Around Classical and Intuitionistic Linear Logics

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Abstract

We revisit many aspects of the syntactic relations between (variants of) classical linear logic (LL) and (variants of) intuitionistic linear logic (ILL) in the propositional setting.

On the one hand, we study different (parametric) "negative" translations from LL to ILL: their expressiveness, the relations with extensions of LL and their use in the proof theory of LL (cut elimination and focusing). In particular, this bridges the intuitionistic restriction on sequents (at most one conclusion) and the focusing property of linear logic. On the other hand, we generalise the known partial results about conservativity of LL over ILL, leading for example to a conservativity proof for LL over tensor logic (TL).

CCS Concepts • **Theory of computation → Proof theory**; **Linear logic**;

Keywords Linear logic, Intuitionistic linear logic, Tensor logic, Negative translations, Double negation, Focusing, Conservativity

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1 Introduction

Linear logic (LL) [9] has become a key element of the toolbox in different areas of computer science in particular in the theory of programming languages (type systems, denotational semantics, quantum computing, concurrency theory, implicit complexity, higher-order model checking, etc).

A key property of linear logic, stressed when it was introduced, is the ability of conciliating an involutive negation (as in classical logic) with constructivity (in terms of confluent normalization, or more generally denotational semantics, but also through disjunction and existence properties). However soon after, an intuitionistic variant ILL was also presented [11]. It is defined in sequent calculus as the restriction of LL under the *intuitionistic constraint*: "exactly one formula on the right-hand side of sequents". It relies on a restricted subset of connectives (in particular the involutive negation is lost) with a focus on linear implication $\multimap$. ILL appears to be more natural than its classical version in various contexts such as typing systems and the analysis of the λ-calculus, categorical semantics, or for game interpretations [1].

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The goal of this paper is to revisit various ways of relating LL and ILL to get a better understanding of the specificities of each of these logics. The central question is then to understand what makes these two systems different (or not).

Conservativity. In studying the relations between LL and ILL, a first natural question is the conservativity property: since any ILL formula can be seen as an LL formula (by unfolding $A \multimap B$ into $A^\perp \parr B$), can we compare provability in ILL and provability in LL? A first direction is easy since any ILL sequent can be turned into an LL sequent (by translating $\Gamma \vdash^i A$ into $\vdash \Gamma^\perp, A$) and any ILL associated proof into an LL proof. The key question is the converse: can we find ILL formulas which are provable in LL but not in ILL?

Let us first mention that, if this question looks similar to the more standard one of comparing provability in classical logic (in the LK sequent calculus for example) and in intuitionistic logic (in the LJ sequent calculus for example), it is rather different. Indeed, even with implication as the only connective, conservativity of LK over LJ fails, as shown by Peirce's law $((A \to B) \to A) \to A$. This is because the intuitionistic restriction ("at most one formula on the right-hand side of sequents") of LJ has a direct impact on the use of structural rules (contraction and weakening). However in a linear setting, these rules are already controlled: that is the purpose of the exponential connectives ! and ?. This makes LL and ILL much closer to each other than LK and LJ. The main (positive and negative) results on conservativity of LL over ILL have been obtained by H. Schellinx [22]:

- in the presence of both $\multimap$ and 0, conservativity may fail as shown by the counterexample $(X \multimap (0 \multimap Y) \multimap Y') \multimap ((X \multimap X') \multimap 0) \multimap Y'$;
- for formulas without $\multimap$ or without 0, conservativity holds;
- for formulas in the image of Girard's translation of LJ into LL ($A \to B \mapsto !A \multimap B$) [9], conservativity holds.

Parametric Negative Translation. While conservativity is about understanding the trivial embedding of ILL into LL, one can also study translations of LL into ILL.

Double-negation translations or negative translations or continuation-passing style (CPS) translations are well known tools to map classical logic into intuitionistic logic which led to computational understandings of classical logic [12, 20]. This logical analysis of control operators has early been related with linear logic [10]. Moreover it has been stressed that, while implication is usually considered as the central connective of intuitionistic logic (in relation with the λ-calculus in particular), in the context of intuitionistic logic used as a target of translations from classical logic, the key connective is negation [14, 24].

These translations have inspired works on translating LL into ILL [3, 8, 15, 18, 25] since they give ways to enforce the intuitionistic constraint on sequents. Once such a translation is settled, one can wonder what it says about the starting system, and what is its expressiveness. This last point can be stated through faithfulness analysis: are the sequents/formulas provable in the image of the translation, the images of provable sequents of the source system? or can we express more through the translation than in the original system? This question has shown to be very fruitful in the context of CPS-translations leading to the introduction of delimited control operators [7].

We follow Chang-Chaudhuri-Pfenning [3] which propose to use a parametric negative translation of LL into ILL and to analyse, for different values of the parameter, the extensions of linear logic represented by the translation (*i.e.* for which provability is exactly the provability in the image). While they use a specific extension JILL of ILL, we work here with standard ILL. The translation $(\_)^\bullet$ we use is close to the one of [4] (parametrized by an arbitrary ILL formula R) with the idea that it uses the minimal number of negations required to get a decoration of proofs of LL into proofs of ILL: during the translation, each rule is macro-expanded into a corresponding rule of ILL together with some negation rules (but no introduction of fresh cut for example is required, and in particular cut-free proofs are turned into cut-free proofs). We make a detailed analysis of the relations between LL and ILL through the translation $(\_)^\bullet$ (in particular regarding the expressiveness of the image). The extension of LL with the equivalence $\mathsf{R} \dashv\vdash \bot$ happens to play a key role.

Focusing. The negations used in the translation $(\_)^\bullet$ are those required to get a decoration of LL proofs as ILL proofs. However fewer negations are required if we just want to preserve provability. This comes from a precise analysis of polarities of formulas as defined by the theory of focusing [2]. Indeed the work on focusing and polarization [2, 10] has put forward a partition of connectives into two classes: positive and negative ones (a.k.a. synchronous and asynchronous). Each class happens to group connectives which share common proof-theoretical properties and which can cluster into macro-connectives. Concretely it led to the definition of focused systems for LL which structurally impose that consecutive connectives of a given polarity are used as a cluster. The focusing property then states the completeness of these focused systems with respect to LL provability: any LL proof can be turned into a focused one with the same conclusion.

We present the optimised translation with tensor logic (TL) [18] as target. Tensor logic is the fragment of intuitionistic linear logic which focuses on positive connectives ($\otimes$, $\oplus$ and !) together with negation (as a restriction of $\multimap$). The idea of tight links between polarization, focusing and double-negation translations is not new [10, 18, 19]. It has been developed in both classical and linear settings. We prove the stronger result that the focusing property of linear logic can be directly deduced from the optimised negative translation from LL to TL. This shows how the "at most one positive formula in the focus" constraint of focused linear systems is a particular case of the "at most one formula on the right" of intuitionistic ones.

CPS-transformations are used in particular in compiling because of the strong structural properties they provide on the generated terms [21]. Similarly focused proofs are providing structural constraints on proofs (important in their use in proof search for example). We thus show these constraints (focusing and CPS) to be of the same nature since focusing happens to be obtained from CPS.

Contributions. Starting from the definition of the parametric translation $(\_)^\bullet$ from LL to ILL, Section 2 studies the expressiveness of $(\_)^\bullet$ while making the parameter vary. We introduce and discuss the parametric logic $\mathsf{RLL}(\mathcal{R})$ (extending LL with $\mathcal{R} \dashv\vdash \bot$) which provides an upper bound on the expressiveness of $(\_)^\bullet$. We then give a sufficient condition on the parameter for this bound to be lower as well. Finally we consider some important possible values for the parameter leading to equivalences between: provability in the image of $(\_)^\bullet$, provability in RLL, provability in LL with some additional rules, and provability in LL of enriched sequents. A typical example being: $\Gamma^\bullet \vdash^i !\Phi \multimap \Phi$ in ILL (Φ atomic) $\iff \vdash^r_{?1} \Gamma$ in RLL(?1) $\iff \vdash^0 \Gamma$ in LL with $(mix_0) \iff \vdash \Gamma, ?1$ in LL. On the way, we prove some proof-theoretical properties of LL and ILL: cut-elimination of LL can be simply deduced from cut-elimination in ILL (by using $(\_)^\bullet$), and the connectives $\bot$ and ? cannot be defined in propositional ILL.

In Section 3, we generalise the work of H. Schellinx on conservativity of LL over ILL [22]. First we give a new counterexample which has implicative order 2 (the minimal possible value): $(((X \otimes \top) \,\&\, (Y \otimes \top)) \multimap 0) \multimap ((X \multimap X') \oplus (Y \multimap Y'))$. On the positive side, we extend the conservativity result for the image of Girard's translation of intuitionistic logic: we prove that it comes from the fact that $\multimap$ is only used in the shape $!\_ \multimap \_$, and we generalise this $!\_$ pattern to a more general set of formulas. Concerning the conservativity for formulas without $\multimap$ or 0, we extend it to formulas without $\_ \multimap 0$ (up to linear equivalence). This gives us the conservativity of LL over TL (with no restriction).

By choosing the parameter value for $(\_)^\bullet$ to be a fixed propositional variable, its target becomes tensor logic. In Section 4, we define the polarized optimisations $(\_)^-$ and $(\_)^+$ of $(\_)^\bullet$. By analysing the image of an LL proof through the induced translation in TL, we show the obtained proof is exactly a proof of the starting sequent in the focused system $\mathsf{LL}_{\mathrm{foc}}$, thus proving the focusing property for linear logic.

Most of the presented results are formalised in the Coq proof assistant with the help of the `Yalla` library [17]:
https://perso.ens-lyon.fr/olivier.laurent/yalla/acaill/

2 From LL to ILL

In this section, we study a parametric negation-based translation from LL to ILL. Formulas of LL [9] are denoted F, G, H, etc:

$$F ::= X \mid X^\perp \mid 1 \mid \bot \mid F \otimes F \mid F \parr F \mid 0 \mid \top \mid F \oplus F \mid F \,\&\, F \mid !F \mid ?F\ .$$

Formulas of ILL [11] are denoted I, J, K, etc:

$$I ::= X \mid 1 \mid I \otimes I \mid I \multimap I \mid 0 \mid \top \mid I \oplus I \mid I \,\&\, I \mid !I\ .$$

Sequents are denoted $\vdash \Gamma$ for LL and $\Gamma \vdash^i I$ for ILL. The logical systems we consider come with exchange rules which allow us to permute formulas in sequents. To make derivations shorter, we omit exchange rules in proof trees. They are very easy to reconstruct if needed.

2.1 Negative Translation

Let us fix an arbitrary formula R of ILL which will be used as a parameter for translating LL into ILL. We use $\_ \multimap \mathsf{R}$ as a *defined* negation connective in ILL denoted $\neg_\mathsf{R}$. The following two rules are derivable:

$$\frac{\Gamma, I \vdash^i \mathsf{R}}{\Gamma \vdash^i \neg_\mathsf{R} I}\ \neg_\mathsf{R} R \qquad\qquad \frac{\Gamma \vdash^i I}{\Gamma, \neg_\mathsf{R} I \vdash^i \mathsf{R}}\ \neg_\mathsf{R} L$$

Lemma 2.1. *In* ILL*, for all formulas I, J, I' and J', we have:*

(i) $I, \neg_\mathsf{R} I' \vdash^i \mathsf{R}$ *and* $J, \neg_\mathsf{R} J' \vdash^i \mathsf{R}$ *implies* $I \otimes J, \neg_\mathsf{R}(I' \otimes J') \vdash^i \mathsf{R}$

(ii) *$I, \neg_{\mathsf{R}} I' \vdash^i \mathsf{R}$ and $J, \neg_{\mathsf{R}} J' \vdash^i \mathsf{R}$ implies $I \oplus J, \neg_{\mathsf{R}}(I' \oplus J') \vdash^i \mathsf{R}$*

Definition 2.2 (Translation of Formulas)**.** The translation $F^\bullet$ of a formula F of LL is a formula of ILL:

$$
\begin{array}{rclrcl}
X^\bullet & = & \neg_{\mathsf{R}} X & (X^\perp)^\bullet & = & X \\
1^\bullet & = & \neg_{\mathsf{R}} 1 & \perp^\bullet & = & 1 \\
0^\bullet & = & \neg_{\mathsf{R}} 0 & \top^\bullet & = & 0 \\
(F \otimes G)^\bullet & = & \neg_{\mathsf{R}}(\neg_{\mathsf{R}} F^\bullet \otimes \neg_{\mathsf{R}} G^\bullet) & (F \parr G)^\bullet & = & F^\bullet \otimes G^\bullet \\
(F \oplus G)^\bullet & = & \neg_{\mathsf{R}}(\neg_{\mathsf{R}} F^\bullet \oplus \neg_{\mathsf{R}} G^\bullet) & (F \,\&\, G)^\bullet & = & F^\bullet \oplus G^\bullet \\
(!F)^\bullet & = & \neg_{\mathsf{R}} ! \neg_{\mathsf{R}} F^\bullet & (?F)^\bullet & = & ! \neg_{\mathsf{R}} \neg_{\mathsf{R}} F^\bullet
\end{array}
$$

When the value of R needs to be explicitly mentioned, we use the notation $(\_)^{\bullet[\mathsf{R}]}$. Almost the same translation is considered in [4, 18]. The main difference is in $(?F)^\bullet$ (to be discussed later).

Lemma 2.3 (Translation and Dual)**.**
For all F, $\neg_{\mathsf{R}} F^\bullet, \neg_{\mathsf{R}}(F^\perp)^\bullet \vdash^i \mathsf{R}$ is derivable in ILL.

Proposition 2.4 (Translation of Proofs)**.**
If $\vdash \Gamma$ is provable in LL then $\Gamma^\bullet \vdash^i \mathsf{R}$ is provable in ILL.

The resulting proof is obtained by "decorating" (in the spirit of [23]) the original one. This means that the structure of the proof is preserved: each LL rule is macro-expanded into a corresponding rule in ILL together with some additional $(\neg R)$ and $(\neg L)$ rules. Only the case of the (cut) rule is more involved since a call to Lemma 2.3 is required as a medium between the two cut proofs (but still a cut in the source induces a cut in the target). Concerning the additive connectives, it is also possible to keep a decoration while decreasing the number of negations by defining $(F \oplus G)^\bullet = F^\bullet \,\&\, G^\bullet$. However this would break a polarity policy which is central in Section 4 for removing more negations. Concerning the exponentials, for any I in ILL, we have $!I \vdash^i !\neg_{\mathsf{R}}\neg_{\mathsf{R}} I$. As a consequence, one can optimise $(?F)^\bullet$ into $(?F)^\bullet = !F^\bullet$ as it is defined in [4, 18]. This would preserve the validity of Proposition 2.4 (since $!F^\bullet \vdash^i !\neg_{\mathsf{R}}\neg_{\mathsf{R}} F^\bullet$), except that we would have to eliminate cuts in LL before translating. Indeed this optimisation breaks Lemma 2.3. Lemma 2.3 is also necessary in the presence of quantifiers for Lemma 5.1 to hold.

Following Proposition 2.4, a natural question is then to understand if more than LL can be encoded into ILL through this translation. We study this now while making the parameter R vary.

We start with a result about the (mix_n) rules ($n \geq 0$):

$$
\frac{\vdash \Gamma_1 \qquad \cdots \qquad \vdash \Gamma_n}{\vdash \Gamma_1, \cdots, \Gamma_n}\ mix_n \qquad \left(\text{in particular } \frac{}{\vdash}\ mix_0\right)
$$

We use the notation $\vdash^n \Gamma$ for sequents in LL extended with (mix_n) (and $\vdash^{02} \Gamma$ for LL with both (mix_0) and (mix_2)). If F is a formula, $\bigotimes_n F$ is defined by: $\bigotimes_0 F = 1$ and $\bigotimes_{n+1} F = F \otimes \bigotimes_n F$.

Lemma 2.5. *Let R be an intuitionistic linear formula, for all $n \geq 0$, if $\bigotimes_n \mathsf{R} \vdash^i \mathsf{R}$ is provable in ILL then LL with (mix_n) can be translated by $(\_)^\bullet$ into ILL.*

2.2 Response Linear Logic

We introduce an extension of linear logic which will happen to be directly related with the image of the translation $(\_)^\bullet$. Given a formula $\mathcal{R}$ of LL, *response linear logic* $\mathsf{RLL}(\mathcal{R})$ is the extension of LL incorporating the two axioms:

$$
\frac{}{\vdash^r_{\mathcal{R}} \mathcal{R}^\perp}\ \perp_{\mathcal{R}} \qquad\qquad \frac{}{\vdash^r_{\mathcal{R}} \mathcal{R}, 1}\ 1_{\mathcal{R}}
$$

This corresponds to adding the equivalence $\mathcal{R} \dashv\vdash \perp$ to LL. Be aware that, because of the introduction of non-trivial axioms in $\mathsf{RLL}(\mathcal{R})$, cut elimination does not hold in general. Sequents of $\mathsf{RLL}(\mathcal{R})$ are denoted $\vdash^r_{\mathcal{R}} \Gamma$.

Lemma 2.6 (Substitution)**.** *If $\vdash^r_{\mathcal{R}} \Gamma$ is provable in $\mathsf{RLL}(\mathcal{R})$ then, for all F and X, $\vdash^r_{\mathcal{R}[F/X]} \Gamma[F/X]$ is provable in $\mathsf{RLL}(\mathcal{R}[F/X])$.*

2.2.1 Alternative Presentations

If the definition of $\mathsf{RLL}(\mathcal{R})$ above does not satisfy cut elimination, it is possible to give alternative presentations with equivalent provability power but better proof-theoretical properties.

Lemma 2.7. *$\vdash^r_{\mathcal{R}} \Gamma$ in $\mathsf{RLL}(\mathcal{R})$ if and only if $\vdash \Gamma, ?\mathcal{R}, ?(\mathcal{R}^\perp \otimes \perp)$ in LL.*

This presentation has the advantage of having an admissible cut rule since cut is admissible in LL and we can build:

$$
\dfrac{\dfrac{\vdash \Gamma, A, ?\mathcal{R}, ?(\mathcal{R}^\perp \otimes \perp) \qquad \vdash \Delta, A^\perp, ?\mathcal{R}, ?(\mathcal{R}^\perp \otimes \perp)}{\vdash \Gamma, \Delta, ?\mathcal{R}, ?\mathcal{R}, ?(\mathcal{R}^\perp \otimes \perp), ?(\mathcal{R}^\perp \otimes \perp)}\ cut}{\vdash \Gamma, \Delta, ?\mathcal{R}, ?(\mathcal{R}^\perp \otimes \perp)}\ ?c
$$

We now consider some more specific values of $\mathcal{R}$.

Lemma 2.8. *$\vdash^r_{?\mathcal{R}} \Gamma$ in $\mathsf{RLL}(?\mathcal{R})$ if and only if $\vdash \Gamma, ?\mathcal{R}$ in LL.*

2.2.2 The Particular Case $\mathcal{R} = \underline{\mathsf{R}}$

We denote by $\underline{I}$ the canonical embedding of the ILL formula I into LL based on $\underline{J \multimap K} = (\underline{J})^\perp \parr \underline{K}$. Given a formula R of ILL, it is possible to relate $\mathsf{RLL}(\underline{\mathsf{R}})$ and the image of the translation $(\_)^\bullet$.

Lemma 2.9. *Given a formula F of LL, $\vdash^r_{\underline{\mathsf{R}}} \underline{F^\bullet}, F$ is provable in $\mathsf{RLL}(\underline{\mathsf{R}})$.*

Proposition 2.10. *If $\Gamma^\bullet \vdash^i \mathsf{R}$ is provable in ILL then $\vdash^r_{\underline{\mathsf{R}}} \Gamma$ is provable in $\mathsf{RLL}(\underline{\mathsf{R}})$.*

Proof. We have $\vdash (\underline{\Gamma^\bullet})^\perp, \underline{\mathsf{R}}$ provable in LL thus in $\mathsf{RLL}(\underline{\mathsf{R}})$. Then if $\Gamma = F_1, \ldots, F_k$, we introduce a cut with $\vdash^r_{\underline{\mathsf{R}}} \underline{\mathsf{R}}^\perp$ and k cuts with proofs from Lemma 2.9. □

We have presented a general pattern:

$$
\vdash \Gamma \text{ in } \mathsf{LL} \quad \Longrightarrow \quad \Gamma^\bullet \vdash^i \mathsf{R} \text{ in } \mathsf{ILL} \quad \Longrightarrow \quad \vdash^r_{\underline{\mathsf{R}}} \Gamma \text{ in } \mathsf{RLL}(\underline{\mathsf{R}})
$$

valid for any R. We thus have general lower and upper bounds on the expressiveness of $(\_)^\bullet$.

2.3 From $\mathsf{RLL}(\underline{\mathsf{R}})$ to ILL

We have seen $\Gamma^\bullet \vdash^i \mathsf{R}$ in $\mathsf{ILL} \Longrightarrow \vdash^r_{\underline{\mathsf{R}}} \Gamma$ in $\mathsf{RLL}(\underline{\mathsf{R}})$. We can try to find values R such that the converse holds. In this case, the upper bound on the expressiveness of $(\_)^\bullet$ will happen to be an exact characterisation. In order to translate $\vdash^r_{\underline{\mathsf{R}}} \Gamma$ into ILL with $(\_)^\bullet$, we have to extend Proposition 2.4 with the translations through $(\_)^\bullet$ of the two additional axioms. This requires the provability of:

$$
(\underline{\mathsf{R}}^\perp)^\bullet \vdash^i \mathsf{R} \qquad\qquad \text{and} \qquad\qquad (\underline{\mathsf{R}})^\bullet, \mathsf{R} \vdash^i \mathsf{R}\ .
$$

We do not currently know which are precisely the formulas R making these two sequents provable, but here are some partial positive and negative results.

On the negative side, one can see that counterexamples exist: $X \multimap X$, $X \multimap Y$, $X \multimap 1$, $X \,\&\, Y$, $(X \,\&\, X') \multimap Y$ or $!X$ for example.

On the positive side, we introduce a notion of *purely positive affine formula* (rejecting $!$, $\multimap$ and $\&$):

$$
E ::= X \mid 1 \mid E \otimes E \mid 0 \mid E \oplus E \mid \top\ .
$$

Note that $\underline{E} = E$ since it does not contain the connective $\multimap$.

The properties $(E^\perp)^{\bullet[E]} \vdash^i E$ and $E^{\bullet[E]}, E \vdash^i E$ are rather badly behaved for a proof by induction on E. This is why we are moving to more general statements first.

Lemma 2.11. *If E is a purely positive affine formula and R is an intuitionistic linear formula, $(E^\perp)^{\bullet[\mathsf{R}]} \vdash^i E$ and $E \vdash^i \neg_{\mathsf{R}} E^{\bullet[\mathsf{R}]}$ are provable in* ILL.

Proposition 2.12. *Let E be a purely positive affine formula, $\Gamma^{\bullet[E]} \vdash^i E$ in* ILL *if and only if $\vdash^r_E \Gamma$ in* RLL(E).

Let us now consider some of the formulas E, and try to see whether we can go further than Section 2.2.1 in understanding RLL(E) and $(\_)^{\bullet[E]}$.

2.3.1 R = 1

We are going to use the (mix_0) and (mix_2) rules in LL to characterise RLL(1) as suggested by Lemma 2.5.

Lemma 2.13. *If $\vdash^{02} \Gamma$ is provable in* LL *with (mix_0) and (mix_2) then $!\mathsf{R}, \Gamma^{\bullet[!\mathsf{R}]} \vdash^i !\mathsf{R}$ is provable in* ILL *for any* R.

Proposition 2.14. *The following statements are equivalent:*

(i) $\vdash^{02} \Gamma$ *in* LL *with (mix_0) and (mix_2)*
(ii) $\Gamma^{\bullet[1]} \vdash^i 1$ *in* ILL
(iii) $\vdash^r_1 \Gamma$ *in* RLL(1)

Proof. We prove the following three implications:

- *(i)⇒(ii)*: we apply Lemma 2.5 or we can use Lemma 2.13 with $\mathsf{R} = \top$ (since $1 \dashv\vdash\, !\top$ in ILL).
- *(ii)⇒(iii)*: this is Proposition 2.10.
- *(iii)⇒(i)*: the two additional rules of RLL(1) are derivable in LL extended with (mix_0) and (mix_2). □

Using linear equivalence with 1, we immediately get the same results for $\mathsf{R} = !\top$, $\mathsf{R} = !1$, $\mathsf{R} = 1 \multimap 1$, etc.

2.3.2 R = ⊤

RLL(⊤) is uninteresting since provability is trivial: any sequent becomes provable.

$$\dfrac{\dfrac{}{\vdash^r_\top 0}\,\bot_{\mathcal{R}} \qquad \dfrac{}{\vdash^r_\top \top, \Gamma}\,\top}{\vdash^r_\top \Gamma}\,cut$$

2.3.3 R = 0

Affine logic AL is LL extended with the rule: $\dfrac{\vdash^a \Gamma}{\vdash^a \Gamma, F}\,wk$.

Proposition 2.15. *The following statements are equivalent:*

(i) $\vdash^a \Gamma$ *in* AL
(ii) $\Gamma^{\bullet[0]} \vdash^i 0$ *in* ILL
(iii) $\vdash^r_0 \Gamma$ *in* RLL(0)

Proof. We prove the following three implications:

- *(i)⇒(ii)*: this is Proposition 2.4 with $\mathsf{R} = 0$ for the standard LL rules. We then have to consider:

$$\dfrac{\overset{\text{IH}}{\Gamma^{\bullet[0]} \vdash^i 0} \qquad \dfrac{}{F^{\bullet[0]}, 0 \vdash^i 0}\,0L}{\Gamma^{\bullet[0]}, F^{\bullet[0]} \vdash^i 0}\,cut$$

- *(ii)⇒(iii)*: this is Proposition 2.10.
- *(iii)⇒(i)*: the two additional rules of RLL(0) are derivable in LL extended with (wk). □

2.3.4 R = Φ

We consider Φ to be a(n intuitionistic) propositional variable.

