)$ when $m \geq 2$

We obtained in Proposition 4.1 some general upper bounds for the $c_m(n)$ and an improvement of these bounds when $-m + 1 \leq n \leq -\frac{me^{2\pi}}{1728}$ in Proposition 4.2 and when $nm \geq 1000$ and $n \geq 4m\ln^2 m$ in Theorem 1.1. Nevertheless, some cases remain where the bounds $c_m(n) \leq 1200e^{4\pi\sqrt{m(m+n)}}$ given in [17] are better than the ones we gave until now. The aim of this section is, first, to give upper bounds for the $c_m(n)$ that improve on Mahler's ones in these remaining cases and then, to give a general statement providing the best upper bounds known for the coefficients $c_m(n)$ when $m \geq 2$.

We start by another general estimate.

Proposition 6.1. *For all $m \in \mathbb{N}^*$, $n \in \mathbb{Z}$, we have*

$$c_m(n) \leq e^{2\pi\sqrt{m(m+n)}} \left(\frac{1728}{e^{2\pi}}\right)^{m+n} \text{ if } -m+1 \leq n \leq 0$$

and

$$c_m(n) \leq e^{4\pi\sqrt{m(m+n)}} \text{ if } n \geq 0.$$

Proof. Let us recall that we have for all $y > 0$

$$c_m(n) = e^{2\pi n y} \int_{-1/2}^{1/2} j^m(x+iy) e^{-2i\pi n x} \mathrm{d}x.$$

We fix $y = \sqrt{\frac{m}{m+n}}$.

First, we assume that $n \geq 0$, Hence, we have $y \leq 1$ and Lemma 3.1 gives

$$\begin{aligned} c_m(n) &\leq e^{2\pi n\sqrt{\frac{m}{m+n}}} \left(e^{2\pi\sqrt{\frac{m+n}{m}}} + 1728 - e^{2\pi}\right)^m \\ &\leq e^{2\pi n\sqrt{\frac{m}{m+n}} + 2\pi\sqrt{m(m+n)}} \left(1 + (1728 - e^{2\pi}) e^{-2\pi\sqrt{\frac{m+n}{m}}}\right)^m \\ &\leq e^{4\pi\sqrt{m(m+n)}} \left(e^{-2\pi\sqrt{\frac{m}{m+n}}} \left(1 + (1728 - e^{2\pi}) e^{-2\pi\sqrt{\frac{m+n}{m}}}\right)\right)^m. \end{aligned}$$

Let $f : x \in]0,1] \longmapsto e^{-2\pi x}(1 + (1728 - e^{2\pi}) e^{-2\pi/x})$. The derivative of f is negative on $]0,1]$. As $\lim_{x \to 0^+} f(x) = 1$, we thus have $f(x) < 1$ for all $x \in]0,1]$. Consequently, $c_m(n) \leq e^{4\pi\sqrt{m(m+n)}} f(y)^m \leq e^{4\pi\sqrt{m(m+n)}}$ since $y \in]0,1]$.

Then, we assume $-m+1 \leq n \leq 0$. We have $y \geq 1$. We get from Lemma 3.1

$$\begin{aligned} c_m(n) &\leq e^{2\pi n\sqrt{\frac{m}{m+n}}} \left(e^{2\pi\sqrt{\frac{m}{m+n}}} + 1728 - e^{2\pi}\right)^m \\ &\leq e^{2\pi\sqrt{m(m+n)}} \left(1 + (1728 - e^{2\pi}) e^{-2\pi\sqrt{\frac{m}{m+n}}}\right)^m. \end{aligned}$$

We study $\left(1 + (1728 - e^{2\pi}) e^{-2\pi\sqrt{\frac{m}{m+n}}}\right)^m$ when $n+m$ is fixed. Therefore, we study $g : x \in [m+n, +\infty[\longmapsto \left(1 + (1728 - e^{2\pi}) e^{-2\pi\sqrt{\frac{x}{m+n}}}\right)^x$. The derivative of g is nonpositive on $[m+n, +\infty]$. Hence, $g(x) \leq g(m+n) = \left(\frac{1728}{e^{2\pi}}\right)^{m+n}$. It follows that

$$c_m(n) \leq e^{2\pi\sqrt{m(m+n)}} \left(\frac{1728}{e^{2\pi}}\right)^{m+n}.$$

Note that $e^{2\pi\sqrt{m(m+n)}} \left(\frac{1728}{e^{2\pi}}\right)^{m+n} \leq e^{4\pi\sqrt{m(m+n)}}$. □

Let $\alpha = 2 - \frac{\ln(1728)}{2\pi} + 2\sqrt{2 - \frac{\ln(1728)}{2\pi}} = 2.617\ldots$. We have, for $0 \leq n \leq \alpha m$,

$$e^{2\pi n} 1728^m \leq e^{4\pi\sqrt{m(m+n)}}.$$

Moreover, for $-m+1 \leq n \leq 0$, the bounds given in Propositions 4.1 and 4.2 are smaller than those from Proposition 6.1.