Proposition 2.16. *The following statements are equivalent:*

(i) $\vdash \Gamma$ *in* LL
(ii) *for all* R, $\Gamma^{\bullet[\mathsf{R}]} \vdash^i \mathsf{R}$ *in* ILL
(iii) *there exists $\Phi \notin \Gamma$ such that $\Gamma^{\bullet[\Phi]} \vdash^i \Phi$ in* ILL
(iv) *there exists $\Phi \notin \Gamma$ such that $\vdash^r_\Phi \Gamma$ in* RLL(Φ)
(v) $\vdash^r_\bot \Gamma$ *in* RLL$(\bot)$

Proof. We prove the following five implications:

- *(i)⇒(ii)*: this is Proposition 2.4.
- *(ii)⇒(iii)*: let Φ be a propositional variable not free in Γ, we simply instantiate R with Φ.
- *(iii)⇒(iv)*: this is Proposition 2.10 with $\underline{\Phi} = \Phi$.
- *(iv)⇒(v)*: by Lemma 2.6, $\vdash^r_\bot \Gamma$ in RLL$(\bot)$ since $\Phi \notin \Gamma$.
- *(v)⇒(i)*: the two additional rules of RLL$(\bot)$ are derivable in LL. □

Using linear equivalence, we immediately get the same results for $1 \multimap \Phi$, etc.

We can also apply this result to prove cut elimination in LL from the corresponding result for ILL:

Corollary 2.17 (Cut Elimination)**.** *Cut elimination in* ILL *entails cut elimination in* LL.

Proof. If $\vdash \Gamma, F$ and $\vdash \Delta, F^\perp$ are provable in LL, by Proposition 2.4, we have $\Gamma^\bullet, F^\bullet \vdash^i \mathsf{R}$ and $\Delta^\bullet, (F^\perp)^\bullet \vdash^i \mathsf{R}$ and thus $\Gamma^\bullet, \Delta^\bullet \vdash^i \mathsf{R}$ in ILL by using Lemma 2.3. By cut elimination in ILL, one can build a cut-free proof of $\Gamma^\bullet, \Delta^\bullet \vdash^i \mathsf{R}$, and thus a cut-free proof of $\vdash \underline{\Gamma^\bullet}^\perp, \underline{\Delta^\bullet}^\perp, \mathsf{R}$ in LL. We choose R to be a fresh propositional variable Φ and, by substitution, we have $\vdash \underline{\Gamma^\bullet}^\perp[^\bot/_\Phi], \underline{\Delta^\bullet}^\perp[^\bot/_\Phi], \bot$.

We can check by induction on F that $\underline{F^\bullet}^\perp[^\bot/_\Phi]$ is obtained from F by adding some $\_ \parr \bot$ and $\_ \otimes 1$ in it. We conclude by induction on the proof of $\vdash \underline{\Gamma^\bullet}^\perp[^\bot/_\Phi], \underline{\Delta^\bullet}^\perp[^\bot/_\Phi], \bot$ that $\vdash \Gamma, \Delta$ in LL. □

2.4 Relating RLL($\mathcal{R}$) and ILL

We have seen in Section 2.3.4 that, in the study of $(\_)^\bullet$, not only RLL($\underline{\mathsf{R}}$) naturally appears, but also RLL($\mathcal{R}$) with $\mathcal{R}$ not an intuitionistic formula ($\bot$ for example in Proposition 2.16). This really adds something since, for example, $\bot$ could not be replaced directly by an intuitionistic formula: there is no R in ILL such that $\underline{\mathsf{R}} \dashv\vdash \bot$ (Lemma 2.19).

Lemma 2.18. *It is not possible to have both $\vdash^0 (\underline{I})^\perp$ and $\vdash^0 !F, \underline{I}$ provable in* LL *with (mix_0), for any I of* ILL *and any F of* LL.

Proof. By induction on I. The key cases are:

- If both $\vdash^0 !F, \underline{J} \otimes \underline{K}$ and $\vdash^0 (\underline{J})^\perp \parr (\underline{K})^\perp$ are provable in LL with (mix_0) then $\vdash^0 (\underline{J})^\perp, (\underline{K})^\perp$ as well. Moreover either $\vdash^0 !F, \underline{J}$ and $\vdash^0 \underline{K}$, or $\vdash^0 !F, \underline{K}$ and $\vdash^0 \underline{J}$ are provable. In the first case for example, we can build:

$$\dfrac{\vdash^0 (\underline{J})^\perp, (\underline{K})^\perp \qquad \vdash^0 \underline{K}}{\vdash^0 (\underline{J})^\perp}\,cut$$

and we apply the induction hypothesis on J.

- If both $\vdash^0 !F, (\underline{J})^\perp \parr \underline{K}$ and $\vdash^0 \underline{J} \otimes (\underline{K})^\perp$ are provable in LL with (mix_0) then $\vdash^0 !F, (\underline{J})^\perp, \underline{K}$, $\vdash^0 \underline{J}$, and $\vdash^0 (\underline{K})^\perp$ as well. So that we can build:

$$\dfrac{\vdash^0 !F, (\underline{J})^\perp, \underline{K} \qquad \vdash^0 \underline{J}}{\vdash^0 !F, \underline{K}}\ cut$$

and we apply the induction hypothesis on K. □

Lemma 2.19. *It is not possible to have both* $\vdash (\underline{I})^\perp$ *and* $\vdash 1, \underline{I}$ *provable in* LL*, for any* I *of* ILL*.*

As a consequence, there is also no formula W in ILL such that for any I, $\underline{W[^I/_X]} \dashv\vdash ?\underline{I}$ in LL (otherwise one would have $\underline{W[^0/_X]} \dashv\vdash ?0 \dashv\vdash \bot$ in LL): this means the connective ? is not definable in ILL.

Let us now consider other values of $\mathcal{R}$ which are also not purely positive affine formulas, or even not equivalent to any ILL formula.

2.4.1 $\mathcal{R} = ?1$ and $\mathsf{R} = !\Phi \multimap \Phi$

As we have seen for $\bot$, $?1$ is also a formula which is "out of the scope" of ILL: there is no R in ILL such that $\underline{\mathsf{R}} \dashv\vdash ?1$ in LL (Lemma 2.20).

Lemma 2.20. *It is not possible to have both* $\vdash (\underline{I})^\perp, ?1$ *and* $\vdash !\bot, \underline{I}$ *provable in* LL*, for any* I *of* ILL*.*

We can however relate $(\_)^\bullet$ and RLL(?1):

Proposition 2.21. *The following statements are equivalent:*

(i) $\vdash^0 \Gamma$ *in* LL *with* (mix_0)
(ii) *for all* R *provable in* ILL, $\Gamma^{\bullet[\mathsf{R}]} \vdash^i \mathsf{R}$ *in* ILL
(iii) *there exists* $\Phi \notin \Gamma$ *such that* $\Gamma^{\bullet[!\Phi \multimap \Phi]} \vdash^i !\Phi \multimap \Phi$ *in* ILL
(iv) *there exists* $\Phi \notin \Gamma$ *such that* $\vdash^r_{?\Phi^\perp \parr \Phi} \Gamma$ *in* RLL$(?\Phi^\perp \parr \Phi)$
(v) $\vdash^r_{?1} \Gamma$ *in* RLL(?1)
(vi) $\vdash \Gamma, ?1$ *in* LL

2.4.2 $\mathcal{R} = !\bot$ and $\mathsf{R} = !\Phi$

First note there is no R in ILL such that $\underline{\mathsf{R}} \dashv\vdash !\bot$ in LL (Lemma 2.23).

Lemma 2.22. *It is not possible to have both* $\vdash (\underline{\Gamma})^\perp$ *provable in* LL *and* $\vdash^0 \underline{I}$ *provable in* LL *with* (mix_0) *for each* $I \in \Gamma$*, for any* Γ *of* ILL*.*

Proof. By induction on the proof of $\vdash (\underline{\Gamma})^\perp$ in LL, we look at the last rule. The key case is the $(\otimes)$ rule. We have $\Gamma = \Gamma', \Gamma''$ with $\vdash (\underline{\Gamma'})^\perp, \underline{J}$ and $\vdash (\underline{\Gamma''})^\perp, (\underline{K})^\perp$, and $\vdash^0 (\underline{J})^\perp, \underline{K}$, as well as $\vdash^0 \underline{I}$ for any I in Γ', Γ'' thus, using cuts, we can build a proof of $\vdash^0 \underline{J}$ and then a proof of $\vdash^0 \underline{K}$. We conclude with the induction hypothesis applied to the proof of $\vdash (\underline{\Gamma''})^\perp, (\underline{K})^\perp$. □

Lemma 2.23. *It is not possible to have both* $\vdash (\underline{I})^\perp, !\bot$ *and* $\vdash ?1, \underline{I}$ *provable in* LL*, for any* I *of* ILL*.*

It is however possible to relate $(\_)^\bullet$ and RLL$(!\bot)$, and to give an alternative characterisation of RLL$(!\bot)$. The system $\mathsf{LL}^{!\bot}$ is the extension of LL obtained by adding to the LL rules, the following one: $\dfrac{\vdash^{!\bot} \Gamma \qquad \vdash^{02} \Delta}{\vdash^{!\bot} \Gamma, \Delta}\ mix^{!\bot}$.

Lemma 2.24. *If* $\vdash^{!\bot} \Gamma$ *is provable in* $\mathsf{LL}^{!\bot}$ *then* $\vdash^{02} \Gamma$ *is provable in* LL *with both* (mix_0) *and* (mix_2)*.*

$\mathsf{LL}^{!\bot}$ is thus intermediate between LL with (mix_2) and LL with both (mix_0) and (mix_2), since (mix_2) is derivable but (mix_0) is constrained to occur only above a $(mix^{!\bot})$ rule. Indeed, we have:

$$\dfrac{\vdash^{!\bot} \Gamma \qquad \dfrac{\vdash^{!\bot} \Delta}{\vdash^{02} \Delta}\ \text{Lemma 2.24}}{\vdash^{!\bot} \Gamma, \Delta}\ mix^{!\bot}$$

but $\vdash^{!\bot}$ is not provable in $\mathsf{LL}^{!\bot}$.

Proposition 2.25 (Cut elimination in $\mathsf{LL}^{!\bot}$)**.**
$\mathsf{LL}^{!\bot}$ *has the cut-elimination property.*

Proposition 2.26. *The following statements are equivalent:*

(i) $\vdash^{!\bot} \Gamma$ *in* $\mathsf{LL}^{!\bot}$
(ii) *for all* R, $\Gamma^{\bullet[!\mathsf{R}]} \vdash^i !\mathsf{R}$ *in* ILL
(iii) *there exists* $\Phi \notin \Gamma$ *such that* $\Gamma^{\bullet[!\Phi]} \vdash^i !\Phi$ *in* ILL
(iv) *there exists* $\Phi \notin \Gamma$ *such that* $\vdash^r_{!\Phi} \Gamma$ *in* RLL$(!\Phi)$
(v) $\vdash^r_{!\bot} \Gamma$ *in* RLL$(!\bot)$

Proof. The main implications are:

- *(i)⇒(ii)*: this is Proposition 2.4 for the standard LL rules. We then have to consider:

$$\dfrac{\overset{\text{IH}}{\Gamma^{\bullet[!\mathsf{R}]} \vdash^i !\mathsf{R}} \qquad \overset{\text{Lemma 2.13}}{!\mathsf{R}, \Delta^{\bullet[!\mathsf{R}]} \vdash^i !\mathsf{R}}}{\Gamma^{\bullet[!\mathsf{R}]}, \Delta^{\bullet[!\mathsf{R}]} \vdash^i !\mathsf{R}}\ cut$$

- *(v)⇒(i)*: the additional rules of RLL$(!\bot)$ are derivable in $\mathsf{LL}^{!\bot}$:

$$\dfrac{\dfrac{}{\vdash^{!\bot} 1}\ 1}{\vdash^{!\bot} ?1}\ ?d \qquad\qquad \dfrac{\dfrac{}{\vdash^{!\bot} 1}\ 1 \qquad \dfrac{\dfrac{\dfrac{}{\vdash^{02}}\ mix_0}{\vdash^{02} \bot}\ \bot}{\vdash^{02} !\bot}\ !}{\vdash^{!\bot} 1, !\bot}\ mix^{!\bot}$$

□

In the derivations just above, the fact that we allow not only (mix_0) but also (mix_2) in the right premise of the $(mix^{!\bot})$ rule does not seem to be required. However it happens to be necessary for the cut-elimination property to hold in $\mathsf{LL}^{!\bot}$ (Proposition 2.25).

In [3], the cases $\mathcal{R} = 1$, $\mathcal{R} = \bot$, $\mathcal{R} = \top$, $\mathcal{R} = 0$, $\mathcal{R} = ?1$ and $\mathcal{R} = !\bot$ are considered in the slightly different context of the logic JILL. While the authors give the same characterisation as here for RLL$(\mathcal{R})$ in terms of extensions of LL in the first five cases, they left open the case of RLL$(!\bot)$.

2.4.3 $\mathcal{R} = ?\bigotimes_n \bot$ and $\mathsf{R} = !((\bigotimes_n \Phi) \multimap \Phi) \multimap \Phi$

We have already seen characterisations of LL extended with (mix_0) in Section 2.4.1 and of LL extended with both (mix_0) and (mix_2) in Section 2.3.1. We turn our attention to LL extended with (mix_n).

Proposition 2.27. *For any* $n \geq 0$*, the following statements are equivalent:*

(i) $\vdash^n \Gamma$ *in* LL *with* (mix_n)
(ii) *for all* R *such that* $\bigotimes_n \mathsf{R} \vdash^i \mathsf{R}$ *is provable in* ILL, $\Gamma^{\bullet[\mathsf{R}]} \vdash^i \mathsf{R}$ *in* ILL
(iii) *there exists* $\Phi \notin \Gamma$ *such that* $\Gamma^{\bullet[!((\bigotimes_n \Phi) \multimap \Phi) \multimap \Phi]} \vdash^i !((\bigotimes_n \Phi) \multimap \Phi) \multimap \Phi$ *in* ILL
(iv) *there exists* $\Phi \notin \Gamma$ *such that* $\vdash^r_{?((\bigotimes_n \Phi) \otimes \Phi^\perp) \parr \Phi} \Gamma$ *is provable in* RLL$(?((\bigotimes_n \Phi) \otimes \Phi^\perp) \parr \Phi)$
(v) $\vdash^r_{?\bigotimes_n \bot} \Gamma$ *in* RLL$(?\bigotimes_n \bot)$
(vi) $\vdash \Gamma, ?\bigotimes_n \bot$ *in* LL

The key point is to use $(?((\bigotimes_n \Phi) \otimes \Phi^\perp) \parr \Phi)[^\bot/_\Phi] \dashv\vdash ?\bigotimes_n \bot$ in *(iv)⇒(v)*.

Let us consider a few particular cases. For $n = 0$, $!(1 \multimap \Phi) \multimap \Phi \dashv\vdash !\Phi \multimap \Phi$ and we find back the results of Section 2.4.1. For $n = 1$, $!((\Phi \otimes 1) \multimap \Phi) \multimap \Phi \dashv\vdash \Phi$ and we find back the results of Section 2.3.4. For $n = 2$, $\mathcal{R} = ?(\bot \otimes \bot)$ and $\mathsf{R} = !((\Phi \otimes \Phi) \multimap \Phi) \multimap \Phi$

give characterisations of LL with (mix_2) alone (in contrast with Section 2.3.1).

2.4.4 $\mathcal{R} = 1$, and $\mathsf{R} = \Phi \multimap \Phi$ or $\mathsf{R} = !\Phi \multimap 1$ or $\mathsf{R} = !((!\Phi \multimap \Phi) \multimap (!\Phi \multimap \Phi) \multimap \Phi) \multimap \Phi$

We can revisit Section 2.3.1 with some variations on R.

Proposition 2.28. *The following statements are equivalent:*

(i) $\vdash^{02} \Gamma$ *in* LL *with (mix_0) and (mix_2)*
(ii) $\Gamma^{\bullet[\Phi\multimap\Phi]} \vdash^{i} \Phi \multimap \Phi$ *in* ILL *with* $\Phi \notin \Gamma$
(iii) $\Gamma^{\bullet[!\Phi\multimap 1]} \vdash^{i} !\Phi \multimap 1$ *in* ILL *with* $\Phi \notin \Gamma$
(iv) $\Gamma^{\bullet[!((!\Phi\multimap\Phi)\multimap(!\Phi\multimap\Phi)\multimap\Phi)\multimap\Phi]} \vdash^{i} !((!\Phi \multimap \Phi) \multimap (!\Phi \multimap \Phi) \multimap \Phi) \multimap \Phi$ *in* ILL *with* $\Phi \notin \Gamma$
(v) $\vdash \Gamma, ?(?1 \otimes ?1)$ *in* LL

3 Back to LL: Conservativity

Through the translation $I \mapsto \underline{I}$, ILL can be seen as a subsystem of LL (simply replace each $I \multimap J$ by $I^{\perp} \parr J$). We often identify I and $\underline{I}$ in the present section. Whether LL is a conservative extension of ILL (*i.e.* formulas or sequents from ILL provable in LL are the same as in ILL) has been studied in [22]. The property does not hold in full generality as shown by H. Schellinx's counterexample [22]:

$$(X \multimap (0 \multimap Y) \multimap Y') \multimap ((X \multimap X') \multimap 0) \multimap Y' \ .$$

One can give more compact variants of this example by using $\otimes$ and $\top$: $((X \multimap X') \multimap 0) \multimap (X \otimes (0 \multimap Y))$ and $((X \multimap X') \multimap 0) \multimap (X \otimes \top)$ [13] (possibly the shortest existing counterexample). If one considers the *implicative order* of formulas (0 for formulas without $\multimap$, and add 1 each time you go to the left of a $\multimap$ in the formula tree), these formulas have order 3. One can also give a new counterexample of order 2 (as we will see below, this is minimal since there is no counterexample of order 1):

$$(((X \otimes \top) \,\&\, (Y \otimes \top)) \multimap 0) \multimap ((X \multimap X') \oplus (Y \multimap Y')) \ .$$

In [22], two constraints are given on formulas in order to ensure conservativity. We are going to generalise both. Another result is proposed in [25, Exercise 2 page 39]: *for all* Γ *in* ILL, $\vdash \underline{\Gamma}^{\perp}, 0$ *in* LL *if and only if* $\Gamma \vdash^{i} 0$ *in* ILL. However if I is a formula such that $\vdash \underline{I}$ in LL and $\nvdash^{i} I$ in ILL (such as H. Schellinx's counterexample), then $I \multimap 0 \vdash^{i} 0$ is not provable in ILL while $\vdash \underline{I} \otimes \top, 0$ is provable in LL.

3.1 Looking to the Left of $\multimap$

In [22], it is proved that conservativity holds for the image of Girard's translation of intuitionistic logic into linear logic. That is for formulas in the following grammar:

$$\mathcal{G} ::= X \mid !\mathcal{G} \multimap \mathcal{G} \mid 0 \mid \mathcal{G} \,\&\, \mathcal{G} \mid !\mathcal{G} \oplus !\mathcal{G} \ .$$

Proposition 3.1 ([22, Corollary 3.7])**.** *A formula in the grammar* $\mathcal{G}$ *is provable in* LL *if and only if it is provable in* ILL*.*

We are going to show that the key point here is the constraint induced on the left-hand side of $\multimap$. Moreover we generalise the shape $!\_ \multimap \_$ into $O \multimap \_$ where O is what we call a *!-like formula* :

$$O ::= X \mid 1 \mid O \otimes O \mid 0 \mid O \oplus O \mid O \,\&\, I \mid I \,\&\, O \mid !I \ .$$

Lemma 3.2. *If* $\vdash \underline{\Gamma}^{\perp}, \underline{\Omega}, \underline{I}$ *is provable in* LL *(where* Ω *contains !-like formulas only, and all sub-formulas of* Γ *and* I *of the shape* $J \multimap K$ *are such that* J *is !-like), then* $\Gamma \vdash^{i} I$ *is provable in* ILL*, and if* Ω *is not empty then* $\Gamma, \Sigma \vdash^{i} I$ *in* ILL *for any* Σ*.*

Proof. By induction on a cut-free proof of $\vdash \underline{\Gamma}^{\perp}, \underline{\Omega}, \underline{I}$ with atomic axioms. The key cases are:

- ($\parr$) rule:
  - If $\underline{I} = F \parr G$, then $I = J \multimap K$ with $\underline{J} = F^{\perp}$ and $\underline{K} = G$. We have:
  $$\dfrac{\overset{\text{IH}}{\Gamma, J, (\Sigma) \vdash^{i} K}}{\Gamma, (\Sigma) \vdash^{i} J \multimap K} \multimap R$$
  - If $\underline{\Gamma}^{\perp}$ contains $F \parr G$ then $\Gamma = \Gamma', J \otimes K$ with $\underline{J} = F^{\perp}$ and $\underline{K} = G^{\perp}$. We have:
  $$\dfrac{\overset{\text{IH}}{\Gamma', J, K, (\Sigma) \vdash^{i} I}}{\Gamma', J \otimes K, (\Sigma) \vdash^{i} I} \otimes L$$
  - Finally, by definition of !-like, $\underline{\Omega}$ cannot contain a formula of the shape $F \parr G$.
- ($\otimes$) rule:
  - If $\underline{I} = F \otimes G$, then $I = J \otimes K$ with $\underline{J} = F$ and $\underline{K} = G$. In the contexts of the premises, Γ is split into Δ' and Δ'' and Ω is split into Ω' and Ω''. We have:
  $$\dfrac{\overset{\text{IH}}{\Delta' \vdash^{i} J} \qquad \overset{\text{IH}}{\Delta'' \vdash^{i} K}}{\Delta', \Delta'' \vdash^{i} J \otimes K} \otimes R$$
  If Ω is not empty, then at least one of Ω' and Ω'' as well. If, for example, $\Omega' \neq \emptyset$ then we have, for all Σ:
  $$\dfrac{\overset{\text{IH}}{\Delta', \Sigma \vdash^{i} J} \qquad \overset{\text{IH}}{\Delta'' \vdash^{i} K}}{\Delta', \Delta'', \Sigma \vdash^{i} J \otimes K} \otimes R$$
  - If $\underline{\Gamma}^{\perp}$ contains $F \otimes G$ then $\Gamma = \Gamma', J \multimap K$ with $\underline{J} = F$ and $\underline{K} = G^{\perp}$. If $\underline{I}$ belongs to the same premise as F, we have $\Gamma = \Delta', \Delta'', J \multimap K$ with $\vdash \underline{\Delta'}^{\perp}, \underline{J}, \underline{I}$ and, since J is !-like, for all Σ:
  $$\overset{\text{IH}}{\Delta', \Delta'', J \multimap K, \Sigma \vdash^{i} I}$$
  If $\underline{I}$ belongs to the same premise as G, we have $\Gamma = \Delta', \Delta'', J \multimap K$ and:
  $$\dfrac{\overset{\text{IH}}{\Delta' \vdash^{i} J} \qquad \overset{\text{IH}}{\Delta'', K \vdash^{i} I}}{\Delta', \Delta'', J \multimap K \vdash^{i} I} \multimap L$$
  If Ω is not empty and splits into Ω' and Ω'' in the premises, then at least one of Ω' and Ω'' is not empty. If, for example, $\Omega' \neq \emptyset$ then we have, for all Σ:
  $$\dfrac{\overset{\text{IH}}{\Delta', \Sigma \vdash^{i} J} \qquad \overset{\text{IH}}{\Delta'', K \vdash^{i} I}}{\Delta', \Delta'', J \multimap K, \Sigma \vdash^{i} I} \multimap L$$
  - If $\underline{\Omega}$ contains $F \otimes G$ then $\Omega = \Omega', J \otimes K$ with $\underline{J} = F$ and $\underline{K} = G$. In the contexts of the premises, Γ is split into Δ' and Δ'' and $\underline{I}$ must be in the same premise as F or G and we have, for all Σ:
  $$\overset{\text{IH}}{\Delta', \Delta'', \Sigma \vdash^{i} I}$$

□

Theorem 3.3 (!-Like Conservativity)**.** *If* I *is a formula of* ILL *such that any formula on the left-hand side of a* $\multimap$ *in* I *is !-like, then* $\vdash \underline{I}$ *is provable in* LL *if and only if* $\vdash^{i} I$ *is provable in* ILL*.*

As a consequence, we get the conservativity of LL over ILL for the fragment corresponding to formulas obtained by Girard's translation from intuitionistic logic into LL (as already shown in [22]) but also for the image of the so-called *call-by-value Girard's translation* ($A \to B \mapsto\ !(A \multimap B)$) [9], and for other variants as soon as they only use the connective $\multimap$ in the $O \multimap \_$ shape. As a corollary, this gives us the faithfulness of these translations: provability of the image implies provability of the source (move from LL to ILL by conservativity, and then to intuitionistic logic using the notion of skeleton [6]).

3.2 Looking to the Right of $\multimap$

The other conservativity result from [22] relies on the study of the interaction between $\multimap$ and 0:

Proposition 3.4 ([22, Proposition 3.8]). *If I is a formula of* ILL *which does not contain* $\multimap$ or *does not contain* 0*, then* $\vdash \underline{I}$ *is provable in* LL *if and only if* $\vdash^i I$ *is provable in* ILL.

We are interested in refining this result since, for example, the translation $(\_)^\bullet$ uses both 0 and $\multimap$, thus Proposition 3.4 does not allow us to get a conservativity result for the image of this translation.

Definition 3.5 (Almost Zero). A formula of ILL is *almost* 0 if it belongs to the following grammar:

$$Z ::= 0 \mid Z \otimes I \mid I \otimes Z \mid Z \,\&\, I \mid I \,\&\, Z \mid Z \oplus Z \mid\ !Z\ .$$

Lemma 3.6. *Let Z be an almost* 0 *formula, we have* $Z \dashv\vdash 0$ *in* ILL.

Definition 3.7 (Zero Clean). A formula of ILL is 0*-clean* if it does not contain any subformula of the shape $\_ \multimap Z$ with Z an almost 0 formula.

Lemma 3.8. *If Γ contains* 0*-clean formulas only, and if* $\vdash \underline{\Gamma}^\perp$ *is provable in* LL*, then Γ contains an almost* 0 *formula.*

Proof. By induction on a cut-free proof with atomic axioms. The key case is the ($\otimes$) rule: if $\underline{I}^\perp = F \otimes G$ then $I = J \multimap K$ with $\underline{J} = F$ and $\underline{K} = G^\perp$. By induction hypothesis applied to the premise $\vdash \underline{\Delta}^\perp, \underline{K}^\perp$, we have an almost 0 formula in $\Delta \subseteq \Gamma$ since I is 0-clean. □

Theorem 3.9 (Zero-Clean Conservativity). *Let I be a* 0*-clean formula,* $\vdash \underline{I}$ *is provable in* LL *if and only if* $\vdash^i I$ *is provable in* ILL.

Proof. The interesting direction is the left-to-right one. We prove by induction on a cut-free proof that, if $\vdash \underline{\Gamma}^\perp, \underline{I}$ containing 0-clean formulas only is provable in LL, then $\Gamma \vdash^i I$ is provable in ILL. The key case is the ($\otimes$) rule.

If $\underline{I} = F \otimes G$, then $I = J \otimes K$ with $\underline{J} = F$ and $\underline{K} = G$. The contexts of the premises are $\underline{\Delta}^\perp$ and $\underline{\Sigma}^\perp$ with $\Gamma = \Delta, \Sigma$. We have:

$$\dfrac{\overset{\text{IH}}{\Delta \vdash^i J} \qquad \overset{\text{IH}}{\Sigma \vdash^i K}}{\Delta, \Sigma \vdash^i J \otimes K} \otimes R$$

If $\underline{\Gamma}^\perp$ contains $F \otimes G$ then $\Gamma = \Gamma', J \multimap K$ with $\underline{J} = F$ and $\underline{K} = G^\perp$. If $\underline{I}$ belongs to the same premise as F, the other premise is of the shape $\vdash \underline{\Sigma}^\perp, G$ with 0-clean formulas only. By Lemma 3.8, Σ, K contains an almost 0 formula Z. We cannot have $K = Z$ otherwise $J \multimap K$ is not 0-clean, thus Z belongs to $\Sigma = \Sigma', Z$. We have $\Gamma = \Delta, \Sigma', Z, J \multimap K$ and:

$$\dfrac{\overset{\text{Lemma 3.6}}{Z \vdash^i 0} \qquad \dfrac{}{0, \Delta, \Sigma', J \multimap K \vdash^i I}\,0L}{Z, \Delta, \Sigma', J \multimap K \vdash^i I}\,cut$$

If $\underline{I}$ belongs to the same premise as G, we have $\Gamma = \Delta, \Sigma, J \multimap K$ and:

$$\dfrac{\overset{\text{IH}}{\Sigma \vdash^i J} \qquad \overset{\text{IH}}{\Delta, K \vdash^i I}}{\Delta, \Sigma, J \multimap K \vdash^i I} \multimap L$$

□

Lemma 3.10. *If* $\vdash \underline{\Gamma}^\perp, \underline{\Delta}$ *is provable in* LL*, with Δ containing at least two formulas, then* $\underline{\Gamma}^\perp, \underline{\Delta}$ *contains* $\top$.

By looking at the key case of Theorem 3.9 (and similarly for Lemma 3.8):

$$\dfrac{\vdash \underline{\Delta}^\perp, \underline{J}, \underline{I} \qquad \vdash \underline{\Sigma}^\perp, \underline{K}^\perp}{\vdash \underline{\Delta}^\perp, \underline{\Sigma}^\perp, \underline{J} \otimes \underline{K}^\perp, \underline{I}} \otimes$$

we can see that we can refine these results: conservativity can only fail if $\_ \multimap Z$ appears in negative position (an odd number of times on the left of a $\multimap$ connective in the formula tree), and if the formula contains (outside this Z) a $\top$ in positive position or a 0 in negative position (thanks to Lemma 3.10 applied to the premise $\vdash \underline{\Delta}^\perp, \underline{J}, \underline{I}$).

Similarly, Lemma 3.2 and Theorem 3.3 can be strengthened by only constraining $\multimap$ to be of the shape $O \multimap \_$ when it occurs in negative position.

To sum up Sections 3.1 and 3.2, a counterexample to the conservativity of LL over ILL must contain:

- a negative sub-formula $I \multimap J$ with I not !-like (which implies that the implicative order of the global formula is at least 2);
- a negative sub-formula $I \multimap Z$ (this also entails implicative order at least 2);
- a positive sub-formula $\top$ or a negative sub-formula 0 outside this Z.

Moreover, for each such counterexample, thanks to Lemma 3.6, one can obtain a possibly shorter one by replacing Z with 0. One can check these three constraints are true for the formulas mentioned in the beginning of Section 3.

3.3 Tensor Logic

Tensor logic (TL) [18] is a variant of ILL based on a *primitive* negation connective. *Tensor formulas* are generated by:

$$U ::= X \mid 1 \mid U \otimes U \mid 0 \mid U \oplus U \mid\ !U \mid \neg U\ .$$

Sequents are $\Gamma \vdash^t \Pi$ where Π is either empty or a single formula U. Rules are given on Figure 1.

Since if we add a primitive negation $\neg$ to ILL, it is possible to prove that $\neg$ and $\neg_\Phi$ are linearly equivalent (for a propositional variable Φ), and since the rules of TL for the other connectives are the same as in ILL, we can work with TL as being the fragment of ILL restricted to tensor formulas (with $\neg$ interpreted as $\neg_\Phi$ and $\Gamma \vdash^t$ as $\Gamma \vdash^i \Phi$). Theorem 3.9 is strong enough to prove a conservativity result of LL over TL.

Theorem 3.11 (Conservativity of LL over TL). *Let U be a formula of* TL*,* $\vdash \underline{U}$ *is provable in* LL *if and only if* $\vdash^t U$ *is provable in* TL.

$$\dfrac{}{U \vdash^t U}\,ax \qquad \dfrac{\Gamma \vdash^t U \qquad \Delta, U \vdash^t \Pi}{\Gamma, \Delta \vdash^t \Pi}\,cut$$

$$\dfrac{\Gamma \vdash^t U}{\Gamma, \neg U \vdash^t}\,\neg L \qquad \dfrac{\Gamma, U \vdash^t}{\Gamma \vdash^t \neg U}\,\neg R$$

$$\dfrac{\Gamma, U, V \vdash^t \Pi}{\Gamma, U \otimes V \vdash^t \Pi}\,\otimes L \qquad \dfrac{\Gamma \vdash^t U \qquad \Delta \vdash^t V}{\Gamma, \Delta \vdash^t U \otimes V}\,\otimes R$$

$$\dfrac{\Gamma \vdash^t \Pi}{\Gamma, 1 \vdash^t \Pi}\,1L \qquad \dfrac{}{\vdash^t 1}\,1R$$

$$\dfrac{\Gamma, U \vdash^t \Pi \qquad \Gamma, V \vdash^t \Pi}{\Gamma, U \oplus V \vdash^t \Pi}\,\oplus L \qquad \dfrac{}{\Gamma, 0 \vdash^t \Pi}\,0L$$

$$\dfrac{\Gamma \vdash^t U}{\Gamma \vdash^t U \oplus V}\,\oplus_1 R \qquad \dfrac{\Gamma \vdash^t V}{\Gamma \vdash^t U \oplus V}\,\oplus_2 R$$

$$\dfrac{\Gamma, U \vdash^t \Pi}{\Gamma, !U \vdash^t \Pi}\,!L \qquad \dfrac{!\Gamma \vdash^t U}{!\Gamma \vdash^t !U}\,!R$$

$$\dfrac{\Gamma \vdash^t \Pi}{\Gamma, !U \vdash^t \Pi}\,!wL \qquad \dfrac{\Gamma, !U, !U \vdash^t \Pi}{\Gamma, !U \vdash^t \Pi}\,!cL$$

Figure 1. Tensor logic (TL)

Proof. This relies on the fact that the only use of the $\multimap$ connective in TL seen as a fragment of ILL, is of the shape $\_ \multimap \Phi$ with Φ a propositional variable. This means that, inside ILL, tensor formulas are 0-clean and we can apply Theorem 3.9. □

This shows that TL can be seen as a *fragment* of LL (a subsystem defined by restricting the set of formulas) and, as a consequence, it can be studied inside LL.

4 Focusing

By using $\neg$ instead of $\neg_{\mathsf{R}}$, the image of $(\_)^\bullet$ belongs to TL and one can see that Proposition 2.4 still holds: if $\vdash \Gamma$ is provable in LL, then $\Gamma^\bullet \vdash^t$ in TL. We are now going to optimise this translation by trying to minimise the use of the $\neg$ connective.

4.1 Polarized Translation

First we partition linear formulas into two classes: *synchronous* and *asynchronous* formulas [2].

$$\begin{array}{rl} S ::= & X \mid 1 \mid F \otimes F \mid 0 \mid F \oplus F \mid !F \\ A ::= & X^\perp \mid \perp \mid F \parr F \mid \top \mid F \,\&\, F \mid ?F \,. \end{array}$$

We consider two mutually defined translations $(\_)^+$ and $(\_)^-$ from LL formulas to TL formulas:

$$\begin{array}{rclcrcl} X^+ & = & X & \quad & (X^\perp)^- & = & X \\ 1^+ & = & 1 & & \perp^- & = & 1 \\ (F \otimes G)^+ & = & F^+ \otimes G^+ & & (F \parr G)^- & = & F^- \otimes G^- \\ 0^+ & = & 0 & & \top^- & = & 0 \\ (F \oplus G)^+ & = & F^+ \oplus G^+ & & (F \,\&\, G)^- & = & F^- \oplus G^- \\ (!F)^+ & = & !\neg F^- & & (?F)^- & = & !\neg F^+ \\ A^+ & = & \neg A^- & & S^- & = & \neg S^+ \end{array}$$

This is similar to the focalized translation of [18], but $(!S)^+ = !\neg\neg S^+$ is crucial here (as also remarked in [5]). The case of $(?A)^-$ can be discussed (see below).

Lemma 4.1. *For all F, $(F^\perp)^+ = F^-$ and $(F^\perp)^- = F^+$.*

Lemma 4.2. *For all F, $F^- \vdash^t F^\bullet$ in* TL.

Proof. We first prove that for any F, $F^- \vdash^t F^\bullet$ implies $\neg F^+, \neg F^\bullet \vdash^t$. We consider the two cases, F synchronous and F asynchronous:

$$\dfrac{S^- \vdash^t S^\bullet}{\neg S^+, \neg S^\bullet \vdash^t}\,\neg L \qquad \dfrac{\dfrac{\dfrac{A^- \vdash^t A^\bullet}{A^-, \neg A^\bullet \vdash^t}\,\neg L}{\neg A^\bullet \vdash^t A^+}\,\neg R}{\neg A^+, \neg A^\bullet \vdash^t}\,\neg L$$

We now prove the statement by induction on the formula F. Here are the main cases:

- $(F \otimes G)^- = \neg(F \otimes G)^+ = \neg(F^+ \otimes G^+)$ and $(F \otimes G)^\bullet = \neg(\neg F^\bullet \otimes \neg G^\bullet)$, and we have:

$$\dfrac{\dfrac{\dfrac{\overset{\text{IH}}{F^- \vdash^t F^\bullet}}{\neg F^+, \neg F^\bullet \vdash^t} \qquad \dfrac{\overset{\text{IH}}{G^- \vdash^t G^\bullet}}{\neg G^+, \neg G^\bullet \vdash^t}}{\neg(F^+ \otimes G^+), \neg F^\bullet \otimes \neg G^\bullet \vdash^t}\,\text{Lemma 2.1 }(i)}{\neg(F^+ \otimes G^+) \vdash^t \neg(\neg F^\bullet \otimes \neg G^\bullet)}\,\neg R$$

- $(!F)^- = \neg(!F)^+ = \neg!\neg F^-$ and $(!F)^\bullet = \neg!\neg F^\bullet$, and we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{IH}}{F^- \vdash^t F^\bullet}}{F^-, \neg F^\bullet \vdash^t}\,\neg L}{\neg F^\bullet \vdash^t \neg F^-}\,\neg R}{!\neg F^\bullet \vdash^t \neg F^-}\,!L}{!\neg F^\bullet \vdash^t !\neg F^-}\,!R}{\neg!\neg F^-, !\neg F^\bullet \vdash^t}\,\neg L}{\neg!\neg F^- \vdash^t \neg!\neg F^\bullet}\,\neg R$$

- $(?F)^- = !\neg F^+$ and $(?F)^\bullet = !\neg\neg F^\bullet$, and we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{IH}}{F^- \vdash^t F^\bullet}}{\neg F^+, \neg F^\bullet \vdash^t}}{\neg F^+ \vdash^t \neg\neg F^\bullet}\,\neg R}{!\neg F^+ \vdash^t \neg\neg F^\bullet}\,!L}{!\neg F^+ \vdash^t !\neg\neg F^\bullet}\,!R$$

□

Theorem 4.3 (Polarized Translation)**.** *If $\vdash \Gamma$ is provable in* LL *then $\Gamma^- \vdash^t$ is provable in* TL.

Proof. By Proposition 2.4, we have $\Gamma^\bullet \vdash^t$ in TL. We then use cuts with the proofs from Lemma 4.2 to turn every $F^\bullet$ into F^-. □

While it is *not possible* to have $(!S)^+ = !S^+$ (it would break Theorem 4.3), for similar reasons as in Section 2.1, one can optimise $(?A)^- = !A^-$ while preserving Lemma 4.2 and Theorem 4.3, but we would loose symmetry (in particular it would lead to $(F^\perp)^- \neq F^+$ and thus the correspondence of Theorem 4.5 would not work so nicely for axioms and cuts). Moreover the additional use of $\neg$ we impose here in $(?A)^-$ is necessary for the focusing analysis to come (it is responsible for the constraint of an empty focus in the asynchronous $(?d)$ rule).

The construction of the translations $(\_)^-/(\_)^+$ by first defining $(\_)^\bullet$ and then using Lemma 4.2 is directly inspired by the focusing proof of S. Zimmerman for differential linear logic [26] which uses a similar decomposition in two steps.

$$\dfrac{}{\vdash^f S^\perp \mid S}\;ax \qquad \dfrac{\vdash^f \Gamma \mid S}{\vdash^f \Gamma, S \mid}\;foc \qquad \dfrac{\vdash^f \Gamma \mid F \qquad \vdash^f \Delta, F^\perp \mid \Pi}{\vdash^f \Gamma, \Delta \mid \Pi}\;cut$$

$$\dfrac{\vdash^f \Gamma, F, G \mid \Pi}{\vdash^f \Gamma, F \parr G \mid \Pi}\;\parr \qquad \dfrac{\vdash^f \Gamma \mid F \qquad \vdash^f \Delta \mid G}{\vdash^f \Gamma, \Delta \mid F \otimes G}\;\otimes$$

$$\dfrac{\vdash^f \Gamma \mid \Pi}{\vdash^f \Gamma, \bot \mid \Pi}\;\bot \qquad \dfrac{}{\vdash^f \ \mid 1}\;1$$

$$\dfrac{\vdash^f \Gamma, F \mid \Pi \qquad \vdash^f \Gamma, G \mid \Pi}{\vdash^f \Gamma, F \mathbin{\&} G \mid \Pi}\;\& \qquad \dfrac{}{\vdash^f \Gamma, \top \mid \Pi}\;\top$$

$$\dfrac{\vdash^f \Gamma \mid F}{\vdash^f \Gamma \mid F \oplus G}\;\oplus_1 \qquad \dfrac{\vdash^f \Gamma \mid G}{\vdash^f \Gamma \mid F \oplus G}\;\oplus_2$$

$$\dfrac{\vdash^f ?\Gamma, F \mid}{\vdash^f ?\Gamma \mid !F}\;! \qquad \dfrac{\vdash^f \Gamma \mid F}{\vdash^f \Gamma, ?F \mid}\;?d$$

$$\dfrac{\vdash^f \Gamma \mid \Pi}{\vdash^f \Gamma, ?F \mid \Pi}\;?w \qquad \dfrac{\vdash^f \Gamma, ?F, ?F \mid \Pi}{\vdash^f \Gamma, ?F \mid \Pi}\;?c$$

Figure 2. The LL_{foc} system

4.2 From TL to LL_{foc}

Weakly focused linear logic (LL_{foc}) [16] is a focused system for LL inspired from [10] and [2]. It relies on the distinction between *synchronous* and *asynchronous* formulas. Sequents are $\vdash^f \Gamma \mid \Pi$ where Π is either empty or a single synchronous formula S (called the *focus*). This means in particular that $\vdash^f \Gamma \mid A$ with A asynchronous is not a sequent of the system. However it is useful to define $\vdash^f \Gamma \mid A$ as a *notation* for $\vdash^f \Gamma, A \mid$, so that we can use the notation:

$$\vdash^f \Gamma \mid F \quad = \quad \begin{cases} \vdash^f \Gamma \mid F & \text{if } F \text{ is synchronous} \\ \vdash^f \Gamma, F \mid & \text{if } F \text{ is asynchronous} \end{cases}$$

for an arbitrary linear formula F. Rules are given on Figure 2.

Let us consider a proof of $\Gamma^- \vdash^t \Pi^+$ in TL. We can rely on the fact that all the ! connectives come with a $\neg$, to constrain slightly the structure of proofs. The system TL' is obtained from TL by replacing the rules ($!L$) and ($!R$) by:

$$\dfrac{\Gamma \vdash^t U}{\Gamma, !\neg U \vdash^t}\;!L' \qquad \dfrac{!\Gamma, U \vdash^t}{!\Gamma \vdash^t !\neg U}\;!R'$$

Lemma 4.4. *If* $\Gamma^- \vdash^t \Pi^+$ *is provable in* TL*, then it is provable in* TL' *as well.*

Proof. We start from a proof of $\Gamma^- \vdash^t \Pi^+$ in TL with atomic axioms. A ($!L$) rule must be of the shape:

$$\dfrac{\Gamma^-, \neg F^+ \vdash^t \Pi^+}{\Gamma^-, !\neg F^+ \vdash^t \Pi^+}\;!L$$

We can see that such an occurrence of rule can be commuted up with all rules until we reach the point where the $\neg$ of $\neg F^+$ has been introduced, so that we can turn it into a ($!L'$) rule.

Let us now look at a ($!R$) rule. It must be of the shape:

$$\dfrac{\overset{\pi}{!\neg\Gamma^+ \vdash^t \neg F^-}}{!\neg\Gamma^+ \vdash^t !\neg F^-}\;!R$$

Since π has atomic axioms, we can find in π the rules ($\neg R$) introducing the negation $\neg$ of $\neg F^-$. They can be commuted down until we reach the ($!R$) rule (this corresponds to the reversibility of the ($\neg R$) rule). We then turn the sequence ($\neg R$)–($!R$) into ($!R'$). □

The transformations of proofs of TL into proofs of TL' described in the proof of Lemma 4.4 can also be obtained by introducing cuts and by eliminating them appropriately in TL:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{F^+ \vdash^t F^+}\;ax}{\neg F^+, F^+ \vdash^t}\;\neg L}{!\neg F^+, F^+ \vdash^t}\;!L}{!\neg F^+ \vdash^t \neg F^+}\;\neg R}{!\neg F^+ \vdash^t !\neg F^+}\;!R \qquad \dfrac{\Gamma^-, \neg F^+ \vdash^t \Pi^+}{\Gamma^-, !\neg F^+ \vdash^t \Pi^+}\;!L}{\Gamma^-, !\neg F^+ \vdash^t \Pi^+}\;cut$$

and

$$\dfrac{\dfrac{!\neg\Gamma^+ \vdash^t \neg F^-}{!\neg\Gamma^+ \vdash^t !\neg F^-}\;!R \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{F^- \vdash^t F^-}\;ax}{\neg F^-, F^- \vdash^t}\;\neg L}{!\neg F^-, F^- \vdash^t}\;!L}{!\neg F^- \vdash^t \neg F^-}\;\neg R}{!\neg F^- \vdash^t !\neg F^-}\;!R}{!\neg\Gamma^+ \vdash^t !\neg F^-}\;cut$$

Theorem 4.5. $\Gamma^- \vdash^t \Pi^+$ *is provable in* TL' *if and only if* $\vdash^f \Gamma \mid \Pi$ *is provable in* LL_{foc}.

Proof. There is a one-to-one correspondence for almost all rules. We focus on the most tricky cases:

- The (ax) rule $S^+ \vdash^t S^+$ exactly corresponds to $\vdash^f S^\perp \mid S$ since $(S^\perp)^- = S^+$.
- The ($\neg R$) rule of TL':
  $$\dfrac{\Gamma^-, A^- \vdash^t}{\Gamma^- \vdash^t \neg A^-}\;\neg R$$
  is the identity in LL_{foc} since it corresponds to the notation $\vdash^f \Gamma \mid A = \vdash^f \Gamma, A \mid$ by $\neg A^- = A^+$.
- The ($\neg L$) rule of TL':
  $$\dfrac{\Gamma^- \vdash^t S^+}{\Gamma^-, \neg S^+ \vdash^t}\;\neg L$$
  corresponds to the (foc) rule of LL_{foc} since $\neg S^+ = S^-$.
- The ($!L'$) rule of TL':
  $$\dfrac{\Gamma^- \vdash^t F^+}{\Gamma^-, !\neg F^+ \vdash^t}\;!L'$$
  corresponds to the ($?d$) rule of LL_{foc}.
- The ($!R'$) rule of TL':
  $$\dfrac{(?\Gamma)^-, F^- \vdash^t}{(?\Gamma)^- \vdash^t !\neg F^-}\;!R'$$
  corresponds to the (!) rule of LL_{foc} since $(?\Gamma)^- = !\neg\Gamma^+$. □

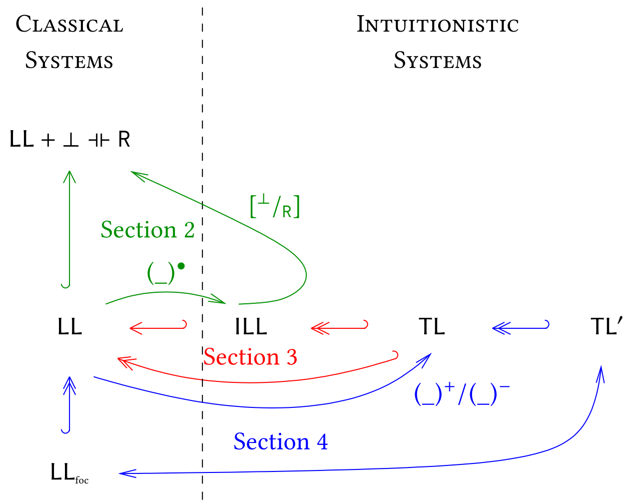
Once restricted to ! connectives only used in association with $\neg$: $!\neg\_$, TL (optimised into TL') can thus be seen as a focused system for LL and the "at most one active synchronous formula" constraint of focused sequents occurs as a particular case of the "at most one formula on the right" constraint of intuitionistic sequents.

Corollary 4.6 (Weak Focusing)**.** *If* $\vdash \Gamma$ *is provable in* LL *then* $\vdash^f \Gamma \mid$ *is provable in* LL_{foc}.

For more discussions about various notions of focusing and how to deduce traditional focusing from this weak focusing property, see [16]. However weak focusing is the key step in focusing.

5 Conclusion

We can sum up the main results in the following picture:



We use $\hookrightarrow$ for (almost) identity-on-formulas translations and $\hookrightarrow\!\!\!\!\rightarrow$ for conservative such embeddings.

We have focused on propositional systems, but the results we have presented can be extended to first-order and second-order quantifiers (except Section 3 which breaks for second-order). The key ingredients to add, in second-order for example, are:

$$\begin{array}{rclcrcl}(\exists X.F)^\bullet & = & \neg_R\exists X.\neg_R F^\bullet & & (\forall X.F)^\bullet & = & \exists X.F^\bullet \\ (\exists X.F)^+ & = & \exists X.F^+ & & (\forall X.F)^- & = & \exists X.F^-\end{array}$$

with the following lemma which allows us to extend Proposition 2.4:

Lemma 5.1. *For all F, G and X,*
if $\mathsf{R}[{}^{\neg_R G^\bullet}/_X] = \mathsf{R}$ *then* $F^\bullet[{}^{\neg_R G^\bullet}/_X], \neg_R(F[{}^G/_X])^\bullet \vdash^i \mathsf{R}$ *in* ILL.

In polarization and focusing, the status of the exponential connectives has always been more difficult to understand than for the other connectives. The negations we are forced to introduce in translating the connective ! of LL into ILL or TL, can be justified by associating to ! the polarity $- \mapsto +$ (meaning that it turns a negative (asynchronous) formula into a positive (synchronous) formula) in LL, but the polarity $+ \mapsto +$ in TL. The ! of LL is then decomposed into the two operations ! *and* ¬ of TL (which is coherent with a polarity $\neg : - \mapsto +$). Moreover when ! is applied to a synchronous formula, another negation has to be added. Such a decomposition of the ! of LL is suggested in [5]. This could also be described in systems using shift operators (↓ and ↑, see for example [18]).