We collect the upper bounds that we obtained for the $c_m(n)$ with $m \geq 2$ in the following assertion.

Corollary 6.2. *For all $m \in \mathbb{N}$, $m \geq 2$, we have*

- *for $-m+1 \leq n \leq -m\frac{e^{2\pi}}{1728}$,*

$$c_m(n) \leq (1728 - e^{2\pi})^{m+n} \left(\frac{-n}{m+n}\right)^n \left(\frac{m}{m+n}\right)^m;$$

- *for $-m\frac{e^{2\pi}}{1728} \leq n \leq \alpha m$,*

$$c_m(n) \leq e^{2\pi n} 1728^m;$$

- *for $\alpha m \leq n \leq \max\left(\frac{1000}{m}, 4m \ln^2 m\right)$,*

$$c_m(n) \leq e^{4\pi\sqrt{m(m+n)}};$$

- *for $n \geq \max\left(\frac{1000}{m}, 4m \ln^2 m\right)$,*

$$c_m(n) \leq \frac{1}{\sqrt{2}} \frac{m^{1/4}}{n^{3/4}} e^{4\pi\sqrt{nm}} \left(1 - \frac{3}{32\pi} \frac{1}{\sqrt{nm}} + \frac{0.055}{nm}\right).$$

These upper bounds improve on Mahler's ones [17]. The reader should also keep in mind the exact expression given in Theorem 5.2 and the estimate (9).

7. Some numerical experiments

First, we deal with the case $m = 1$. In Table 1, we compare the first ten values of $c(n)$ with the bounds obtained from (17) : for $n \geq 1$, let

$$\mathrm{Min}(n) = \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}} \left(1 - \frac{3}{32\pi\sqrt{n}} - \frac{15}{2048\pi^2 n} - \frac{3 \cdot 10^{-4}}{n^{3/2}}\right)$$

and

$$\mathrm{Maj}(n) = \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}} \left(1 - \frac{3}{32\pi\sqrt{n}} - \frac{15}{2048\pi^2 n} + \frac{3 \cdot 10^{-4}}{n^{3/2}}\right).$$

For $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

Table 1. .

n	$c(n)$	$c(n) - \lceil \mathrm{Min}(n) \rceil$	$\lfloor \mathrm{Maj}(n) \rfloor - c(n)$
1	196884	382	−261
2	21493760	−217	4876
3	864299970	52342	49232
4	20245856256	582090	959653
5	333202640600	7600670	10525532
6	4252023300096	71384200	104367475
7	44656994071935	601906544	861509359
8	401490886656000	4409439269	6351251025
9	3176440229784420	29283637831	42019206145
10	22567393309593600	177496731279	254801748021

Then, we assume $m \geq 2$ and we want to check if the condition $n \geq 4m\ln^2 m$ is necessary in Theorem 1.1 and, in case of a negative answer, if the condition (6) is sufficient. It is easy to see that the condition $nm \geq 1000$ is stronger than the condition $n \geq 4m\ln^2 m$ if and only if $m \leq 7$. Hence, we only examine the case $m \geq 8$. We put, from condition (6), $f(m) = m\left(\frac{1}{2\pi}\ln((1728 - e^{2\pi})(m-1))\right)^2$ for $m \geq 2$. In Table 2, $n_{\min}(m)$ denotes the least rational integer such that $c_m(n)$ satisfies the result of Theorem 1.1 for all $n \geq n_{\min}(m)$. For $m \in \{30, 35, 40, 45, 50\}$, because of a lack of memory of our computer, we can only give a lower bound for $n_{\min}(m)$ (we believe that these bounds are indeed the correct values). We see in this table that, for none of the values of m considered, condition $n \geq 4m\ln^2 m$ is a necessary one and condition (6) is a sufficient one.

Table 2. .

m	$\lceil f(m) \rceil$	$\lceil 4m\ln^2 m \rceil$	$n_{\min}(m)$
8	17	139	60
9	20	174	71
10	22	213	83
11	25	253	95
15	36	441	146
20	51	718	215
25	67	1037	290
30	84	1389	≥ 369
35	100	1770	≥ 451
40	118	2178	≥ 537
45	135	2609	≥ 625
50	153	3061	≥ 716

The GP programs used and tables of $c(n)$ and $c_m(n)$ computed can be obtained from the web page

http:
//perso.ens-lyon.fr/nicolas.brisebarre/fuji.html.

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