As a final remark, let us mention that reintroducing the parameter R in negations, so that !_ in LL is mapped to !(_ ⅋ R) is what G. Munch proposed for working with delimited continuations [19].

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References

[1] Samson Abramsky, Radha Jagadeesan, and Pasquale Malacaria. 2000. Full Abstraction for PCF. *Information and Computation* 163, 2 (2000), 409–470.
[2] Jean-Marc Andreoli. 1992. Logic Programming with Focusing Proofs in Linear Logic. *Journal of Logic and Computation* 2, 3 (1992), 297–347.
[3] Boy-Yuh Evan Chang, Kaustuv Chaudhuri, and Frank Pfenning. 2003. *A judgmental analysis of linear logic.* Technical Report CMU-CS-03-131R. Department of Computer Science, Carnegie Mellon University. Available at http://chaudhuri.info/papers/tr03jill.pdf .
[4] Boy-Yuh Evan Chang, Kaustuv Chaudhuri, and Frank Pfenning. 2003. *A judgmental analysis of linear logic.* Technical Report CMU-CS-03-131. Department of Computer Science, Carnegie Mellon University. Available at http://reports-archive.adm.cs.cmu.edu/anon/2003/CMU-CS-03-131.pdf .
[5] Pierre-Louis Curien, Marcelo Fiore, and Guillaume Munch-Maccagnoni. 2016. A theory of effects and resources: adjunction models and polarised calculi. In *Proceedings of Principles of Programming Languages*, R. Bodík and R. Majumdar (Eds.). ACM, 44–56.
[6] Vincent Danos, Jean-Baptiste Joinet, and Harold Schellinx. 1995. On the linear decoration of intuitionistic derivations. *Archive for Mathematical Logic* 33, 6 (1995), 387–412.
[7] Olivier Danvy and Andrzej Filinski. 1990. Abstracting Control. In *Proceedings of LISP and Functional Programming*. ACM, 151–160.
[8] Kosta Dosen. 1992. Nonmodal Classical Linear Predicate Logic is a Fragment of Intuitionistic Linear Logic. *Theoretical Computer Science* 102, 1 (1992), 207–214.
[9] Jean-Yves Girard. 1987. Linear logic. *Theoretical Computer Science* 50 (1987), 1–102.
[10] Jean-Yves Girard. 1991. A new constructive logic: classical logic. *Mathematical Structures in Computer Science* 1, 3 (1991), 255–296.
[11] Jean-Yves Girard and Yves Lafont. 1987. Linear Logic and Lazy Computation. In *Proceedings of Theory and Practice of Software Development (LNCS)*, H. Ehrig, R. Kowalski, G. Levi, and U. Montanari (Eds.), Vol. 250. Springer, 52–66.
[12] Timothy Griffin. 1990. A Formulae-as-Types notion of control. In *Proceedings of Principles of Programming Languages*. ACM, 47–58.
[13] Yves Lafont. 1999. The Linear Logic Pages. (1999). Available at http://iml.univ-mrs.fr/~lafont/pub/llpages.pdf .
[14] Yves Lafont, Bernhard Reus, and Thomas Streicher. 1993. *Continuation Semantics or Expressing Implication by Negation.* Technical Report 93-21. Ludwig-Maximilians-Universität, München. Available at http://iml.univ-mrs.fr/~lafont/pub/continuation.ps .
[15] François Lamarche. 1995. Games Semantics for Full Propositional Linear Logic. In *Proceedings of Logic In Computer Science*. IEEE, 464–473.
[16] Olivier Laurent. 2004–2017. A Proof of the Focusing Property of Linear Logic. (2004–2017). Available at https://perso.ens-lyon.fr/olivier.laurent/llfoc2.pdf .
[17] Olivier Laurent. 2017. Yalla: Yet Another deep embedding of Linear Logic in Coq. Coq library. (2017). Available at https://perso.ens-lyon.fr/olivier.laurent/yalla/ .
[18] Paul-André Melliès and Nicolas Tabareau. 2010. Resource modalities in tensor logic. *Annals of Pure and Applied Logic* 161, 5 (2010), 632–653.
[19] Guillaume Munch-Maccagnoni. 2011. From delimited CPS to polarisation. (2011). Available at https://hal.inria.fr/inria-00587597/ .
[20] Chetan Murthy. 1992. Classical proofs as programs: How, what and why. In *Constructivity in Computer Science*, P. Myers and M. O'Donnell (Eds.). Springer, 71–88.
[21] Amr Sabry and Matthias Felleisen. 1993. Reasoning about programs in continuation-passing style. *LISP and Symbolic Computation* 6, 3 (1993), 289–360.
[22] Harold Schellinx. 1991. Some Syntactical Observations on Linear Logic. *Journal of Logic and Computation* 1, 4 (1991), 537–559.
[23] Harold Schellinx. 1994. *The noble art of linear decorating.* ILLC Dissertation series. Universiteit van Amsterdam.
[24] Hayo Thielecke. 1997. Continuation semantics and self-adjointness. *Electronic Notes in Theoretical Computer Science* 6 (1997), 348–364.
[25] Anne Troelstra. 1992. *Lectures in Linear Logic.* CSLI Lecture Notes, Vol. 29. University of Chicago Press.
[26] Stéphane Zimmermann. 2013. *Vers une ludique différentielle.* Thèse de Doctorat. Université Paris VII.

Appendix

A Notations

Main notations	**Definition**
$\vdash$ and LL: Classical Linear Logic	page 11
$\vdash^n$: LL with (mix_n)	page 3
$\vdash^{02}$: LL with (mix_0) and (mix_2)	page 3
$\vdash^a$ and AL: Affine Logic	page 4
$\vdash^i$ and ILL: Intuitionistic Linear Logic	page 12
$\vdash^t$ and TL: Tensor Logic	page 8
$\vdash^f$ and $\mathsf{LL}_{\mathrm{foc}}$: Weakly Focused Linear Logic	page 9
$\vdash^{!\perp}$ and $\mathsf{LL}^{!\perp}$	page 5
$\vdash^r_{\mathcal{R}}$ and RLL($\mathcal{R}$): Response Linear Logic	page 3
X, Y: propositional variables	
F, G, H: linear formulas	page 2
I, J, K: intuitionistic linear formulas	page 2
U, V: tensor formulas	page 7
A, B: asynchronous formulas (in LL)	page 8
S, T: synchronous formulas (in LL)	page 8
$\mathcal{R}$: "fixed" linear formula (in LL)	page 3
R: "fixed" intuitionistic linear formula (in ILL)	
Φ: "fixed" intuitionistic linear propositional variable (in ILL)	
$\mathcal{G}$: image of Girard's translation (in ILL)	page 6
E: purely positive affine formula (in ILL $\cap$ LL)	page 3
O: !-like formula (in ILL)	page 6
Z: almost zero formula (in ILL)	page 7
$\underline{I}$: LL formula associated to an ILL one	page 3
$(\_)^\bullet$: translation from LL to ILL	page 3
$(\_)^+$ and $(\_)^-$: translations from LL to TL	page 8

B Linear Logic (LL)

Given a countable set of propositional variables denoted X, Y, etc, *linear formulas* are generated by:

$$F ::= X \mid X^\perp \mid 1 \mid \perp \mid F \otimes F \mid F \parr F \mid 0 \mid \top \mid F \oplus F \mid F \,\&\, F \mid\, !F \mid ?F\ .$$

Linear negation $(\_)^\perp$ is extended to an involution on all formulas by:

$$\begin{array}{rclcrcl}
(X^\perp)^\perp & = & X & & & & \\
1^\perp & = & \perp & & \perp^\perp & = & 1 \\
(F \otimes G)^\perp & = & F^\perp \parr G^\perp & & (F \parr G)^\perp & = & F^\perp \otimes G^\perp \\
0^\perp & = & \top & & \top^\perp & = & 0 \\
(F \oplus G)^\perp & = & F^\perp \,\&\, G^\perp & & (F \,\&\, G)^\perp & = & F^\perp \oplus G^\perp \\
(!F)^\perp & = & ?F^\perp & & (?F)^\perp & = & !F^\perp
\end{array}$$

Sequents are $\vdash \Gamma$ where Γ is a list of linear formulas.

$$\frac{}{\vdash F, F^\perp}\ ax \qquad \frac{\vdash \Gamma, F \qquad \vdash \Delta, F^\perp}{\vdash \Gamma, \Delta}\ cut \qquad \left(\frac{\vdash \Gamma}{\vdash \sigma(\Gamma)}\ ex\right)$$

$$\frac{\vdash \Gamma, F, G}{\vdash \Gamma, F \parr G}\ \parr \qquad \frac{\vdash \Gamma, F \qquad \vdash \Delta, G}{\vdash \Gamma, \Delta, F \otimes G}\ \otimes \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\ \perp \qquad \frac{}{\vdash 1}\ 1$$

$$\frac{\vdash \Gamma, F \qquad \vdash \Gamma, G}{\vdash \Gamma, F \,\&\, G}\ \& \qquad \frac{\vdash \Gamma, F}{\vdash \Gamma, F \oplus G}\ \oplus_1 \qquad \frac{\vdash \Gamma, G}{\vdash \Gamma, F \oplus G}\ \oplus_2 \qquad \frac{}{\vdash \Gamma, \top}\ \top$$

$$\frac{\vdash \Gamma, F}{\vdash \Gamma, ?F}\ ?d \qquad \frac{\vdash \Gamma, ?F, ?F}{\vdash \Gamma, ?F}\ ?c \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?F}\ ?w \qquad \frac{\vdash ?\Gamma, F}{\vdash ?\Gamma, !F}\ !$$

Lemma B.1 (Deduction). *$\vdash \Gamma$ is provable in* LL *extended with axioms $\frac{}{\vdash F}$ for each $F \in \Sigma$ if and only if $\vdash \Gamma, ?\Sigma^\perp$ is provable in* LL.

Proof. See for example Lemmas 3.2 and 3.3 of *Decision Problems for Propositional Linear Logic* by Lincoln–Mitchell–Scedrov–Shankar (APAL 56(1–3) pages 239–311, 1992).

From left to right, we first check by induction on a proof of LL (with or without additional axioms) that it is possible to add $?\Sigma$ to each sequent (by using $(?w)$ rules on axioms in particular). In a second step, we replace:

$$\text{each}\quad \dfrac{\dfrac{}{\vdash F}}{\vdash F, ?\Sigma^{\perp}}\,?w \qquad \text{by} \qquad \dfrac{\dfrac{\dfrac{}{\vdash F, F^{\perp}}\,ax}{\vdash F, ?F^{\perp}}\,?d}{\vdash F, ?\Sigma^{\perp}}\,?w$$

From right to left, we introduce a cut for each formula F in Σ:

$$\begin{array}{c} \vdash \Gamma, ?\Sigma^{\perp} \\ \vdots \\ \dfrac{\vdots \qquad \dfrac{\dfrac{}{\vdash F}}{\vdash !F}\,!}{}\,cut \\ \vdots \\ \vdash \Gamma \end{array}$$

□

C Intuitionistic Linear Logic (ILL)

Given a countable set of propositional variables denoted X, Y, etc, *intuitionistic linear formulas* are generated by:

$$I ::= X \mid 1 \mid I \otimes I \mid I \multimap I \mid 0 \mid \top \mid I \oplus I \mid I \,\&\, I \mid !I\,.$$

Sequents are $\Gamma \vdash^i I$ where Γ is a list of intuitionistic linear formulas.

$$\dfrac{}{I \vdash^i I}\,ax \qquad \dfrac{\Gamma \vdash^i I \qquad \Delta, I \vdash^i K}{\Gamma, \Delta \vdash^i K}\,cut \qquad \left(\dfrac{\Gamma \vdash^i K}{\sigma(\Gamma) \vdash^i K}\,ex\right)$$

$$\dfrac{\Gamma, I, J \vdash^i K}{\Gamma, I \otimes J \vdash^i K}\,\otimes L \qquad \dfrac{\Gamma \vdash^i I \qquad \Delta \vdash^i J}{\Gamma, \Delta \vdash^i I \otimes J}\,\otimes R \qquad \dfrac{\Gamma \vdash^i K}{\Gamma, 1 \vdash^i K}\,1L \qquad \dfrac{}{\vdash^i 1}\,1R$$

$$\dfrac{\Gamma \vdash^i I \qquad \Delta, J \vdash^i K}{\Gamma, \Delta, I \multimap J \vdash^i K}\multimap L \qquad \dfrac{\Gamma, I \vdash^i J}{\Gamma \vdash^i I \multimap J}\multimap R$$

$$\dfrac{\Gamma, I \vdash^i K}{\Gamma, I \,\&\, J \vdash^i K}\,\&_1 L \qquad \dfrac{\Gamma, J \vdash^i K}{\Gamma, I \,\&\, J \vdash^i K}\,\&_2 L \qquad \dfrac{\Gamma \vdash^i I \qquad \Gamma \vdash^i J}{\Gamma \vdash^i I \,\&\, J}\,\& R \qquad \dfrac{}{\Gamma \vdash^i \top}\,\top R$$

$$\dfrac{\Gamma, I \vdash^i K \qquad \Gamma, J \vdash^i K}{\Gamma, I \oplus J \vdash^i K}\,\oplus L \qquad \dfrac{\Gamma \vdash^i I}{\Gamma \vdash^i I \oplus J}\,\oplus_1 R \qquad \dfrac{\Gamma \vdash^i J}{\Gamma \vdash^i I \oplus J}\,\oplus_2 R \qquad \dfrac{}{\Gamma, 0 \vdash^i K}\,0L$$

$$\dfrac{\Gamma, I \vdash^i K}{\Gamma, !I \vdash^i K}\,!L \qquad \dfrac{\Gamma, !I, !I \vdash^i K}{\Gamma, !I \vdash^i K}\,!cL \qquad \dfrac{\Gamma \vdash^i K}{\Gamma, !I \vdash^i K}\,!wL \qquad \dfrac{!\Gamma \vdash^i I}{!\Gamma \vdash^i !I}\,!R$$

Lemma C.1 (! Decoration)**.** *Given a list of formulas Σ, any open derivation from $\Gamma_1 \vdash^i I_1, \ldots, \Gamma_k \vdash^i I_k$ to $\Gamma \vdash^i I$ can be decorated by adding some $!\Sigma$ and some $(!cL)$ and $(!wL)$ rules into a derivation from $!\Sigma, \Gamma_1 \vdash^i I_1, \ldots, !\Sigma, \Gamma_k \vdash^i I_k$ to $!\Sigma, \Gamma \vdash^i I$.*

Proof. We use $(!wL)$ rules at each (ax) and $(1R)$ rule:

$$\dfrac{\dfrac{}{I \vdash^i I}\,ax}{!\Sigma, I \vdash^i I}\,!wL \qquad \dfrac{\dfrac{}{\vdash^i 1}\,1R}{!\Sigma \vdash^i 1}\,!wL$$

We use $(!cL)$ rules at each (cut), $(\otimes R)$ and $(\multimap L)$ rule:

$$\dfrac{\dfrac{!\Sigma, \Gamma \vdash^i I \qquad !\Sigma, \Delta, I \vdash^i K}{!\Sigma, !\Sigma, \Gamma, \Delta \vdash^i K}\,cut}{!\Sigma, \Gamma, \Delta \vdash^i K}\,!cL \qquad \dfrac{\dfrac{!\Sigma, \Gamma \vdash^i I \qquad !\Sigma, \Delta \vdash^i J}{!\Sigma, !\Sigma, \Gamma, \Delta \vdash^i I \otimes J}\,\otimes R}{!\Sigma, \Gamma, \Delta \vdash^i I \otimes J}\,!cL \qquad \dfrac{\dfrac{!\Sigma, \Gamma \vdash^i I \qquad !\Sigma, \Delta, J \vdash^i K}{!\Sigma, !\Sigma, \Gamma, \Delta, I \multimap J \vdash^i K}\multimap L}{!\Sigma, \Gamma, \Delta, I \multimap J \vdash^i K}\,!cL$$

Note the context constraint on $(!R)$ rules is preserved. □

By interpreting $I \multimap J$ as $I^{\perp} ⅋ J$, any ILL formula can be seen as an LL formula. More formally, one can define the following (almost identity) translation of ILL formulas into LL formulas:

$$\begin{array}{rcl@{\qquad}rcl} \underline{X} & = & X & \underline{1} & = & 1 \\ \underline{I \otimes J} & = & \underline{I} \otimes \underline{J} & \underline{I \multimap J} & = & (\underline{I})^{\perp} ⅋ \underline{J} \\ \underline{0} & = & 0 & \underline{\top} & = & \top \\ \underline{I \oplus J} & = & \underline{I} \oplus \underline{J} & \underline{I \mathbin{\&} J} & = & \underline{I} \mathbin{\&} \underline{J} \\ \underline{!I} & = & !\underline{I} & & & \end{array}$$

This relies on the fact that, for simplicity, we use the same set of propositional variables for linear formulas and for intuitionistic linear formulas, and some shared notations for connectives.

Lemma C.2. *If* $\Gamma \vdash^i I$ *is provable in* ILL *then* $\vdash (\underline{\Gamma})^{\perp}, \underline{I}$ *is provable in* LL.

Proof. By induction on the proof of $\Gamma \vdash^i I$. □

C.1 Defined Parametric Negation

As defined above, ILL does not contain a negation connective. However it is possible to use the construction _ $\multimap$ R as a defined negation in ILL. Let R be a formula of ILL, we use the notation $\neg_{\mathsf{R}} I$ (which is more associative than binary connectives) for $I \multimap \mathsf{R}$ so that the following two rules are derivable in ILL:

$$\frac{\Gamma, I \vdash^i \mathsf{R}}{\Gamma \vdash^i \neg_{\mathsf{R}} I}\ \neg_{\mathsf{R}} R \qquad\qquad \frac{\Gamma \vdash^i I}{\Gamma, \neg_{\mathsf{R}} I \vdash^i \mathsf{R}}\ \neg_{\mathsf{R}} L$$

Proof.

$$\frac{\Gamma, I \vdash^i \mathsf{R}}{\Gamma \vdash^i \neg_{\mathsf{R}} I} \multimap R \qquad\qquad \frac{\Gamma \vdash^i I \qquad \dfrac{}{\mathsf{R} \vdash^i \mathsf{R}}\ ax}{\Gamma, \neg_{\mathsf{R}} I \vdash^i \mathsf{R}} \multimap L$$

□

C.2 Primitive Negation

Given a fixed propositional variable Φ of ILL, it is also possible to extend ILL with a new connective $\neg$:

$$I ::= \cdots \mid \neg I$$

and the two rules:

$$\frac{\Gamma, I \vdash^i \Phi}{\Gamma \vdash^i \neg I}\ \neg R \qquad\qquad \frac{\Gamma \vdash^i I}{\Gamma, \neg I \vdash^i \Phi}\ \neg L$$

We denote by $\mathsf{ILL}^{\neg}$ the obtained system.

Theorem C.3 (Cut Elimination for $\mathsf{ILL}^{\neg}$)**.** *Provability in* $\mathsf{ILL}^{\neg}$ *is the same with and without the (cut) rule.*

Proof. As for cut elimination of ILL with the additional key case:

$$\frac{\dfrac{\Gamma, I \vdash^i \Phi}{\Gamma \vdash^i \neg I}\ \neg R \qquad \dfrac{\Delta \vdash^i I}{\Delta, \neg I \vdash^i \Phi}\ \neg L}{\Gamma, \Delta \vdash^i \Phi}\ cut \qquad \mapsto \qquad \frac{\Delta \vdash^i I \qquad \Gamma, I \vdash^i \Phi}{\Gamma, \Delta \vdash^i \Phi}\ cut$$

□

The translation $I \mapsto \underline{I}$ can be extended with $\underline{\neg I} = \underline{I}^{\perp} ⅋ \Phi$.

Lemma C.4. *If* $\Gamma \vdash^i I$ *is provable in* $\mathsf{ILL}^{\neg}$ *then* $\vdash \underline{\Gamma}^{\perp}, \underline{I}$ *is provable in* LL.

Lemma C.5 (Definable Negation)**.** *In* $\mathsf{ILL}^{\neg}$, $\neg I \dashv\vdash \neg_{\Phi} I$.

Proof.

$$\frac{\dfrac{\dfrac{}{I \vdash^i I}\ ax}{\neg I, I \vdash^i \Phi}\ \neg L}{\neg I \vdash^i \neg_{\Phi} I} \multimap R \qquad\qquad \frac{\dfrac{\dfrac{}{I \vdash^i I}\ ax \qquad \dfrac{}{\Phi \vdash^i \Phi}\ ax}{\neg_{\Phi} I, I \vdash^i \Phi} \multimap L}{\neg_{\Phi} I \vdash^i \neg I}\ \neg R$$

□

Proposition C.6. ILL *proofs can be modified (without changing the conclusion) in such a way that all occurrences of ($\multimap L$) rules introducing* $\neg_{\Phi} A$ *are replaced by ($\neg_{\Phi} L$) rules.*

Proof. Let π be such a proof, we can see it as well as a proof in $\mathsf{ILL}^{\neg}$ since this is an extension of ILL. We can introduce cuts in π in $\mathsf{ILL}^{\neg}$ using the proofs of Lemma C.5 to replace any $\neg_{\Phi} I$ by $\neg I$. We eliminate cuts (Theorem C.3) in the obtained proof to obtain a proof π'. If we replace every $\neg I$ in π' by $\neg_{\Phi} I$ and the rules ($\neg L$) and ($\neg R$) by ($\neg_{\Phi} L$) and ($\neg_{\Phi} R$), we obtain a proof in ILL with the same conclusion as π. □

D Detailed Proofs

Section 2

Lemma 2.1. In ILL, for all formulas I, J, I' and J', we have:

(i) $I, \neg_{\mathsf{R}} I' \vdash^i \mathsf{R}$ and $J, \neg_{\mathsf{R}} J' \vdash^i \mathsf{R}$ implies $I \otimes J, \neg_{\mathsf{R}}(I' \otimes J') \vdash^i \mathsf{R}$

(ii) $I, \neg_{\mathsf{R}} I' \vdash^i \mathsf{R}$ and $J, \neg_{\mathsf{R}} J' \vdash^i \mathsf{R}$ implies $I \oplus J, \neg_{\mathsf{R}}(I' \oplus J') \vdash^i \mathsf{R}$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{I, \neg_{\mathsf{R}} I' \vdash^i \mathsf{R}}{I \vdash^i \neg_{\mathsf{R}}\neg_{\mathsf{R}} I'}\,\neg_{\mathsf{R}}R \qquad \dfrac{J, \neg_{\mathsf{R}} J' \vdash^i \mathsf{R}}{J \vdash^i \neg_{\mathsf{R}}\neg_{\mathsf{R}} J'}\,\neg_{\mathsf{R}}R}{I, J \vdash^i \neg_{\mathsf{R}}\neg_{\mathsf{R}} I' \otimes \neg_{\mathsf{R}}\neg_{\mathsf{R}} J'}\,\otimes R}{I \otimes J \vdash^i \neg_{\mathsf{R}}\neg_{\mathsf{R}} I' \otimes \neg_{\mathsf{R}}\neg_{\mathsf{R}} J'}\,\otimes L \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{I' \vdash^i I'}\,ax \qquad \dfrac{}{J' \vdash^i J'}\,ax}{I', J' \vdash^i I' \otimes J'}\,\otimes R}{I', J', \neg_{\mathsf{R}}(I' \otimes J') \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}}L}{J', \neg_{\mathsf{R}}(I' \otimes J') \vdash^i \neg_{\mathsf{R}} I'}\,\neg_{\mathsf{R}}R}{\neg_{\mathsf{R}}\neg_{\mathsf{R}} I', J', \neg_{\mathsf{R}}(I' \otimes J') \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}}L}{\neg_{\mathsf{R}}\neg_{\mathsf{R}} I', \neg_{\mathsf{R}}(I' \otimes J') \vdash^i \neg_{\mathsf{R}} J'}\,\neg_{\mathsf{R}}R}{\neg_{\mathsf{R}}\neg_{\mathsf{R}} I', \neg_{\mathsf{R}}\neg_{\mathsf{R}} J', \neg_{\mathsf{R}}(I' \otimes J') \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}}L}{\neg_{\mathsf{R}}\neg_{\mathsf{R}} I' \otimes \neg_{\mathsf{R}}\neg_{\mathsf{R}} J', \neg_{\mathsf{R}}(I' \otimes J') \vdash^i \mathsf{R}}\,\otimes L}{I \otimes J, \neg_{\mathsf{R}}(I' \otimes J') \vdash^i \mathsf{R}}\,cut$$

$$\dfrac{\dfrac{\dfrac{\dfrac{I, \neg_{\mathsf{R}} I' \vdash^i \mathsf{R}}{I \vdash^i \neg_{\mathsf{R}}\neg_{\mathsf{R}} I'}\,\neg_{\mathsf{R}}R}{I \vdash^i \neg_{\mathsf{R}}\neg_{\mathsf{R}} I' \oplus \neg_{\mathsf{R}}\neg_{\mathsf{R}} J'}\,\oplus_1 R \qquad \dfrac{\dfrac{J, \neg_{\mathsf{R}} J' \vdash^i \mathsf{R}}{J \vdash^i \neg_{\mathsf{R}}\neg_{\mathsf{R}} J'}\,\neg_{\mathsf{R}}R}{J \vdash^i \neg_{\mathsf{R}}\neg_{\mathsf{R}} I' \oplus \neg_{\mathsf{R}}\neg_{\mathsf{R}} J'}\,\oplus_2 R}{I \oplus J \vdash^i \neg_{\mathsf{R}}\neg_{\mathsf{R}} I' \oplus \neg_{\mathsf{R}}\neg_{\mathsf{R}} J'}\,\oplus L \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{I' \vdash^i I'}\,ax}{I' \vdash^i I' \oplus J'}\,\oplus_1 R}{I', \neg_{\mathsf{R}}(I' \oplus J') \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}}L}{\neg_{\mathsf{R}}(I' \oplus J') \vdash^i \neg_{\mathsf{R}} I'}\,\neg_{\mathsf{R}}R}{\neg_{\mathsf{R}}\neg_{\mathsf{R}} I', \neg_{\mathsf{R}}(I' \oplus J') \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}}L \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{J' \vdash^i J'}\,ax}{J' \vdash^i I' \oplus J'}\,\oplus_2 R}{J', \neg_{\mathsf{R}}(I' \oplus J') \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}}L}{\neg_{\mathsf{R}}(I' \oplus J') \vdash^i \neg_{\mathsf{R}} J'}\,\neg_{\mathsf{R}}R}{\neg_{\mathsf{R}}\neg_{\mathsf{R}} J', \neg_{\mathsf{R}}(I' \oplus J') \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}}L}{\neg_{\mathsf{R}}\neg_{\mathsf{R}} I' \oplus \neg_{\mathsf{R}}\neg_{\mathsf{R}} J', \neg_{\mathsf{R}}(I' \oplus J') \vdash^i \mathsf{R}}\,\oplus L}{I \oplus J, \neg_{\mathsf{R}}(I' \oplus J') \vdash^i \mathsf{R}}\,cut$$

□

Lemma 2.3. For all F, $\neg_{\mathsf{R}} F^\bullet, \neg_{\mathsf{R}}(F^\perp)^\bullet \vdash^i \mathsf{R}$ is derivable in ILL.

Proof. By induction on F (using the fact that the statement is symmetric between F and $F^\perp$):

$$\dfrac{\dfrac{}{\neg_{\mathsf{R}} X \vdash^i \neg_{\mathsf{R}} X}\,ax}{\neg_{\mathsf{R}}\neg_{\mathsf{R}} X, \neg_{\mathsf{R}} X \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}}L \qquad \dfrac{\dfrac{}{\neg_{\mathsf{R}} 1 \vdash^i \neg_{\mathsf{R}} 1}\,ax}{\neg_{\mathsf{R}}\neg_{\mathsf{R}} 1, \neg_{\mathsf{R}} 1 \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}}L \qquad \dfrac{\dfrac{}{\neg_{\mathsf{R}} 0 \vdash^i \neg_{\mathsf{R}} 0}\,ax}{\neg_{\mathsf{R}}\neg_{\mathsf{R}} 0, \neg_{\mathsf{R}} 0 \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}}L$$

$$\dfrac{\dfrac{\dfrac{\overset{\text{IH}}{\neg_{\mathsf{R}} F^\bullet, \neg_{\mathsf{R}}(F^\perp)^\bullet \vdash^i \mathsf{R}} \qquad \overset{\text{IH}}{\neg_{\mathsf{R}} G^\bullet, \neg_{\mathsf{R}}(G^\perp)^\bullet \vdash^i \mathsf{R}}}{\neg_{\mathsf{R}} F^\bullet \otimes \neg_{\mathsf{R}} G^\bullet, \neg_{\mathsf{R}}((F^\perp)^\bullet \otimes (G^\perp)^\bullet) \vdash^i \mathsf{R}}\,\text{Lemma 2.1 }(i)}{\neg_{\mathsf{R}}((F^\perp)^\bullet \otimes (G^\perp)^\bullet) \vdash^i \neg_{\mathsf{R}}(\neg_{\mathsf{R}} F^\bullet \otimes \neg_{\mathsf{R}} G^\bullet)}\,\neg_{\mathsf{R}}R}{\neg_{\mathsf{R}}\neg_{\mathsf{R}}(\neg_{\mathsf{R}} F^\bullet \otimes \neg_{\mathsf{R}} G^\bullet), \neg_{\mathsf{R}}((F^\perp)^\bullet \otimes (G^\perp)^\bullet) \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}}L \qquad \dfrac{\dfrac{\dfrac{\overset{\text{IH}}{\neg_{\mathsf{R}} F^\bullet, \neg_{\mathsf{R}}(F^\perp)^\bullet \vdash^i \mathsf{R}} \qquad \overset{\text{IH}}{\neg_{\mathsf{R}} G^\bullet, \neg_{\mathsf{R}}(G^\perp)^\bullet \vdash^i \mathsf{R}}}{\neg_{\mathsf{R}} F^\bullet \oplus \neg_{\mathsf{R}} G^\bullet, \neg_{\mathsf{R}}((F^\perp)^\bullet \oplus (G^\perp)^\bullet) \vdash^i \mathsf{R}}\,\text{Lemma 2.1 }(ii)}{\neg_{\mathsf{R}}((F^\perp)^\bullet \oplus (G^\perp)^\bullet) \vdash^i \neg_{\mathsf{R}}(\neg_{\mathsf{R}} F^\bullet \oplus \neg_{\mathsf{R}} G^\bullet)}\,\neg_{\mathsf{R}}R}{\neg_{\mathsf{R}}\neg_{\mathsf{R}}(\neg_{\mathsf{R}} F^\bullet \oplus \neg_{\mathsf{R}} G^\bullet), \neg_{\mathsf{R}}((F^\perp)^\bullet \oplus (G^\perp)^\bullet) \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}}L$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{IH}}{\neg_{\mathsf{R}} F^\bullet, \neg_{\mathsf{R}}(F^\perp)^\bullet \vdash^i \mathsf{R}}}{\neg_{\mathsf{R}} F^\bullet \vdash^i \neg_{\mathsf{R}}\neg_{\mathsf{R}}(F^\perp)^\bullet}\,\neg_{\mathsf{R}}R}{!\neg_{\mathsf{R}} F^\bullet \vdash^i \neg_{\mathsf{R}}\neg_{\mathsf{R}}(F^\perp)^\bullet}\,!L}{!\neg_{\mathsf{R}} F^\bullet \vdash^i !\neg_{\mathsf{R}}\neg_{\mathsf{R}}(F^\perp)^\bullet}\,!R}{!\neg_{\mathsf{R}} F^\bullet, \neg_{\mathsf{R}}!\neg_{\mathsf{R}}\neg_{\mathsf{R}}(F^\perp)^\bullet \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}}L}{\neg_{\mathsf{R}}!\neg_{\mathsf{R}}\neg_{\mathsf{R}}(F^\perp)^\bullet \vdash^i \neg_{\mathsf{R}}!\neg_{\mathsf{R}} F^\bullet}\,\neg_{\mathsf{R}}R}{\neg_{\mathsf{R}}\neg_{\mathsf{R}}!\neg_{\mathsf{R}} F^\bullet, \neg_{\mathsf{R}}!\neg_{\mathsf{R}}\neg_{\mathsf{R}}(F^\perp)^\bullet \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}}L$$

□

Proposition 2.4. If $\vdash \Gamma$ is provable in LL then $\Gamma^\bullet \vdash^i \mathsf{R}$ is provable in ILL.

Proof. We can consider a proof with atomic axioms only and we build a decoration of each rule:

$$\dfrac{\dfrac{}{X \vdash^i X}\,ax}{\neg_{\mathsf{R}} X, X \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}} L \qquad \dfrac{\dfrac{\overset{\text{IH}}{\Gamma^\bullet, F^\bullet \vdash^i \mathsf{R}}}{\Gamma^\bullet \vdash^i \neg_{\mathsf{R}} F^\bullet}\,\neg R \qquad \dfrac{\dfrac{\overset{\text{IH}}{\Delta^\bullet, (F^\perp)^\bullet \vdash^i \mathsf{R}}}{\Delta^\bullet \vdash^i \neg_{\mathsf{R}} (F^\perp)^\bullet}\,\neg R \qquad \overset{\text{Lemma 2.3}}{\neg_{\mathsf{R}} F^\bullet, \neg_{\mathsf{R}} (F^\perp)^\bullet \vdash^i \mathsf{R}}}{\neg_{\mathsf{R}} F^\bullet, \Delta^\bullet \vdash^i \mathsf{R}}\,cut}{\Gamma^\bullet, \Delta^\bullet \vdash^i \mathsf{R}}\,cut$$

$$\dfrac{\overset{\text{IH}}{\Gamma^\bullet, F^\bullet, G^\bullet \vdash^i \mathsf{R}}}{\Gamma^\bullet, F^\bullet \otimes G^\bullet \vdash^i \mathsf{R}}\,\otimes L \qquad \dfrac{\dfrac{\dfrac{\overset{\text{IH}}{\Gamma^\bullet, F^\bullet \vdash^i \mathsf{R}}}{\Gamma^\bullet \vdash^i \neg_{\mathsf{R}} F^\bullet}\,\neg R \qquad \dfrac{\overset{\text{IH}}{\Delta^\bullet, G^\bullet \vdash^i \mathsf{R}}}{\Delta^\bullet \vdash^i \neg_{\mathsf{R}} G^\bullet}\,\neg R}{\Gamma^\bullet, \Delta^\bullet \vdash^i \neg_{\mathsf{R}} F^\bullet \otimes \neg_{\mathsf{R}} G^\bullet}\,\otimes R}{\Gamma^\bullet, \Delta^\bullet, \neg_{\mathsf{R}} (\neg_{\mathsf{R}} F^\bullet \otimes \neg_{\mathsf{R}} G^\bullet) \vdash^i \mathsf{R}}\,\neg L \qquad \dfrac{\overset{\text{IH}}{\Gamma^\bullet \vdash^i \mathsf{R}}}{\Gamma^\bullet, 1 \vdash^i \mathsf{R}}\,1L \qquad \dfrac{\dfrac{}{\vdash^i 1}\,1R}{\neg_{\mathsf{R}} 1 \vdash^i \mathsf{R}}\,\neg L$$

$$\dfrac{\overset{\text{IH}}{\Gamma^\bullet, F^\bullet \vdash^i \mathsf{R}} \qquad \overset{\text{IH}}{\Gamma^\bullet, G^\bullet \vdash^i \mathsf{R}}}{\Gamma^\bullet, F^\bullet \oplus G^\bullet \vdash^i \mathsf{R}}\,\oplus L \qquad \dfrac{\dfrac{\dfrac{\overset{\text{IH}}{\Gamma^\bullet, F^\bullet \vdash^i \mathsf{R}}}{\Gamma^\bullet \vdash^i \neg_{\mathsf{R}} F^\bullet}\,\neg R}{\Gamma^\bullet \vdash^i \neg_{\mathsf{R}} F^\bullet \oplus \neg_{\mathsf{R}} G^\bullet}\,\oplus_1 R}{\Gamma^\bullet, \neg_{\mathsf{R}} (\neg_{\mathsf{R}} F^\bullet \oplus \neg_{\mathsf{R}} G^\bullet) \vdash^i \mathsf{R}}\,\neg L \qquad \dfrac{\dfrac{\dfrac{\overset{\text{IH}}{\Gamma^\bullet, G^\bullet \vdash^i \mathsf{R}}}{\Gamma^\bullet \vdash^i \neg_{\mathsf{R}} G^\bullet}\,\neg R}{\Gamma^\bullet \vdash^i \neg_{\mathsf{R}} F^\bullet \oplus \neg_{\mathsf{R}} G^\bullet}\,\oplus_2 R}{\Gamma^\bullet, \neg_{\mathsf{R}} (\neg_{\mathsf{R}} F^\bullet \oplus \neg_{\mathsf{R}} G^\bullet) \vdash^i \mathsf{R}}\,\neg L \qquad \dfrac{}{\Gamma^\bullet, 0 \vdash^i \mathsf{R}}\,0L$$

$$\dfrac{\dfrac{\dfrac{\overset{\text{IH}}{\Gamma^\bullet, F^\bullet \vdash^i \mathsf{R}}}{\Gamma^\bullet \vdash^i \neg_{\mathsf{R}} F^\bullet}\,\neg R}{\Gamma^\bullet, \neg_{\mathsf{R}} \neg_{\mathsf{R}} F^\bullet \vdash^i \mathsf{R}}\,\neg L}{\Gamma^\bullet, !\neg_{\mathsf{R}} \neg_{\mathsf{R}} F^\bullet \vdash^i \mathsf{R}}\,!L \qquad \dfrac{\overset{\text{IH}}{\Gamma^\bullet, !\neg_{\mathsf{R}} \neg_{\mathsf{R}} F^\bullet, !\neg_{\mathsf{R}} \neg_{\mathsf{R}} F^\bullet \vdash^i \mathsf{R}}}{\Gamma^\bullet, !\neg_{\mathsf{R}} \neg_{\mathsf{R}} F^\bullet \vdash^i \mathsf{R}}\,!cL \qquad \dfrac{\overset{\text{IH}}{\Gamma^\bullet \vdash^i \mathsf{R}}}{\Gamma^\bullet, !\neg_{\mathsf{R}} \neg_{\mathsf{R}} F^\bullet \vdash^i \mathsf{R}}\,!wL \qquad \dfrac{\dfrac{\dfrac{\overset{\text{IH}}{!\neg_{\mathsf{R}} \neg_{\mathsf{R}} \Gamma^\bullet, F^\bullet \vdash^i \mathsf{R}}}{!\neg_{\mathsf{R}} \neg_{\mathsf{R}} \Gamma^\bullet \vdash^i \neg_{\mathsf{R}} F^\bullet}\,\neg R}{!\neg_{\mathsf{R}} \neg_{\mathsf{R}} \Gamma^\bullet \vdash^i !\neg_{\mathsf{R}} F^\bullet}\,!R}{!\neg_{\mathsf{R}} \neg_{\mathsf{R}} \Gamma^\bullet, \neg_{\mathsf{R}} !\neg_{\mathsf{R}} F^\bullet \vdash^i \mathsf{R}}\,\neg L$$

□

Lemma 2.5. Let R be an intuitionistic linear formula, for all $n \geq 0$, if $\bigotimes_n \mathsf{R} \vdash^i \mathsf{R}$ is provable in ILL then LL with (mix_n) can be translated by $(\_)^\bullet$ into ILL.

Proof. We follow the pattern of Proposition 2.4 and we just have to consider:

$$\dfrac{\dfrac{\overset{\text{IH}}{\Gamma_1^\bullet \vdash^i \mathsf{R}} \qquad \dfrac{\overset{\text{IH}}{\Gamma_2^\bullet \vdash^i \mathsf{R}} \qquad \overset{\dfrac{\overset{\text{IH}}{\Gamma_n^\bullet \vdash^i \mathsf{R}} \qquad \dfrac{}{\vdash^i 1}\,1R}{\Gamma_n^\bullet \vdash^i \bigotimes_1 \mathsf{R}}\,\otimes R}{\vdots}}{\Gamma_2^\bullet, \ldots, \Gamma_n^\bullet \vdash^i \bigotimes_{n-1} \mathsf{R}}\,\otimes R}{\Gamma_1^\bullet, \ldots, \Gamma_n^\bullet \vdash^i \bigotimes_n \mathsf{R}}\,\otimes R \qquad \bigotimes_n \mathsf{R} \vdash^i \mathsf{R}}{\Gamma_1^\bullet, \ldots, \Gamma_n^\bullet \vdash^i \mathsf{R}}\,cut$$

□

Lemma 2.6. If $\vdash^r_{\mathcal{R}} \Gamma$ is provable in $\mathsf{RLL}(\mathcal{R})$ then, for all F and X, $\vdash^r_{\mathcal{R}[F/X]} \Gamma[F/X]$ is provable in $\mathsf{RLL}(\mathcal{R}[F/X])$.

Proof. By induction on the proof, the key cases are the specific axioms of $\mathsf{RLL}(\mathcal{R})$:

$$\dfrac{}{\vdash^r_{\mathcal{R}[F/X]} \mathcal{R}^\perp[F/X]}\,\perp_{\mathcal{R}} \qquad \dfrac{}{\vdash^r_{\mathcal{R}[F/X]} \mathcal{R}[F/X], 1}\,1_{\mathcal{R}}$$

□

Lemma 2.7. $\vdash^r_{\mathcal{R}} \Gamma$ in $\mathsf{RLL}(\mathcal{R})$ if and only if $\vdash \Gamma, ?\mathcal{R}, ?(\mathcal{R}^\perp \otimes \perp)$ in LL.

Proof. This comes directly from Lemma B.1. □

Lemma 2.8. $\vdash^r_{?\mathcal{R}} \Gamma$ in $\mathsf{RLL}(?\mathcal{R})$ if and only if $\vdash \Gamma, ?\mathcal{R}$ in LL.

Proof. Since $\vdash !(?\mathcal{R} \parr 1)$ is provable in LL:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\,1}{\vdash ?\mathcal{R}, 1}\,?w}{\vdash ?\mathcal{R} \parr 1}\,\parr}{\vdash !(?\mathcal{R} \parr 1)}\,!$$

The result comes from Lemma 2.7 by using $??\mathcal{R} \dashv\vdash ?\mathcal{R}$ and a cut with the just given proof. □

Lemma 2.9. Given a formula F of LL, $\vdash^r_{\underline{\mathsf{R}}} \underline{F^\bullet}, F$ is provable in RLL($\underline{\mathsf{R}}$).

Proof. First note the rule $\dfrac{\vdash^r_{\mathcal{R}} \Gamma}{\vdash^r_{\mathcal{R}} \Gamma, \mathcal{R}}\,wk_{\mathcal{R}}$ is admissible in RLL($\mathcal{R}$):

$$\dfrac{\dfrac{\vdash^r_{\mathcal{R}} \Gamma}{\vdash^r_{\mathcal{R}} \Gamma, \bot}\,\bot \qquad \dfrac{}{\vdash^r_{\mathcal{R}} \mathcal{R}, 1}\,1_{\mathcal{R}}}{\vdash^r_{\mathcal{R}} \Gamma, \mathcal{R}}\,cut$$

Then we go by induction on F:

$$\dfrac{\dfrac{\dfrac{}{\vdash^r_{\underline{\mathsf{R}}} X^\bot, X}\,ax}{\vdash^r_{\underline{\mathsf{R}}} X^\bot, \underline{\mathsf{R}}, X}\,wk_{\mathcal{R}}}{\vdash^r_{\underline{\mathsf{R}}} X^\bot \parr \underline{\mathsf{R}}, X}\,\parr \qquad \dfrac{}{\vdash^r_{\underline{\mathsf{R}}} X, X^\bot}\,ax \qquad \dfrac{\dfrac{\dfrac{}{\vdash^r_{\underline{\mathsf{R}}} \bot, 1}\,ax}{\vdash^r_{\underline{\mathsf{R}}} \bot, \underline{\mathsf{R}}, 1}\,wk_{\mathcal{R}}}{\vdash^r_{\underline{\mathsf{R}}} \bot \parr \underline{\mathsf{R}}, 1}\,\parr \qquad \dfrac{}{\vdash^r_{\underline{\mathsf{R}}} 1, \bot}\,ax \qquad \dfrac{\dfrac{\dfrac{}{\vdash^r_{\underline{\mathsf{R}}} \top, 0}\,ax}{\vdash^r_{\underline{\mathsf{R}}} \top, \underline{\mathsf{R}}, 0}\,wk_{\mathcal{R}}}{\vdash^r_{\underline{\mathsf{R}}} \top \parr \underline{\mathsf{R}}, 0}\,\parr \qquad \dfrac{}{\vdash^r_{\underline{\mathsf{R}}} 0, \top}\,ax$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{IH}}{\vdash^r_{\underline{\mathsf{R}}} \underline{F^\bullet}, F} \qquad \dfrac{}{\vdash^r_{\underline{\mathsf{R}}} \underline{\mathsf{R}}^\bot}\,\bot_{\mathcal{R}}}{\vdash^r_{\underline{\mathsf{R}}} \underline{F^\bullet} \otimes \underline{\mathsf{R}}^\bot, F}\,\otimes \qquad \dfrac{\overset{\text{IH}}{\vdash^r_{\underline{\mathsf{R}}} \underline{G^\bullet}, G} \qquad \dfrac{}{\vdash^r_{\underline{\mathsf{R}}} \underline{\mathsf{R}}^\bot}\,\bot_{\mathcal{R}}}{\vdash^r_{\underline{\mathsf{R}}} \underline{G^\bullet} \otimes \underline{\mathsf{R}}^\bot, G}\,\otimes}{\vdash^r_{\underline{\mathsf{R}}} \underline{F^\bullet} \otimes \underline{\mathsf{R}}^\bot, \underline{G^\bullet} \otimes \underline{\mathsf{R}}^\bot, F \otimes G}\,\otimes}{\vdash^r_{\underline{\mathsf{R}}} (\underline{F^\bullet} \otimes \underline{\mathsf{R}}^\bot) \parr (\underline{G^\bullet} \otimes \underline{\mathsf{R}}^\bot), F \otimes G}\,\parr}{\vdash^r_{\underline{\mathsf{R}}} (\underline{F^\bullet} \otimes \underline{\mathsf{R}}^\bot) \parr (\underline{G^\bullet} \otimes \underline{\mathsf{R}}^\bot), \underline{\mathsf{R}}, F \otimes G}\,wk_{\mathcal{R}}}{\vdash^r_{\underline{\mathsf{R}}} ((\underline{F^\bullet} \otimes \underline{\mathsf{R}}^\bot) \parr (\underline{G^\bullet} \otimes \underline{\mathsf{R}}^\bot)) \parr \underline{\mathsf{R}}, F \otimes G}\,\parr \qquad \dfrac{\dfrac{\overset{\text{IH}}{\vdash^r_{\underline{\mathsf{R}}} \underline{F^\bullet}, F} \qquad \overset{\text{IH}}{\vdash^r_{\underline{\mathsf{R}}} \underline{G^\bullet}, G}}{\vdash^r_{\underline{\mathsf{R}}} \underline{F^\bullet} \otimes \underline{G^\bullet}, F, G}\,\otimes}{\vdash^r_{\underline{\mathsf{R}}} \underline{F^\bullet} \otimes \underline{G^\bullet}, F \parr G}\,\parr$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{IH}}{\vdash^r_{\underline{\mathsf{R}}} \underline{F^\bullet}, F} \qquad \dfrac{}{\vdash^r_{\underline{\mathsf{R}}} \underline{\mathsf{R}}^\bot}\,\bot_{\mathcal{R}}}{\vdash^r_{\underline{\mathsf{R}}} \underline{F^\bullet} \otimes \underline{\mathsf{R}}^\bot, F}\,\otimes}{\vdash^r_{\underline{\mathsf{R}}} \underline{F^\bullet} \otimes \underline{\mathsf{R}}^\bot, F \oplus G}\,\oplus_1 \qquad \dfrac{\dfrac{\overset{\text{IH}}{\vdash^r_{\underline{\mathsf{R}}} \underline{G^\bullet}, G} \qquad \dfrac{}{\vdash^r_{\underline{\mathsf{R}}} \underline{\mathsf{R}}^\bot}\,\bot_{\mathcal{R}}}{\vdash^r_{\underline{\mathsf{R}}} \underline{G^\bullet} \otimes \underline{\mathsf{R}}^\bot, G}\,\otimes}{\vdash^r_{\underline{\mathsf{R}}} \underline{G^\bullet} \otimes \underline{\mathsf{R}}^\bot, F \oplus G}\,\oplus_2}{\vdash^r_{\underline{\mathsf{R}}} (\underline{F^\bullet} \otimes \underline{\mathsf{R}}^\bot) \,\&\, (\underline{G^\bullet} \otimes \underline{\mathsf{R}}^\bot), F \oplus G}\,\&}{\vdash^r_{\underline{\mathsf{R}}} (\underline{F^\bullet} \otimes \underline{\mathsf{R}}^\bot) \,\&\, (\underline{G^\bullet} \otimes \underline{\mathsf{R}}^\bot), \underline{\mathsf{R}}, F \oplus G}\,wk_{\mathcal{R}}}{\vdash^r_{\underline{\mathsf{R}}} ((\underline{F^\bullet} \otimes \underline{\mathsf{R}}^\bot) \,\&\, (\underline{G^\bullet} \otimes \underline{\mathsf{R}}^\bot)) \parr \underline{\mathsf{R}}, F \oplus G}\,\parr \qquad \dfrac{\dfrac{\overset{\text{IH}}{\vdash^r_{\underline{\mathsf{R}}} \underline{F^\bullet}, F}}{\vdash^r_{\underline{\mathsf{R}}} \underline{F^\bullet} \oplus \underline{G^\bullet}, F}\,\oplus_1 \qquad \dfrac{\overset{\text{IH}}{\vdash^r_{\underline{\mathsf{R}}} \underline{G^\bullet}, G}}{\vdash^r_{\underline{\mathsf{R}}} \underline{F^\bullet} \oplus \underline{G^\bullet}, G}\,\oplus_2}{\vdash^r_{\underline{\mathsf{R}}} \underline{F^\bullet} \oplus \underline{G^\bullet}, F \,\&\, G}\,\&$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{IH}}{\vdash^r_{\underline{\mathsf{R}}} \underline{F^\bullet}, F} \qquad \dfrac{}{\vdash^r_{\underline{\mathsf{R}}} \underline{\mathsf{R}}^\bot}\,\bot_{\mathcal{R}}}{\vdash^r_{\underline{\mathsf{R}}} \underline{F^\bullet} \otimes \underline{\mathsf{R}}^\bot, F}\,\otimes}{\vdash^r_{\underline{\mathsf{R}}} ?(\underline{F^\bullet} \otimes \underline{\mathsf{R}}^\bot), F}\,?d}{\vdash^r_{\underline{\mathsf{R}}} ?(\underline{F^\bullet} \otimes \underline{\mathsf{R}}^\bot), !F}\,!}{\vdash^r_{\underline{\mathsf{R}}} ?(\underline{F^\bullet} \otimes \underline{\mathsf{R}}^\bot), \underline{\mathsf{R}}, !F}\,wk_{\mathcal{R}}}{\vdash^r_{\underline{\mathsf{R}}} ?(\underline{F^\bullet} \otimes \underline{\mathsf{R}}^\bot) \parr \underline{\mathsf{R}}, !F}\,\parr \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{IH}}{\vdash^r_{\underline{\mathsf{R}}} \underline{F^\bullet}, F} \qquad \dfrac{}{\vdash^r_{\underline{\mathsf{R}}} \underline{\mathsf{R}}^\bot}\,\bot_{\mathcal{R}}}{\vdash^r_{\underline{\mathsf{R}}} \underline{F^\bullet} \otimes \underline{\mathsf{R}}^\bot, F}\,\otimes}{\vdash^r_{\underline{\mathsf{R}}} \underline{F^\bullet} \otimes \underline{\mathsf{R}}^\bot, \underline{\mathsf{R}}, F}\,wk_{\mathcal{R}}}{\vdash^r_{\underline{\mathsf{R}}} (\underline{F^\bullet} \otimes \underline{\mathsf{R}}^\bot) \parr \underline{\mathsf{R}}, F}\,\parr}{\vdash^r_{\underline{\mathsf{R}}} (\underline{F^\bullet} \otimes \underline{\mathsf{R}}^\bot) \parr \underline{\mathsf{R}}, ?F}\,?d}{\vdash^r_{\underline{\mathsf{R}}} !((\underline{F^\bullet} \otimes \underline{\mathsf{R}}^\bot) \parr \underline{\mathsf{R}}), ?F}\,!$$

□

Proposition 2.10. If $\Gamma^\bullet \vdash^i \mathsf{R}$ is provable in ILL then $\vdash^r_{\underline{\mathsf{R}}} \Gamma$ is provable in RLL($\underline{\mathsf{R}}$).

Proof. First, by Lemma C.2, $\vdash (\underline{\Gamma^\bullet})^\bot, \underline{\mathsf{R}}$ is provable in LL thus in RLL($\underline{\mathsf{R}}$). Then if $\Gamma = F_1, \ldots, F_k$, we introduce $k + 1$ cuts:

$$\dfrac{\dfrac{\vdash^r_{\underline{\mathsf{R}}} (\underline{F_1^\bullet})^\perp,\dots,(\underline{F_k^\bullet})^\perp,\underline{\mathsf{R}} \qquad \dfrac{}{\vdash^r_{\underline{\mathsf{R}}} \underline{\mathsf{R}}^\perp}\,\bot_{\mathcal{R}}}{\vdash^r_{\underline{\mathsf{R}}} (\underline{F_1^\bullet})^\perp,\dots,(\underline{F_k^\bullet})^\perp}\,cut \qquad \overset{\text{Lemma 2.9}}{\vdash^r_{\underline{\mathsf{R}}} \underline{F_1^\bullet}, F_1}}{\vdash^r_{\underline{\mathsf{R}}} F_1, (\underline{F_2^\bullet})^\perp,\dots,(\underline{F_k^\bullet})^\perp}\,cut$$

$$\vdots$$

$$\dfrac{\vdash^r_{\underline{\mathsf{R}}} F_1,\dots,F_{k-1},(\underline{F_k^\bullet})^\perp \qquad \overset{\text{Lemma 2.9}}{\vdash^r_{\underline{\mathsf{R}}} \underline{F_k^\bullet}, F_k}}{\vdash^r_{\underline{\mathsf{R}}} F_1,\dots,F_k}\,cut$$

□

Lemma 2.11. If E is a purely positive affine formula and R is an intuitionistic linear formula, $(E^\perp)^{\bullet[\mathsf{R}]} \vdash^i E$ and $E \vdash^i \neg_{\mathsf{R}} E^{\bullet[\mathsf{R}]}$ are provable in ILL.

Proof. By induction on E with fixed R:

$$\dfrac{}{X \vdash^i X}\,ax \qquad\qquad \dfrac{\dfrac{\dfrac{}{X \vdash^i X}\,ax}{X, \neg_{\mathsf{R}} X \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}} L}{X \vdash^i \neg_{\mathsf{R}}\neg_{\mathsf{R}} X}\,\neg_{\mathsf{R}} R$$

$$\dfrac{}{1 \vdash^i 1}\,ax \qquad\qquad \dfrac{\dfrac{\dfrac{}{1 \vdash^i 1}\,ax}{1, \neg_{\mathsf{R}} 1 \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}} L}{1 \vdash^i \neg_{\mathsf{R}}\neg_{\mathsf{R}} 1}\,\neg_{\mathsf{R}} R$$

$$\dfrac{\dfrac{\overset{\text{IH}}{(E_1^\perp)^\bullet \vdash^i E_1} \qquad \overset{\text{IH}}{(E_2^\perp)^\bullet \vdash^i E_2}}{(E_1^\perp)^\bullet, (E_2^\perp)^\bullet \vdash^i E_1 \otimes E_2}\,\otimes R}{(E_1^\perp)^\bullet \otimes (E_2^\perp)^\bullet \vdash^i E_1 \otimes E_2}\,\otimes L$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{IH}}{E_1 \vdash^i \neg_{\mathsf{R}} E_1^\bullet} \qquad \overset{\text{IH}}{E_2 \vdash^i \neg_{\mathsf{R}} E_2^\bullet}}{E_1, E_2 \vdash^i \neg_{\mathsf{R}} E_1^\bullet \otimes \neg_{\mathsf{R}} E_2^\bullet}\,\otimes R}{E_1 \otimes E_2 \vdash^i \neg_{\mathsf{R}} E_1^\bullet \otimes \neg_{\mathsf{R}} E_2^\bullet}\,\otimes L}{E_1 \otimes E_2, \neg_{\mathsf{R}}(\neg_{\mathsf{R}} E_1^\bullet \otimes \neg_{\mathsf{R}} E_2^\bullet) \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}} L}{E_1 \otimes E_2 \vdash^i \neg_{\mathsf{R}}\neg_{\mathsf{R}}(\neg_{\mathsf{R}} E_1^\bullet \otimes \neg_{\mathsf{R}} E_2^\bullet)}\,\neg_{\mathsf{R}} R$$

$$\dfrac{}{0 \vdash^i 0}\,0L \qquad\qquad \dfrac{}{0 \vdash^i \neg_{\mathsf{R}}\neg_{\mathsf{R}} 0}\,0L$$

$$\dfrac{\dfrac{\overset{\text{IH}}{(E_1^\perp)^\bullet \vdash^i E_1}}{(E_1^\perp)^\bullet \vdash^i E_1 \oplus E_2}\,\oplus_1 R \qquad \dfrac{\overset{\text{IH}}{(E_2^\perp)^\bullet \vdash^i E_2}}{(E_2^\perp)^\bullet \vdash^i E_1 \oplus E_2}\,\oplus_2 R}{(E_1^\perp)^\bullet \oplus (E_2^\perp)^\bullet \vdash^i E_1 \oplus E_2}\,\oplus L$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{IH}}{E_1 \vdash^i \neg_{\mathsf{R}} E_1^\bullet}}{E_1 \vdash^i \neg_{\mathsf{R}} E_1^\bullet \oplus \neg_{\mathsf{R}} E_2^\bullet}\,\oplus_1 R \qquad \dfrac{\overset{\text{IH}}{E_2 \vdash^i \neg_{\mathsf{R}} E_2^\bullet}}{E_2 \vdash^i \neg_{\mathsf{R}} E_1^\bullet \oplus \neg_{\mathsf{R}} E_2^\bullet}\,\oplus_2 R}{E_1 \oplus E_2 \vdash^i \neg_{\mathsf{R}} E_1^\bullet \oplus \neg_{\mathsf{R}} E_2^\bullet}\,\oplus L}{E_1 \oplus E_2, \neg_{\mathsf{R}}(\neg_{\mathsf{R}} E_1^\bullet \oplus \neg_{\mathsf{R}} E_2^\bullet) \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}} L}{E_1 \oplus E_2 \vdash^i \neg_{\mathsf{R}}\neg_{\mathsf{R}}(\neg_{\mathsf{R}} E_1^\bullet \oplus \neg_{\mathsf{R}} E_2^\bullet)}\,\neg_{\mathsf{R}} R$$

$$\dfrac{}{\neg_{\mathsf{R}} 0 \vdash^i \top}\,\top R \qquad\qquad \dfrac{\dfrac{}{\top, 0 \vdash^i \mathsf{R}}\,0L}{\top \vdash^i \neg_{\mathsf{R}} 0}\,\neg_{\mathsf{R}} R$$

□

Proposition 2.12. Let E be a purely positive affine formula, $\Gamma^{\bullet[E]} \vdash^i E$ in ILL if and only if $\vdash^r_E \Gamma$ in $\mathsf{RLL}(E)$.

Proof. Remember that $\underline{E} = E$. The left-to-right implication is Proposition 2.10. The right-to-left implication comes from Lemma 2.11 by choosing $\mathsf{R} = E$ together with:

$$\dfrac{E \vdash^i \neg_E E^{\bullet[E]} \qquad \dfrac{\dfrac{}{E^{\bullet[E]} \vdash^i E^{\bullet[E]}}\,ax \qquad \dfrac{}{E \vdash^i E}\,ax}{E^{\bullet[E]}, \neg_E E^{\bullet[E]} \vdash^i E}\,\multimap L}{E^{\bullet[E]}, E \vdash^i E}\,cut$$

□

Lemma 2.13. If $\vdash^{02} \Gamma$ is provable in LL with (mix_0) and (mix_2) then $!\mathsf{R}, \Gamma^{\bullet[!\mathsf{R}]} \vdash^i !\mathsf{R}$ is provable in ILL for any R.

Proof. This is Proposition 2.4 with !R for the standard LL rules which we decorate with !R according to Lemma C.1. We then have to consider:

$$\dfrac{}{!\mathsf{R} \vdash^i !\mathsf{R}}\ ax \qquad\qquad \dfrac{\overset{\text{IH}}{!\mathsf{R}, \Gamma^{\bullet[!\mathsf{R}]} \vdash^i !\mathsf{R}} \qquad \overset{\text{IH}}{!\mathsf{R}, \Delta^{\bullet[!\mathsf{R}]} \vdash^i !\mathsf{R}}}{!\mathsf{R}, \Gamma^{\bullet[!\mathsf{R}]}, \Delta^{\bullet[!\mathsf{R}]} \vdash^i !\mathsf{R}}\ cut$$

for (mix_0) and (mix_2). □

Proposition 2.14. The following statements are equivalent:

(i) $\vdash^{02} \Gamma$ in LL with (mix_0) and (mix_2)
(ii) $\Gamma^{\bullet[1]} \vdash^i 1$ in ILL
(iii) $\vdash^r_1 \Gamma$ in RLL(1)

Proof. We prove the following three implications:

- *(i)⇒(ii)*: we can apply Lemma 2.5 since:

$$\dfrac{}{\vdash^i 1}\ 1R \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash^i 1}\ 1R}{1 \vdash^i 1}\ 1L}{1, 1 \vdash^i 1}\ 1L}{1 \otimes 1 \vdash^i 1}\ \otimes L$$

  or we can use Lemma 2.13 with $\mathsf{R} = \top$ (since $1 \dashv\vdash\, !\top$ in ILL) by introducing a cut with $\dfrac{\dfrac{}{\vdash^i \top}\ \top R}{\vdash^i !\top}\ !R$.
- *(ii)⇒(iii)*: this is Proposition 2.10.
- *(iii)⇒(i)*: the two additional rules of RLL(1) are derivable in LL extended with (mix_0) and (mix_2):

$$\dfrac{\dfrac{}{\vdash^{02}}\ mix_0}{\vdash^{02} \bot}\ \bot \qquad\qquad \dfrac{\dfrac{}{\vdash^{02} 1}\ 1 \qquad \dfrac{}{\vdash^{02} 1}\ 1}{\vdash^{02} 1, 1}\ mix_2$$

□

Proposition 2.15. The following statements are equivalent:

(i) $\vdash^a \Gamma$ in AL
(ii) $\Gamma^{\bullet[0]} \vdash^i 0$ in ILL
(iii) $\vdash^r_0 \Gamma$ in RLL(0)

Proof. We prove the following three implications:

- *(i)⇒(ii)*: this is Proposition 2.4 with $\mathsf{R} = 0$ for the standard LL rules. We then have to consider:

$$\dfrac{\overset{\text{IH}}{\Gamma^{\bullet[0]} \vdash^i 0} \qquad \dfrac{}{F^{\bullet[0]}, 0 \vdash^i 0}\ 0L}{\Gamma^{\bullet[0]}, F^{\bullet[0]} \vdash^i 0}\ cut$$

- *(ii)⇒(iii)*: this is Proposition 2.10.
- *(iii)⇒(i)*: the two additional rules of RLL(0) are derivable in LL extended with (wk):

$$\dfrac{}{\vdash^a \top}\ \top \qquad\qquad \dfrac{\dfrac{}{\vdash^a 1}\ 1}{\vdash^a 0, 1}\ wk$$

□

Proposition 2.16. The following statements are equivalent:

(i) $\vdash \Gamma$ in LL
(ii) for all R, $\Gamma^{\bullet[\mathsf{R}]} \vdash^i \mathsf{R}$ in ILL
(iii) there exists $\Phi \notin \Gamma$ such that $\Gamma^{\bullet[\Phi]} \vdash^i \Phi$ in ILL
(iv) there exists $\Phi \notin \Gamma$ such that $\vdash^r_\Phi \Gamma$ in RLL(Φ)
(v) $\vdash^r_\bot \Gamma$ in RLL(⊥)

Proof. We prove the following five implications:

- *(i)⇒(ii)*: this is Proposition 2.4.
- *(ii)⇒(iii)*: let Φ be a propositional variable not free in Γ, we simply instantiate R with Φ.
- *(iii)⇒(iv)*: this is Proposition 2.10 with $\underline{\Phi} = \Phi$.
- *(iv)⇒(v)*: by Lemma 2.6, $\vdash^r_\bot \Gamma$ in RLL(⊥) since $\Phi \notin \Gamma$.

- *(v)⇒(i)*: the provability in LL and RLL(⊥) are the same since the two additional rules of RLL(⊥) are derivable in LL:

$$\dfrac{}{\vdash 1}\,1 \qquad\qquad \dfrac{}{\vdash \bot, 1}\,ax$$

□

Corollary 2.17. Cut elimination in ILL entails cut elimination in LL.

Proof. If $\vdash \Gamma, F$ and $\vdash \Delta, F^\perp$ are provable in LL, by Proposition 2.4, we have $\Gamma^\bullet, F^\bullet \vdash^i \mathsf{R}$ and $\Delta^\bullet, (F^\perp)^\bullet \vdash^i \mathsf{R}$ and thus $\Gamma^\bullet, \Delta^\bullet \vdash^i \mathsf{R}$ in ILL by using Lemma 2.3. By cut elimination in ILL, one can build a cut-free proof of $\Gamma^\bullet, \Delta^\bullet \vdash^i \mathsf{R}$, and thus a cut-free proof of $\vdash \underline{\Gamma^\bullet}^\perp, \underline{\Delta^\bullet}^\perp, \mathsf{R}$ in LL. We choose R to be a fresh propositional variable Φ and, by substitution, we have $\vdash \underline{\Gamma^\bullet}^\perp[^\perp/_\Phi], \underline{\Delta^\bullet}^\perp[^\perp/_\Phi], \bot$.

We can check by induction on F that $\underline{F^\bullet}^\perp[^\perp/_\Phi]$ is obtained from F by adding some $\_ \parr \bot$ and $\_ \otimes 1$ in it. We conclude by induction on the proof of $\vdash \underline{\Gamma^\bullet}^\perp[^\perp/_\Phi], \underline{\Delta^\bullet}^\perp[^\perp/_\Phi], \bot$ that $\vdash \Gamma, \Delta$ in LL. This requires us to remove some $\_ \parr \bot$ which can be obtained by reversibility of $\parr$ and $\bot$, but also to remove some $\_ \otimes 1$. We first prove that if $\vdash \Gamma, 1$ is (cut-free) provable in LL, then for any $\vdash \Delta$ (cut-free) provable in LL, $\vdash \Gamma, \Delta$ is (cut-free) provable as well, and then we show that each $(\otimes)$ rule introducing $\_ \otimes 1$:

$$\dfrac{\vdash \Gamma_1, F \qquad \vdash \Gamma_2, 1}{\vdash \Gamma_1, \Gamma_2, F \otimes 1}\,\otimes$$

can be transformed into:

$$\dfrac{\vdash \Gamma_1, F}{\vdash \Gamma_1, \Gamma_2, F}$$

□

Lemma 2.18. It is not possible to have both $\vdash^0 (\underline{I})^\perp$ and $\vdash^0\, !F, \underline{I}$ provable in LL with (mix_0), for any I of ILL and any F of LL.

Proof. By induction on I:

- $\vdash^0 X^\perp$ is not provable in LL with (mix_0).
- $\vdash^0\, !F, 1$ is not provable in LL with (mix_0): there is no possible last rule in a cut-free proof.
- If both $\vdash^0\, !F, \underline{J} \otimes \underline{K}$ and $\vdash^0 (\underline{J})^\perp \parr (\underline{K})^\perp$ are provable in LL with (mix_0) then $\vdash^0 (\underline{J})^\perp, (\underline{K})^\perp$ as well. Moreover either $\vdash^0\, !F, \underline{J}$ and $\vdash^0 \underline{K}$, or $\vdash^0\, !F, \underline{K}$ and $\vdash^0 \underline{J}$ are provable. So that we can build:

$$\dfrac{\vdash^0 (\underline{J})^\perp, (\underline{K})^\perp \qquad \vdash^0 \underline{K}}{\vdash^0 (\underline{J})^\perp}\,cut \qquad \text{or} \qquad \dfrac{\vdash^0 (\underline{J})^\perp, (\underline{K})^\perp \qquad \vdash^0 \underline{J}}{\vdash^0 (\underline{K})^\perp}\,cut$$

  and we apply the corresponding induction hypothesis on J or K.
- If both $\vdash^0\, !F, (\underline{J})^\perp \parr \underline{K}$ and $\vdash^0 \underline{J} \otimes (\underline{K})^\perp$ are provable in LL with (mix_0) then $\vdash^0\, !F, (\underline{J})^\perp, \underline{K}$, $\vdash^0 \underline{J}$, and $\vdash^0 (\underline{K})^\perp$ as well. So that we can build:

$$\dfrac{\vdash^0\, !F, (\underline{J})^\perp, \underline{K} \qquad \vdash^0 \underline{J}}{\vdash^0\, !F, \underline{K}}\,cut$$

  and we apply the induction hypothesis on K.
- $\vdash^0\, !F, 0$ is not provable in LL with (mix_0): there is no possible last rule in a cut-free proof.
- $\vdash^0 0$ is not provable in LL with (mix_0): there is no possible last rule in a cut-free proof.
- If both $\vdash^0\, !F, \underline{J} \oplus \underline{K}$ and $\vdash^0 (\underline{J})^\perp \,\&\, (\underline{K})^\perp$ are provable in LL with (mix_0) then $\vdash^0 (\underline{J})^\perp$ and $\vdash^0 (\underline{K})^\perp$ also, as well as $\vdash^0\, !F, \underline{J}$ or $\vdash^0\, !F, \underline{K}$. And we apply the corresponding induction hypothesis on J or K.
- If both $\vdash^0\, !F, \underline{J} \,\&\, \underline{K}$ and $\vdash^0 (\underline{J})^\perp \oplus (\underline{K})^\perp$ are provable in LL with (mix_0) then $\vdash^0\, !F, \underline{J}$ and $\vdash^0\, !F, \underline{K}$ also, as well as $\vdash^0 (\underline{J})^\perp$ or $\vdash^0 (\underline{K})^\perp$. We apply the corresponding induction hypothesis on J or K.
- $\vdash^0\, !F, !\underline{J}$ is not provable in LL with (mix_0): there is no possible last rule in a cut-free proof. □

Lemma 2.19. It is not possible to have both $\vdash (\underline{I})^\perp$ and $\vdash 1, \underline{I}$ provable in LL, for any I of ILL.

Proof. Otherwise one would have:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \top}\,\top}{\vdash !\top}\,!}{\vdash !\top, \bot}\,\bot \qquad \vdash 1, \underline{I}}{\vdash !\top, \underline{I}}\,cut$$

thus $\vdash^0 (\underline{I})^\perp$ and $\vdash^0\, !\top, \underline{I}$ provable in LL with (mix_0) (since it contains LL), contradicting Lemma 2.18. □

Lemma 2.20. It is not possible to have both $\vdash (\underline{I})^\perp, ?1$ and $\vdash !\bot, \underline{I}$ provable in LL, for any I of ILL.

Proof. Otherwise one would have $\vdash^0 (\underline{I})^\perp, ?1$ and $\vdash^0\, !\bot, \underline{I}$ provable in LL with (mix_0) (since it contains LL). So that:

$$\dfrac{\vdash^0 (\underline{I})^\perp, ?1 \qquad \dfrac{\dfrac{\dfrac{}{\vdash^0}\,mix_0}{\vdash^0 \bot}\,\bot}{\vdash^0 !\bot}\,!}{\vdash^0 (\underline{I})^\perp}\,cut$$

thus contradicting Lemma 2.18. □

Proposition 2.21. The following statements are equivalent:

(i) $\vdash^0 \Gamma$ in LL with (mix_0)
(ii) for all R provable in ILL, $\Gamma^{\bullet[\mathsf{R}]} \vdash^i \mathsf{R}$ in ILL
(iii) there exists $\Phi \notin \Gamma$ such that $\Gamma^{\bullet[!\Phi \multimap \Phi]} \vdash^i !\Phi \multimap \Phi$ in ILL
(iv) there exists $\Phi \notin \Gamma$ such that $\vdash^r_{?\Phi^\perp \parr \Phi} \Gamma$ in RLL($?\Phi^\perp \parr \Phi$)
(v) $\vdash^r_{?1} \Gamma$ in RLL(?1)
(vi) $\vdash \Gamma, ?1$ in LL

Proof. We prove the following six implications:

- *(i)⇒(ii)*: this is Lemma 2.5.
- *(ii)⇒(iii)*: let Φ be a propositional variable not free in Γ, we can instantiate R with $!\Phi \multimap \Phi$ since:

$$\dfrac{\dfrac{\dfrac{}{\Phi \vdash^i \Phi}\,ax}{!\Phi \vdash^i \Phi}\,!L}{\vdash^i !\Phi \multimap \Phi}\,\multimap R$$

- *(iii)⇒(iv)*: this is Proposition 2.10.
- *(iv)⇒(v)*: by Lemma 2.6, $\vdash^r_{?1 \parr \bot} \Gamma$ is provable in RLL($?1 \parr \bot$) since $\Phi \notin \Gamma$. But, since $?1 \parr \bot \dashv\vdash ?1$ in LL, this is equivalent to $\vdash^r_{?1} \Gamma$ provable in RLL(?1).
- *(v)⇒(vi)*: this is Lemma 2.8.
- *(vi)⇒(i)*: this is Lemma B.1 since:

$$\dfrac{\dfrac{}{\vdash^0}\,mix_0}{\vdash^0 \bot}\,\bot$$

□

Lemma 2.22. It is not possible to have both $\vdash (\underline{\Gamma})^\perp$ provable in LL and $\vdash^0 \underline{I}$ provable in LL with (mix_0) for each $I \in \Gamma$, for any Γ of ILL.

Proof. By induction on the proof of $\vdash (\underline{\Gamma})^\perp$ in LL, we look at the last rule:

- ($\parr$) rule: We have $\vdash (\underline{\Gamma'})^\perp, (\underline{J})^\perp, (\underline{K})^\perp$, $\vdash^0 \underline{J}$ and $\vdash^0 \underline{K}$ as well as $\vdash^0 \underline{I}$ for any I in Γ' thus we can apply the induction hypothesis.
- ($\otimes$) rule: We have $\Gamma = \Gamma', \Gamma''$ with $\vdash (\underline{\Gamma'})^\perp, \underline{J}$ and $\vdash (\underline{\Gamma''})^\perp, (\underline{K})^\perp$, and $\vdash^0 (\underline{J})^\perp, \underline{K}$, as well as $\vdash^0 \underline{I}$ for any I in Γ', Γ'' thus we can build:

$$\dfrac{\begin{array}{c}\vdash^0 (\underline{\Gamma'})^\perp, \underline{J} \\ \dfrac{\vdots \qquad \vdash^0 \underline{I}}{}\,cut \\ \vdots \\ \vdash^0 \underline{J}\end{array} \qquad \vdash^0 (\underline{J})^\perp, \underline{K}}{\vdash^0 \underline{K}}\,cut$$

  and we apply the induction hypothesis with the proof of $\vdash (\underline{\Gamma''})^\perp, (\underline{K})^\perp$.
- ($\top$) rule: $\vdash^0 0$ is not provable in LL with (mix_0), there is no possible last rule in a cut-free proof.
- For ($\bot$), (&), ($\oplus_i$), ($?d$), ($?c$) and ($?w$) we simply apply the induction hypothesis.
- The other rules are not possible. □

Lemma 2.23. It is not possible to have both $\vdash (\underline{I})^\perp, !\bot$ and $\vdash ?1, \underline{I}$ provable in LL, for any I of ILL.

Proof. Otherwise one would have $\vdash^0 ?1, \underline{I}$ provable in LL with (mix_0) (since it contains LL), and then:

$$\dfrac{\vdash (\underline{I})^\perp, !\bot \qquad \dfrac{\dfrac{}{\vdash 1}\,1}{\vdash ?1}\,?d}{\vdash (\underline{I})^\perp}\,cut \qquad \text{and} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash^0}\,mix_0}{\vdash^0 \bot}\,\bot}{\vdash^0 !\bot}\,! \qquad \vdash^0 ?1, \underline{I}}{\vdash^0 \underline{I}}\,cut$$

thus contradicting Lemma 2.22. □

Lemma 2.24. If $\vdash^{!\perp} \Gamma$ is provable in $\mathsf{LL}^{!\perp}$ then $\vdash^{02} \Gamma$ is provable in LL with both (mix_0) and (mix_2).

Proof. Turning any sequent $\vdash^{!\perp} \Gamma$ into $\vdash^{02} \Gamma$ and any $(mix^{!\perp})$ rule into a (mix_2) rule transforms a proof in $\mathsf{LL}^{!\perp}$ into a proof in LL with both (mix_0) and (mix_2). □

Proposition 2.25. $\mathsf{LL}^{!\perp}$ has the cut-elimination property.

Proof. Everything goes as for LL cut-elimination extended with the following cases:

$$\dfrac{\dfrac{\vdash^{!\perp} \Gamma, A \qquad \vdash^{02} \Delta}{\vdash^{!\perp} \Gamma, A, \Delta}\, mix^{!\perp} \qquad \vdash^{!\perp} \Sigma, A^{\perp}}{\vdash^{!\perp} \Gamma, \Delta, \Sigma}\, cut \quad \mapsto \quad \dfrac{\dfrac{\vdash^{!\perp} \Gamma, A \qquad \vdash^{!\perp} \Sigma, A^{\perp}}{\vdash^{!\perp} \Gamma, \Sigma}\, cut \qquad \vdash^{02} \Delta}{\vdash^{!\perp} \Gamma, \Delta, \Sigma}\, mix^{!\perp}$$

$$\dfrac{\dfrac{\vdash^{!\perp} \Gamma \qquad \vdash^{02} \Delta, A}{\vdash^{!\perp} \Gamma, \Delta, A}\, mix^{!\perp} \qquad \vdash^{!\perp} \Sigma, A^{\perp}}{\vdash^{!\perp} \Gamma, \Delta, \Sigma}\, cut \quad \mapsto \quad \dfrac{\vdash^{!\perp} \Gamma \qquad \dfrac{\vdash^{02} \Delta, A \qquad \dfrac{\vdash^{!\perp} \Sigma, A^{\perp}}{\vdash^{02} \Sigma, A^{\perp}}\, \text{Lemma 2.24}}{\vdash^{02} \Delta, \Sigma}\, cut}{\vdash^{!\perp} \Gamma, \Delta, \Sigma}\, mix^{!\perp}$$

□

Proposition 2.26. The following statements are equivalent:

(i) $\vdash^{!\perp} \Gamma$ in $\mathsf{LL}^{!\perp}$
(ii) for all R, $\Gamma^{\bullet[!\mathsf{R}]} \vdash^{i} !\mathsf{R}$ in ILL
(iii) there exists $\Phi \notin \Gamma$ such that $\Gamma^{\bullet[!\Phi]} \vdash^{i} !\Phi$ in ILL
(iv) there exists $\Phi \notin \Gamma$ such that $\vdash^{r}_{!\Phi} \Gamma$ in $\mathsf{RLL}(!\Phi)$
(v) $\vdash^{r}_{!\perp} \Gamma$ in $\mathsf{RLL}(!\perp)$

Proof. We prove the following five implications:

- *(i)⇒(ii)*: this is Proposition 2.4 for the standard LL rules. We then have to consider:

$$\dfrac{\overset{\text{IH}}{\Gamma^{\bullet[!\mathsf{R}]} \vdash^{i} !\mathsf{R}} \qquad \overset{\text{Lemma 2.13}}{!\mathsf{R}, \Delta^{\bullet[!\mathsf{R}]} \vdash^{i} !\mathsf{R}}}{\Gamma^{\bullet[!\mathsf{R}]}, \Delta^{\bullet[!\mathsf{R}]} \vdash^{i} !\mathsf{R}}\, cut$$

- *(ii)⇒(iii)*: let Φ be a propositional variable not free in Γ, we simply instantiate R with Φ.
- *(iii)⇒(iv)*: this is Proposition 2.10.
- *(iv)⇒(v)*: by Lemma 2.6, $\vdash^{r}_{!\perp} \Gamma$ is provable in $\mathsf{RLL}(!\perp)$ since $\Phi \notin \Gamma$.
- *(v)⇒(i)*: the two additional rules of $\mathsf{RLL}(!\perp)$ are derivable in $\mathsf{LL}^{!\perp}$:

$$\dfrac{\dfrac{}{\vdash^{!\perp} 1}\, 1}{\vdash^{!\perp} ?1}\, ?d \qquad\qquad \dfrac{\dfrac{}{\vdash^{!\perp} 1}\, 1 \qquad \dfrac{\dfrac{\dfrac{}{\vdash^{02}}\, mix_0}{\vdash^{02} \perp}\, \perp}{\vdash^{02} !\perp}\, !}{\vdash^{!\perp} 1, !\perp}\, mix^{!\perp}$$

□

Proposition 2.27. For any $n \geq 0$, the following statements are equivalent:

(i) $\vdash^{n} \Gamma$ in LL with (mix_n)
(ii) for all R such that $\bigotimes_n \mathsf{R} \vdash^{i} \mathsf{R}$ is provable in ILL, $\Gamma^{\bullet[\mathsf{R}]} \vdash^{i} \mathsf{R}$ in ILL
(iii) there exists $\Phi \notin \Gamma$ such that $\Gamma^{\bullet[!((\bigotimes_n \Phi) \multimap \Phi) \multimap \Phi]} \vdash^{i} !((\bigotimes_n \Phi) \multimap \Phi) \multimap \Phi$ in ILL
(iv) there exists $\Phi \notin \Gamma$ such that $\vdash^{r}_{?((\bigotimes_n \Phi) \otimes \Phi^{\perp}) \parr \Phi} \Gamma$ is provable in $\mathsf{RLL}(?((\bigotimes_n \Phi) \otimes \Phi^{\perp}) \parr \Phi)$
(v) $\vdash^{r}_{?\bigotimes_n \perp} \Gamma$ in $\mathsf{RLL}(?\bigotimes_n \perp)$
(vi) $\vdash \Gamma, ?\bigotimes_n \perp$ in LL

Proof. We prove the following six implications:

- *(i)⇒(ii)*: this is Lemma 2.5.
- *(ii)⇒(iii)*: let Φ be a propositional variable not free in Γ, we can instantiate R with $!((\bigotimes_n \Phi) \multimap \Phi) \multimap \Phi$ since (using the notation $I = !((\bigotimes_n \Phi) \multimap \Phi) \multimap \Phi$):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{!((\bigotimes_n \Phi) \multimap \Phi) \vdash^i !((\bigotimes_n \Phi) \multimap \Phi)}\,ax \quad \dfrac{}{\Phi \vdash^i \Phi}\,ax}{I, !((\bigotimes_n \Phi) \multimap \Phi) \vdash^i \Phi}\,\multimap L \quad \begin{array}{c}\dfrac{\dfrac{\dfrac{}{\vdash^i \bigotimes_0 \Phi}\,1}{!((\bigotimes_n \Phi) \multimap \Phi) \vdash^i \bigotimes_0 \Phi}\,!wL}{\bigotimes_0 I, !((\bigotimes_n \Phi) \multimap \Phi) \vdash^i \bigotimes_0 \Phi}\,1L \\ \vdots \\ \bigotimes_{n-1} I, !((\bigotimes_n \Phi) \multimap \Phi) \vdash^i \bigotimes_{n-1} \Phi\end{array}}{I, \bigotimes_{n-1} I, !((\bigotimes_n \Phi) \multimap \Phi), !((\bigotimes_n \Phi) \multimap \Phi) \vdash^i \bigotimes_n \Phi}\,\otimes R}{\bigotimes_n I, !((\bigotimes_n \Phi) \multimap \Phi), !((\bigotimes_n \Phi) \multimap \Phi) \vdash^i \bigotimes_n \Phi}\,\otimes L}{\bigotimes_n I, !((\bigotimes_n \Phi) \multimap \Phi) \vdash^i \bigotimes_n \Phi}\,!cL \quad \dfrac{}{\Phi \vdash^i \Phi}\,ax}{\bigotimes_n I, !((\bigotimes_n \Phi) \multimap \Phi), (\bigotimes_n \Phi) \multimap \Phi \vdash^i \Phi}\,\multimap L}{\bigotimes_n I, !((\bigotimes_n \Phi) \multimap \Phi), !((\bigotimes_n \Phi) \multimap \Phi) \vdash^i \Phi}\,!L}{\bigotimes_n I, !((\bigotimes_n \Phi) \multimap \Phi) \vdash^i \Phi}\,!cL}{\bigotimes_n I \vdash^i I}\,\multimap R
$$

- *(iii)⇒(iv)*: this is Proposition 2.10.
- *(iv)⇒(v)*: by Lemma 2.6, $\vdash^r_{?((\bigotimes_n \bot) \otimes 1) \parr \bot} \Gamma$ is provable in RLL($?((\bigotimes_n \bot) \otimes 1) \parr \bot$) since $\Phi \notin \Gamma$. But, since $?((\bigotimes_n \bot) \otimes 1) \parr \bot \dashv\vdash ? \bigotimes_n \bot$ in LL, this is equivalent to $\vdash^r_{? \bigotimes_n \bot} \Gamma$ provable in RLL($? \bigotimes_n \bot$).
- *(v)⇒(vi)*: this is Lemma 2.8.
- *(vi)⇒(i)*: we can use a cut since $\vdash^n \, ! \parr_n 1$ is derivable in LL with (mix_n):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash^n 1}\,1 \quad \dots \quad \dfrac{}{\vdash^n 1}\,1}{\vdash^n 1, \dots, 1}\,mix_n}{\vdash^n 1, \dots, 1, \bot}\,\bot}{\vdash^n \parr_n 1}\,\parr}{\vdash^n \, ! \parr_n 1}\,!
$$

□

Proposition 2.28. The following statements are equivalent:

(i) $\vdash^{02} \Gamma$ in LL with (mix_0) and (mix_2)
(ii) $\Gamma^{\bullet[\Phi \multimap \Phi]} \vdash^i \Phi \multimap \Phi$ in ILL with $\Phi \notin \Gamma$
(iii) $\Gamma^{\bullet[!\Phi \multimap 1]} \vdash^i !\Phi \multimap 1$ in ILL with $\Phi \notin \Gamma$
(iv) $\Gamma^{\bullet[!((!\Phi \multimap \Phi) \multimap (!\Phi \multimap \Phi) \multimap \Phi) \multimap \Phi]} \vdash^i !((!\Phi \multimap \Phi) \multimap (!\Phi \multimap \Phi) \multimap \Phi) \multimap \Phi$ in ILL with $\Phi \notin \Gamma$
(v) $\vdash \Gamma, ?(?1 \otimes ?1)$ in LL

Proof. We prove *(i)⇔(ii)*, *(i)⇔(iii)*, and *(i)⇒(iv)⇒(v)⇒(i)*.

- *(i)⇒(ii)*: we can apply Lemma 2.5 since:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Phi \vdash^i \Phi}\,ax \quad \dfrac{}{\Phi \vdash^i \Phi}\,ax}{\Phi \multimap \Phi, \Phi \vdash^i \Phi}\,\multimap L \quad \dfrac{}{\Phi \vdash^i \Phi}\,ax}{\Phi \multimap \Phi, \Phi \multimap \Phi, \Phi \vdash^i \Phi}\,\multimap L}{(\Phi \multimap \Phi) \otimes (\Phi \multimap \Phi), \Phi \vdash^i \Phi}\,\otimes L}{(\Phi \multimap \Phi) \otimes (\Phi \multimap \Phi) \vdash^i \Phi \multimap \Phi}\,\multimap R
\qquad\qquad
\dfrac{\dfrac{}{\Phi \vdash^i \Phi}\,ax}{\vdash^i \Phi \multimap \Phi}\,\multimap R
$$

- *(ii)⇒(i)*: by Proposition 2.10, we have $\vdash^r_{\Phi^\bot \parr \Phi} \Gamma$ is provable in RLL($\Phi^\bot \parr \Phi$). By Lemma 2.6, $\vdash^r_{1 \parr \bot} \Gamma$ is provable in RLL($1 \parr \bot$) since $\Phi \notin \Gamma$. But, since $1 \parr \bot \dashv\vdash 1$ in LL, this is equivalent to $\vdash^r_1 \Gamma$ provable in RLL(1) and we conclude with Proposition 2.14.
- *(i)⇒(iii)*: we can apply Lemma 2.5 since:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{!\Phi \vdash^i !\Phi}\,ax \quad \dfrac{\dfrac{\dfrac{}{!\Phi \vdash^i !\Phi}\,ax \quad \dfrac{}{1 \vdash^i 1}\,ax}{!\Phi \multimap 1, !\Phi \vdash^i 1}\,\multimap L}{1, !\Phi \multimap 1, !\Phi \vdash^i 1}\,1L}{!\Phi \multimap 1, !\Phi \multimap 1, !\Phi, !\Phi \vdash^i 1}\,\multimap L}{\dfrac{!\Phi \multimap 1, !\Phi \multimap 1, !\Phi \vdash^i 1}{(!\Phi \multimap 1) \otimes (!\Phi \multimap 1), !\Phi \vdash^i 1}\,\otimes L}\,!cL \;\Rightarrow\; \dfrac{(!\Phi \multimap 1) \otimes (!\Phi \multimap 1), !\Phi \vdash^i 1}{(!\Phi \multimap 1) \otimes (!\Phi \multimap 1) \vdash^i !\Phi \multimap 1}\,\multimap R
\qquad\qquad
\dfrac{\dfrac{\dfrac{}{\vdash^i 1}\,1R}{!\Phi \vdash^i 1}\,!wL}{\vdash^i !\Phi \multimap 1}\,\multimap R
$$

- *(iii)⇒(i)*: by Proposition 2.10, we have $\vdash^r_{?\Phi^\perp \parr 1} \Gamma$ is provable in RLL($?\Phi^\perp \parr 1$). By Lemma 2.6, $\vdash^r_{?\perp \parr 1} \Gamma$ is provable in RLL($?\perp \parr 1$) since $\Phi \notin \Gamma$. But, since $?\perp \parr 1 \dashv\vdash 1$ in LL, this is equivalent to $\vdash^r_1 \Gamma$ provable in RLL(1) and we conclude with Proposition 2.14.
- *(i)⇒(iv)*: we can apply Lemma 2.5 since (using the notation $I = (!\Phi \multimap \Phi) \multimap (!\Phi \multimap \Phi) \multimap \Phi$):

$$
\dfrac{\dfrac{\dfrac{!I \multimap \Phi, !I, !I, !I, !I \vdash^i \Phi}{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{!I \vdash^i !I}\,ax \quad \dfrac{}{\Phi \vdash^i \Phi}\,ax}{!I \multimap \Phi, !I \vdash^i \Phi}\multimap L}{!I \multimap \Phi, !I, !\Phi \vdash^i \Phi}\,!wL}{!I \multimap \Phi, !I \vdash^i !\Phi \multimap \Phi}\multimap R \quad \dfrac{\dfrac{\dfrac{\dfrac{}{!I \vdash^i !I}\,ax \quad \dfrac{}{\Phi \vdash^i \Phi}\,ax}{!I \multimap \Phi, !I \vdash^i \Phi}\multimap L}{\dfrac{!I \multimap \Phi, !I, !\Phi \vdash^i \Phi}{!I \multimap \Phi, !I \vdash^i !\Phi \multimap \Phi}\multimap R}\,!wL \quad \dfrac{}{\Phi \vdash^i \Phi}\,ax}{!I \multimap \Phi, !I, (!\Phi \multimap \Phi) \multimap \Phi \vdash^i \Phi}\multimap L}{!I \multimap \Phi, !I \multimap \Phi, !I, !I, I \vdash^i \Phi}\multimap L}{!I \multimap \Phi, !I \multimap \Phi, !I, !I, !I \vdash^i \Phi}\,!L}\,!cL}{!I \multimap \Phi, !I \multimap \Phi, !I \vdash^i \Phi}}{\dfrac{(!I \multimap \Phi) \otimes (!I \multimap \Phi), !I \vdash^i \Phi}{(!I \multimap \Phi) \otimes (!I \multimap \Phi) \vdash^i !I \multimap \Phi}\multimap R}\otimes L
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Phi \vdash^i \Phi}\,ax}{!\Phi \vdash^i \Phi}\,!L}{\vdash^i !\Phi \multimap \Phi}\multimap R \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\Phi \vdash^i \Phi}\,ax}{!\Phi \vdash^i \Phi}\,!L}{\vdash^i !\Phi \multimap \Phi}\multimap R \quad \dfrac{}{\Phi \vdash^i \Phi}\,ax}{(!\Phi \multimap \Phi) \multimap \Phi \vdash^i \Phi}\multimap L}{(!\Phi \multimap \Phi) \multimap (!\Phi \multimap \Phi) \multimap \Phi \vdash^i \Phi}\multimap L}{!((!\Phi \multimap \Phi) \multimap (!\Phi \multimap \Phi) \multimap \Phi) \vdash^i \Phi}\,!L}{\vdash^i !((!\Phi \multimap \Phi) \multimap (!\Phi \multimap \Phi) \multimap \Phi) \multimap \Phi}\multimap R
$$

- *(iv)⇒(v)*: by Proposition 2.10, we have $\vdash^r_{?((?\Phi^\perp \parr \Phi) \otimes ((?\Phi^\perp \parr \Phi) \otimes \Phi^\perp)) \parr \Phi} \Gamma$ is provable in RLL($?((?\Phi^\perp \parr \Phi) \otimes ((?\Phi^\perp \parr \Phi) \otimes \Phi^\perp)) \parr \Phi$). By Lemma 2.6, $\vdash^r_{?((?1 \parr \perp) \otimes ((?1 \parr \perp) \otimes 1)) \parr \perp} \Gamma$ is provable in RLL($?((?1 \parr \perp) \otimes ((?1 \parr \perp) \otimes 1)) \parr \perp$) since $\Phi \notin \Gamma$. But, since $?((?1 \parr \perp) \otimes ((?1 \parr \perp) \otimes 1)) \parr \perp \dashv\vdash ?(?1 \otimes ?1)$ in LL, this is equivalent to $\vdash^r_{?(?1 \otimes ?1)} \Gamma$ provable in RLL($?(?1 \otimes ?1)$) and we conclude with Lemma 2.8.
- *(v)⇒(i)*: this is Lemma B.1 since:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash^{02}}\,mix_0}{\vdash^{02} \perp}\,\perp}{\vdash^{02} !\perp}\,! \quad \dfrac{\dfrac{\dfrac{}{\vdash^{02}}\,mix_0}{\vdash^{02} \perp}\,\perp}{\vdash^{02} !\perp}\,!}{\vdash^{02} !\perp, !\perp}\,mix_2}{\vdash^{02} !\perp \parr !\perp}\,\parr
$$

□

Section 3

Counterexample.

$$(((X \otimes \top) \,\&\, (Y \otimes \top)) \multimap 0) \multimap ((X \multimap X') \oplus (Y \multimap Y'))$$

Proof. The proof in LL:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash X, X^\perp}\,ax \quad \dfrac{}{\vdash \top, X'}\,\top}{\vdash X \otimes \top, X^\perp, X'}\otimes}{\vdash X \otimes \top, X^\perp \parr X'}\parr}{\vdash X \otimes \top, (X^\perp \parr X') \oplus (Y^\perp \parr Y')}\oplus_1 \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash Y, Y^\perp}\,ax \quad \dfrac{}{\vdash \top, Y'}\,\top}{\vdash Y \otimes \top, Y^\perp, Y'}\otimes}{\vdash Y \otimes \top, Y^\perp \parr Y'}\parr}{\vdash Y \otimes \top, (X^\perp \parr X') \oplus (Y^\perp \parr Y')}\oplus_2}{\vdash (X \otimes \top) \,\&\, (Y \otimes \top), (X^\perp \parr X') \oplus (Y^\perp \parr Y')}\,\& \quad \dfrac{}{\vdash \top}\,\top}{\vdash ((X \otimes \top) \,\&\, (Y \otimes \top)) \otimes \top, (X^\perp \parr X') \oplus (Y^\perp \parr Y')}\otimes}{\vdash (((X \otimes \top) \,\&\, (Y \otimes \top)) \otimes \top) \parr ((X^\perp \parr X') \oplus (Y^\perp \parr Y'))}\parr
$$

Up to an appropriate use of the reversibility of rules, one can see there are mainly two patterns for tentative proofs in ILL:

$$\dfrac{\dfrac{\dfrac{\vdash^i X \qquad \dfrac{}{\vdash^i \top}\,\top R}{\vdash^i X \otimes \top}\,\otimes R \qquad \dfrac{\vdash^i Y \qquad \dfrac{}{\vdash^i \top}\,\top R}{\vdash^i Y \otimes \top}\,\otimes R}{\vdash^i (X \otimes \top) \,\&\, (Y \otimes \top)}\,\&R \qquad \dfrac{}{0 \vdash^i (X \multimap X') \oplus (Y \multimap Y')}\,0L}{\dfrac{((X \otimes \top) \,\&\, (Y \otimes \top)) \multimap 0 \vdash^i (X \multimap X') \oplus (Y \multimap Y')}{\vdash^i (((X \otimes \top) \,\&\, (Y \otimes \top)) \multimap 0) \multimap ((X \multimap X') \oplus (Y \multimap Y'))}\,\multimap R}\,\multimap L$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{X \vdash^i X}\,ax \qquad \dfrac{}{\vdash^i \top}\,\top R}{X \vdash^i X \otimes \top}\,\otimes R \qquad \dfrac{\vdash^i Y \qquad \dfrac{}{X \vdash^i \top}\,\top R}{X \vdash^i Y \otimes \top}\,\otimes R}{X \vdash^i (X \otimes \top) \,\&\, (Y \otimes \top)}\,\&R \qquad \dfrac{}{0 \vdash^i X'}\,0L}{\dfrac{((X \otimes \top) \,\&\, (Y \otimes \top)) \multimap 0, X \vdash^i X'}{((X \otimes \top) \,\&\, (Y \otimes \top)) \multimap 0 \vdash^i X \multimap X'}\,\multimap R}\,\multimap L}{\dfrac{((X \otimes \top) \,\&\, (Y \otimes \top)) \multimap 0 \vdash^i (X \multimap X') \oplus (Y \multimap Y')}{\vdash^i (((X \otimes \top) \,\&\, (Y \otimes \top)) \multimap 0) \multimap ((X \multimap X') \oplus (Y \multimap Y'))}\,\multimap R}\,\oplus_1 R$$

and both fail. □

Lemma 3.2. If $\vdash \underline{\Gamma}^\perp, \underline{\Omega}, \underline{I}$ is provable in LL (where Ω contains !-like formulas only, and all sub-formulas of Γ and I of the shape $J \multimap K$ are such that J is !-like), then $\Gamma \vdash^i I$ is provable in ILL, and if Ω is not empty then $\Gamma, \Sigma \vdash^i I$ in ILL for any Σ.

Proof. By induction on a cut-free proof of $\vdash \underline{\Gamma}^\perp, \underline{\Omega}, \underline{I}$ with atomic axioms:

- atomic (ax) rule: we have $\Omega = \emptyset$ and $X \vdash^i X$ provable in ILL.
- ($⅋$) rule:
  - If $\underline{I} = F ⅋ G$, then $I = J \multimap K$ with $\underline{J} = F^\perp$ and $\underline{K} = G$. We have:

$$\dfrac{\overset{\text{IH}}{\Gamma, J, (\Sigma) \vdash^i K}}{\Gamma, (\Sigma) \vdash^i J \multimap K}\,\multimap R$$

  - If $\underline{\Gamma}^\perp$ contains $F ⅋ G$ then $\Gamma = \Gamma', J \otimes K$ with $\underline{J} = F^\perp$ and $\underline{K} = G^\perp$. We have:

$$\dfrac{\overset{\text{IH}}{\Gamma', J, K, (\Sigma) \vdash^i I}}{\Gamma', J \otimes K, (\Sigma) \vdash^i I}\,\otimes L$$

  - Finally, by definition of !-like, $\underline{\Omega}$ cannot contain a formula of the shape $F ⅋ G$.
- ($\otimes$) rule:
  - If $\underline{I} = F \otimes G$, then $I = J \otimes K$ with $\underline{J} = F$ and $\underline{K} = G$. In the contexts of the premises, Γ is split into Δ' and Δ'' and Ω is split into Ω' and Ω''. We have:

$$\dfrac{\overset{\text{IH}}{\Delta' \vdash^i J} \qquad \overset{\text{IH}}{\Delta'' \vdash^i K}}{\Delta', \Delta'' \vdash^i J \otimes K}\,\otimes R$$

    If Ω is not empty, then at least one of Ω' and Ω'' as well. If, for example, $\Omega' \neq \emptyset$ then we have, for all Σ:

$$\dfrac{\overset{\text{IH}}{\Delta', \Sigma \vdash^i J} \qquad \overset{\text{IH}}{\Delta'' \vdash^i K}}{\Delta', \Delta'', \Sigma \vdash^i J \otimes K}\,\otimes R$$

  - If $\underline{\Gamma}^\perp$ contains $F \otimes G$ then $\Gamma = \Gamma', J \multimap K$ with $\underline{J} = F$ and $\underline{K} = G^\perp$. If $\underline{I}$ belongs to the same premise as F, we have $\Gamma = \Delta', \Delta'', J \multimap K$ with $\vdash \underline{\Delta'}^\perp, \underline{J}, \underline{I}$ and, since J is !-like, for all Σ:

$$\overset{\text{IH}}{\Delta', \Delta'', J \multimap K, \Sigma \vdash^i I}$$

    If $\underline{I}$ belongs to the same premise as G, we have $\Gamma = \Delta', \Delta'', J \multimap K$ and:

$$\dfrac{\overset{\text{IH}}{\Delta' \vdash^i J} \qquad \overset{\text{IH}}{\Delta'', K \vdash^i I}}{\Delta', \Delta'', J \multimap K \vdash^i I}\,\multimap L$$

    If Ω is not empty and splits into Ω' and Ω'' in the premises, then at least one of Ω' and Ω'' is not empty. If, for example, $\Omega' \neq \emptyset$ then we have, for all Σ:

$$\dfrac{\overset{\text{IH}}{\Delta', \Sigma \vdash^i J} \qquad \overset{\text{IH}}{\Delta'', K \vdash^i I}}{\Delta', \Delta'', J \multimap K, \Sigma \vdash^i I}\,\multimap L$$

  - If $\underline{\Omega}$ contains $F \otimes G$ then $\Omega = \Omega', J \otimes K$ with $\underline{J} = F$ and $\underline{K} = G$. In the contexts of the premises, Γ is split into Δ' and Δ'' and $\underline{I}$ must be in the same premise as F or G and we have, for all Σ:

$$\begin{array}{c}\text{IH}\\ \Delta', \Delta'', \Sigma \vdash^i I\end{array}$$

- (1) rule: $\dfrac{}{\vdash^i 1}\ 1R$
- ($\bot$) rule: We have $\Gamma = \Gamma', 1$ and:

$$\dfrac{\begin{array}{c}\text{IH}\\ \Gamma', (\Sigma) \vdash^i I\end{array}}{\Gamma', 1, (\Sigma) \vdash^i I}\ 1L$$

- (&) rule:
  - If $\underline{I} = F \,\&\, G$, then $I = J \,\&\, K$ with $\underline{J} = F$ and $\underline{K} = G$. We have:

$$\dfrac{\begin{array}{c}\text{IH}\\ \Gamma, (\Sigma) \vdash^i J\end{array} \qquad \begin{array}{c}\text{IH}\\ \Gamma, (\Sigma) \vdash^i K\end{array}}{\Gamma, (\Sigma) \vdash^i J \,\&\, K}\ \&R$$

  - If $\underline{\Gamma}^\perp$ contains $F \,\&\, G$ then $\Gamma = \Gamma', J \oplus K$ with $\underline{J} = F^\perp$ and $\underline{K} = G^\perp$. We have:

$$\dfrac{\begin{array}{c}\text{IH}\\ \Gamma', J, (\Sigma) \vdash^i I\end{array} \qquad \begin{array}{c}\text{IH}\\ \Gamma', K, (\Sigma) \vdash^i I\end{array}}{\Gamma', J \oplus K, (\Sigma) \vdash^i I}\ \oplus L$$

  - If $\underline{\Omega}$ contains $F \,\&\, G$, then $\Omega = \Omega', J \,\&\, K$ with $\underline{J} = F$ and $\underline{K} = G$. By definition of !-like, at least one of J and K is !-like. We have, for all Σ:

$$\begin{array}{c}\text{IH}\\ \Gamma, \Sigma \vdash^i I\end{array}$$

- ($\oplus_1$) rule:
  - If $\underline{I} = F \oplus G$, then $I = J \oplus K$ with $\underline{J} = F$ and $\underline{K} = G$. We have:

$$\dfrac{\begin{array}{c}\text{IH}\\ \Gamma, (\Sigma) \vdash^i J\end{array}}{\Gamma, (\Sigma) \vdash^i J \oplus K}\ \oplus_1 R$$

  - If $\underline{\Gamma}^\perp$ contains $F \oplus G$ then $\Gamma = \Gamma', J \,\&\, K$ with $\underline{J} = F^\perp$ and $\underline{K} = G^\perp$. We have:

$$\dfrac{\begin{array}{c}\text{IH}\\ \Gamma', J, (\Sigma) \vdash^i I\end{array}}{\Gamma', J \,\&\, K, (\Sigma) \vdash^i I}\ \&_1 L$$

  - If $\underline{\Omega}$ contains $F \oplus G$, then $\Omega = \Omega', J \oplus K$ with $\underline{J} = F$ and $\underline{K} = G$. By definition of !-like, J is !-like. We have, for all Σ:

$$\begin{array}{c}\text{IH}\\ \Gamma, \Sigma \vdash^i I\end{array}$$

- ($\top$) rule:
  - If $\underline{I} = \top$, then $I = \top$. We have:

$$\dfrac{}{\Gamma, (\Sigma) \vdash^i \top}\ \top R$$

  - If $\underline{\Gamma}^\perp$ contains $\top$ then $\Gamma = \Gamma', J$ with $\underline{J} = 0$ thus $J = 0$. We have:

$$\dfrac{}{\Gamma', 0, (\Sigma) \vdash^i I}\ 0L$$

  - By definition of !-like, $\underline{\Omega}$ cannot contain $\top$.
- ($?d$) rule: If $\underline{\Gamma}^\perp$ contains $?F$ then $\Gamma = \Gamma', !J$ with $\underline{J} = F^\perp$. We have:

$$\dfrac{\begin{array}{c}\text{IH}\\ \Gamma', J, (\Sigma) \vdash^i I\end{array}}{\Gamma', !J, (\Sigma) \vdash^i I}\ !L$$

- ($?c$) rule: If $\underline{\Gamma}^\perp$ contains $?F$ then $\Gamma = \Gamma', !J$ with $\underline{J} = F^\perp$. We have:

$$\dfrac{\begin{array}{c}\text{IH}\\ \Gamma', !J, !J, (\Sigma) \vdash^i I\end{array}}{\Gamma', !J, (\Sigma) \vdash^i I}\ !cL$$

- ($?w$) rule: If $\underline{\Gamma}^\perp$ contains $?F$ then $\Gamma = \Gamma', !J$ with $\underline{J} = F^\perp$. We have:

$$\dfrac{\dfrac{\text{IH}}{\Gamma', (\Sigma) \vdash^i I}}{\Gamma', !J, (\Sigma) \vdash^i I}\ !wL$$

- (!) rule:
  - If $\underline{I} = !F$ and $\underline{\Gamma}^\perp, \underline{\Omega} = ?\Delta$, then $I = !J$ with $\underline{J} = F$, $\Omega = \emptyset$ and $\Gamma = !\Gamma'$ with $\underline{\Gamma'} = \Delta^\perp$. We have:

$$\dfrac{\dfrac{\text{IH}}{!\Gamma' \vdash^i J}}{!\Gamma' \vdash^i !J}\ !R$$

  - If Ω contains $!F$, we have a contradiction since it is not possible to have $\underline{I} = ?G$. □

Theorem 3.3. If I is a formula of ILL such that any formula on the left-hand side of a $\multimap$ in I is !-like, then $\vdash \underline{I}$ is provable in LL if and only if $\vdash^i I$ is provable in ILL.

Proof. This is Lemma 3.2 in one direction and Lemma C.2 in the other. □

Lemma 3.6. Let Z be an almost 0 formula, we have $Z \dashv\vdash 0$ in ILL.

Proof. For any I, we have $\dfrac{}{0 \vdash^i I}\ 0L$. Concerning the opposite direction we have, by induction on Z:

$$\dfrac{}{0 \vdash^i 0}\ 0L \qquad \dfrac{\dfrac{\dfrac{\text{IH}}{Z \vdash^i 0} \quad \dfrac{}{0, I \vdash^i 0}\ 0L}{Z, I \vdash^i 0}\ cut}{Z \otimes I \vdash^i 0}\ \otimes L$$

$$\dfrac{\dfrac{\text{IH}}{Z \vdash^i 0}}{Z \,\&\, I \vdash^i 0}\ \&_1 L \qquad \dfrac{\dfrac{\text{IH}}{Z_1 \vdash^i 0} \quad \dfrac{\text{IH}}{Z_2 \vdash^i 0}}{Z_1 \oplus Z_2 \vdash^i 0}\ \oplus L \qquad \dfrac{\dfrac{\text{IH}}{Z \vdash^i 0}}{!Z \vdash^i 0}\ !L$$

□

Lemma 3.8. If Γ contains 0-clean formulas only, and if $\vdash \underline{\Gamma}^\perp$ is provable in LL, then Γ contains an almost 0 formula.

Proof. By induction on a cut-free proof with atomic axioms:

- For an atomic (ax) rule, X is not of the shape $\underline{I}^\perp$.
- If $\underline{I}^\perp = F \parr G$ then $I = J \otimes K$ with $\underline{J} = F^\perp$ and $\underline{K} = G^\perp$. By induction hypothesis applied to $\vdash \underline{\Gamma}^\perp, \underline{J}^\perp, \underline{K}^\perp$, either we have an almost 0 formula in Γ and we are done, or J or K is an almost 0 formula, so that I is an almost 0 formula.
- If $\underline{I}^\perp = F \otimes G$ then $I = J \multimap K$ with $\underline{J} = F$ and $\underline{K} = G^\perp$. By induction hypothesis applied to the premise $\vdash \underline{\Delta}^\perp, \underline{K}^\perp$, either we have an almost 0 formula in $\Delta \subseteq \Gamma$ and we are done, or K is an almost 0 formula but this is not possible since I is 0-clean.
- For the ($\perp$) rule, we simply apply the induction hypothesis.
- For the (1) rule, 1 is not of the shape $\underline{I}^\perp$.
- If $\underline{I}^\perp = F \,\&\, G$ then $I = J \oplus K$ with $\underline{J} = F^\perp$ and $\underline{K} = G^\perp$. By induction hypothesis applied to $\vdash \underline{\Gamma}^\perp, \underline{J}^\perp$, either we have an almost 0 formula in Γ or J is an almost 0 formula. Similarly with Γ and K. If we have found an almost 0 formula in Γ we are done, otherwise both J and K are, and I is an almost 0 formula.
- If $\underline{I}^\perp = F \oplus G$ then $I = J \,\&\, K$ with $\underline{J} = F^\perp$ and $\underline{K} = G^\perp$. By induction hypothesis applied to $\vdash \underline{\Gamma}^\perp, \underline{J}^\perp$, either we have an almost 0 formula in Γ and we are done, or J is an almost 0 formula, so that I is an almost 0 formula.
- For the ($\top$) rule, the only formula I such that $\top = \underline{I}^\perp$ is 0 which is an almost 0 formula.
- If $\underline{I}^\perp = ?F$ then $I = !J$ with $\underline{J} = F^\perp$. By induction hypothesis applied to $\vdash \underline{\Gamma}^\perp, \underline{J}^\perp$, either we have an almost 0 formula in Γ and we are done, or J is an almost 0 formula, so that I is an almost 0 formula.
- For the ($?c$) and ($?w$) rules, we simply apply the induction hypothesis.
- For the (!) rule, !_ is not of the shape $\underline{I}^\perp$. □

Theorem 3.9. Let I be a 0-clean formula, $\vdash \underline{I}$ is provable in LL if and only if $\vdash^i I$ is provable in ILL.

Proof. The right-to-left direction is Lemma C.2. For the other direction, we prove by induction on a cut-free proof the slightly more general statement: if $\vdash \underline{\Gamma}^\perp, \underline{I}$ containing 0-clean formulas only is provable in LL then $\Gamma \vdash^i I$ is provable in ILL.

- (ax) rule: $\dfrac{}{I \vdash^i I}\ ax$
- ($\parr$) rule: If $\underline{I} = F \parr G$, then $I = J \multimap K$ with $\underline{J} = F^\perp$ and $\underline{K} = G$. We have:

$$\dfrac{\dfrac{\text{IH}}{\Gamma, J \vdash^i K}}{\Gamma \vdash^i J \multimap K}\ \multimap R$$

  If $\underline{\Gamma}^\perp$ contains $F \parr G$ then $\Gamma = \Gamma', J \otimes K$ with $\underline{J} = F^\perp$ and $\underline{K} = G^\perp$. We have:

$$\dfrac{\overset{\text{IH}}{\Gamma', J, K \vdash^i I}}{\Gamma', J \otimes K \vdash^i I}\ \otimes L$$

- ($\otimes$) rule: If $\underline{I} = F \otimes G$, then $I = J \otimes K$ with $\underline{J} = F$ and $\underline{K} = G$. The contexts of the premises are $\underline{\Delta}^\perp$ and $\underline{\Sigma}^\perp$ with $\Gamma = \Delta, \Sigma$. We have:

$$\dfrac{\overset{\text{IH}}{\Delta \vdash^i J} \qquad \overset{\text{IH}}{\Sigma \vdash^i K}}{\Delta, \Sigma \vdash^i J \otimes K}\ \otimes R$$

If $\underline{\Gamma}^\perp$ contains $F \otimes G$ then $\Gamma = \Gamma', J \multimap K$ with $\underline{J} = F$ and $\underline{K} = G^\perp$. If $\underline{I}$ belongs to the same premise as F, the other premise is of the shape $\vdash \underline{\Sigma}^\perp, G$ with 0-clean formulas only. By Lemma 3.8, Σ, K contains an almost 0 formula Z. We cannot have $K = Z$ otherwise $J \multimap K$ is not 0-clean, thus Z belongs to $\Sigma = \Sigma', Z$. We have $\Gamma = \Delta, \Sigma', Z, J \multimap K$ and:

$$\dfrac{\overset{\text{Lemma 3.6}}{Z \vdash^i 0} \qquad \dfrac{}{0, \Delta, \Sigma', J \multimap K \vdash^i I}\ 0L}{Z, \Delta, \Sigma', J \multimap K \vdash^i I}\ cut$$

If $\underline{I}$ belongs to the same premise as G, we have $\Gamma = \Delta, \Sigma, J \multimap K$ and:

$$\dfrac{\overset{\text{IH}}{\Sigma \vdash^i J} \qquad \overset{\text{IH}}{\Delta, K \vdash^i I}}{\Delta, \Sigma, J \multimap K \vdash^i I}\ \multimap L$$

- (1) rule: $\dfrac{}{\vdash^i 1}\ 1R$
- ($\perp$) rule: We have $\Gamma = \Gamma', 1$ and:

$$\dfrac{\overset{\text{IH}}{\Gamma' \vdash^i I}}{\Gamma', 1 \vdash^i I}\ 1L$$

- ($\&$) rule: If $\underline{I} = F \,\&\, G$, then $I = J \,\&\, K$ with $\underline{J} = F$ and $\underline{K} = G$. We have:

$$\dfrac{\overset{\text{IH}}{\Gamma \vdash^i J} \qquad \overset{\text{IH}}{\Gamma \vdash^i K}}{\Gamma \vdash^i J \,\&\, K}\ \&R$$

If $\underline{\Gamma}^\perp$ contains $F \,\&\, G$ then $\Gamma = \Gamma', J \oplus K$ with $\underline{J} = F^\perp$ and $\underline{K} = G^\perp$. We have:

$$\dfrac{\overset{\text{IH}}{\Gamma', J \vdash^i I} \qquad \overset{\text{IH}}{\Gamma', K \vdash^i I}}{\Gamma', J \oplus K \vdash^i I}\ \oplus L$$

- ($\oplus_1$) rule: If $\underline{I} = F \oplus G$, then $I = J \oplus K$ with $\underline{J} = F$ and $\underline{K} = G$. We have:

$$\dfrac{\overset{\text{IH}}{\Gamma \vdash^i J}}{\Gamma \vdash^i J \oplus K}\ \oplus_1 R$$

If $\underline{\Gamma}^\perp$ contains $F \oplus G$ then $\Gamma = \Gamma', J \,\&\, K$ with $\underline{J} = F^\perp$ and $\underline{K} = G^\perp$. We have:

$$\dfrac{\overset{\text{IH}}{\Gamma', J \vdash^i I}}{\Gamma', J \,\&\, K \vdash^i I}\ \&_1 L$$

- ($\top$) rule: If $\underline{I} = \top$, then $I = \top$. We have:

$$\dfrac{}{\Gamma \vdash^i \top}\ \top R$$

If $\underline{\Gamma}^\perp$ contains $\top$ then $\Gamma = \Gamma', J$ with $\underline{J} = 0$ thus $J = 0$. We have:

$$\dfrac{}{\Gamma', 0 \vdash^i I}\ 0L$$

- ($?d$) rule: If $\underline{\Gamma}^\perp$ contains $?F$ then $\Gamma = \Gamma', !J$ with $\underline{J} = F^\perp$. We have:

$$\dfrac{\overset{\text{IH}}{\Gamma', J \vdash^i I}}{\Gamma', !J \vdash^i I}\ !L$$

- ($?c$) rule: If $\underline{\Gamma}^\perp$ contains $?F$ then $\Gamma = \Gamma', !J$ with $\underline{J} = F^\perp$. We have:

$$\dfrac{\overset{\text{IH}}{\Gamma', !J, !J \vdash^i I}}{\Gamma', !J \vdash^i I}\ !cL$$

- (?w) rule: If $\underline{\Gamma}^\perp$ contains $?F$ then $\Gamma = \Gamma', !J$ with $\underline{J} = F^\perp$. We have:

$$\dfrac{\overset{\text{IH}}{\Gamma' \vdash^i I}}{\Gamma', !J \vdash^i I}\ !wL$$

- (!) rule: If $\underline{I} = !F$ and $\underline{\Gamma}^\perp = ?\Delta$, then $I = !J$ with $\underline{J} = F$ and $\Gamma = !\Gamma'$ with $\underline{\Gamma'} = \Delta^\perp$. We have

$$\dfrac{\overset{\text{IH}}{!\Gamma' \vdash^i J}}{!\Gamma' \vdash^i !J}\ !R$$

□

Lemma 3.10. If $\vdash \underline{\Gamma}^\perp, \underline{\Delta}$ is provable in LL, with Δ containing at least two formulas, then $\underline{\Gamma}^\perp, \underline{\Delta}$ contains $\top$.

Proof. By induction on a cut-free proof with atomic axioms only:

- The atomic axiom rule introduces only one formula of the shape $\underline{I}$.
- The case of the $(\top)$ rule is immediate.
- For all the other rules, if the conclusion contains at least two formulas of the shape $\underline{I}$, then at least one of its premises as well and we can apply the induction hypothesis. □

Theorem 3.11. Let U be a formula of TL, $\vdash \underline{U}$ is provable in LL if and only if $\vdash^t U$ is provable in TL.

Proof. The right-to-left implication is Lemma C.4. Concerning the left-to-right implication, we start with a proof π of $\vdash \underline{U}$ in LL. Since if we see U as an ILL formula, the only use of the connective $\multimap$ is in sub-formulas of the shape $\_ \multimap \Phi$ (with Φ a propositional variable), it is always 0-clean. By Theorem 3.9, $\vdash^i U$ is provable in ILL. By applying Proposition C.6, we turn this proof into a proof of $\vdash^i U$ in $\mathsf{ILL}^\neg$ with $\neg$ being the primitive negation, thus into a proof of $\vdash^t U$ in TL. □

Section 4

Lemma 4.2. For all F, $F^- \vdash^t F^\bullet$ in TL.

Proof. We first prove that for any F, $F^- \vdash^t F^\bullet$ implies $\neg F^+, \neg F^\bullet \vdash^t$. We consider the two cases, F synchronous and F asynchronous:

$$\dfrac{S^- \vdash^t S^\bullet}{\neg S^+, \neg S^\bullet \vdash^t}\ \neg L \qquad\qquad \dfrac{\dfrac{\dfrac{A^- \vdash^t A^\bullet}{A^-, \neg A^\bullet \vdash^t}\ \neg L}{\neg A^\bullet \vdash^t A^+}\ \neg R}{\neg A^+, \neg A^\bullet \vdash^t}$$

We now prove the statement by induction on the formula F.

- $(X^\perp)^- = X = (X^\perp)^\bullet$, $\perp^- = 1 = \perp^\bullet$, and $\top^- = 0 = \top^\bullet$.
- $(F \parr G)^- = F^- \otimes G^-$ and $(F \parr G)^\bullet = F^\bullet \otimes G^\bullet$, and we have:

$$\dfrac{\dfrac{\overset{\text{IH}}{F^- \vdash^t F^\bullet} \qquad \overset{\text{IH}}{G^- \vdash^t G^\bullet}}{F^-, G^- \vdash^t F^\bullet \otimes G^\bullet}\ \otimes R}{F^- \otimes G^- \vdash^t F^\bullet \otimes G^\bullet}\ \otimes L$$

- $(F \& G)^- = F^- \oplus G^-$ and $(F \& G)^\bullet = F^\bullet \oplus G^\bullet$, and we have:

$$\dfrac{\dfrac{\overset{\text{IH}}{F^- \vdash^t F^\bullet}}{F^- \vdash^t F^\bullet \oplus G^\bullet}\ \oplus_1 R \qquad \dfrac{\overset{\text{IH}}{G^- \vdash^t G^\bullet}}{G^- \vdash^t F^\bullet \oplus G^\bullet}\ \oplus_2 R}{F^- \oplus G^- \vdash^t F^\bullet \oplus G^\bullet}\ \oplus L$$

- $(?F)^- = !\neg F^+$ and $(?F)^\bullet = !\neg\neg F^\bullet$, and we have:

$$\dfrac{\dfrac{\dfrac{\overset{\text{IH}}{F^- \vdash^t F^\bullet}}{\neg F^+, \neg F^\bullet \vdash^t}}{\neg F^+ \vdash^t \neg\neg F^\bullet}\ \neg R}{\dfrac{!\neg F^+ \vdash^t \neg\neg F^\bullet}{!\neg F^+ \vdash^t !\neg\neg F^\bullet}\ !R}\ !L$$

- $X^- = \neg X^+ = \neg X = X^\bullet$, $1^- = \neg 1^+ = \neg 1 = 1^\bullet$, and $0^- = \neg 0^+ = \neg 0 = 0^\bullet$.

- $(F \otimes G)^- = \neg(F \otimes G)^+ = \neg(F^+ \otimes G^+)$ and $(F \otimes G)^\bullet = \neg(\neg F^\bullet \otimes \neg G^\bullet)$, and we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{IH}}{F^- \vdash^t F^\bullet}}{\neg F^+, \neg F^\bullet \vdash^t} \qquad \dfrac{\dfrac{\text{IH}}{G^- \vdash^t G^\bullet}}{\neg G^+, \neg G^\bullet \vdash^t}}{\neg(F^+ \otimes G^+), \neg F^\bullet \otimes \neg G^\bullet \vdash^t}\;\text{Lemma 2.1 } (i)}{\neg(F^+ \otimes G^+) \vdash^t \neg(\neg F^\bullet \otimes \neg G^\bullet)}\;\neg R$$

- $(F \oplus G)^- = \neg(F \oplus G)^+ = \neg(F^+ \oplus G^+)$ and $(F \oplus G)^\bullet = \neg(\neg F^\bullet \oplus \neg G^\bullet)$, and we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{IH}}{F^- \vdash^t F^\bullet}}{\neg F^+, \neg F^\bullet \vdash^t} \qquad \dfrac{\dfrac{\text{IH}}{G^- \vdash^t G^\bullet}}{\neg G^+, \neg G^\bullet \vdash^t}}{\neg(F^+ \oplus G^+), \neg F^\bullet \oplus \neg G^\bullet \vdash^t}\;\text{Lemma 2.1 } (ii)}{\neg(F^+ \oplus G^+) \vdash^t \neg(\neg F^\bullet \oplus \neg G^\bullet)}\;\neg R$$

- $(!F)^- = \neg(!F)^+ = \neg!\neg F^-$ and $(!F)^\bullet = \neg!\neg F^\bullet$, and we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{IH}}{F^- \vdash^t F^\bullet}}{F^-, \neg F^\bullet \vdash^t}\;\neg L}{\neg F^\bullet \vdash^t \neg F^-}\;\neg R}{!\neg F^\bullet \vdash^t \neg F^-}\;!L}{!\neg F^\bullet \vdash^t !\neg F^-}\;!R}{\neg!\neg F^-, !\neg F^\bullet \vdash^t}\;\neg L}{\neg!\neg F^- \vdash^t \neg!\neg F^\bullet}\;\neg R$$

□

Remark on Lemma 4.4. The transformations of proofs of TL into proofs of TL′ described in the proof of Lemma 4.4 can also be obtained by introducing and eliminating appropriate cuts in TL:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{F^+ \vdash^t F^+}\;ax}{\neg F^+, F^+ \vdash^t}\;\neg L}{!\neg F^+, F^+ \vdash^t}\;!L}{!\neg F^+ \vdash^t \neg F^+}\;\neg R}{!\neg F^+ \vdash^t !\neg F^+}\;!R \qquad \dfrac{\Gamma^-, \neg F^+ \vdash^t \Pi^+}{\Gamma^-, !\neg F^+ \vdash^t \Pi^+}\;!L}{\Gamma^-, !\neg F^+ \vdash^t \Pi^+}\;cut$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{F^+ \vdash^t F^+}\;ax}{\neg F^+, F^+ \vdash^t}\;\neg L}{!\neg F^+, F^+ \vdash^t}\;!L}{!\neg F^+ \vdash^t \neg F^+}\;\neg R \qquad \Gamma^-, \neg F^+ \vdash^t \Pi^+}{\Gamma^-, !\neg F^+ \vdash^t \Pi^+}\;cut$$

$$\mapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{}{F^+ \vdash^t F^+}\;ax}{!\neg F^+, F^+ \vdash^t}\;!L'}{!\neg F^+ \vdash^t \neg F^+}\;\neg R \qquad \Gamma^-, \neg F^+ \vdash^t \Pi^+}{\Gamma^-, !\neg F^+ \vdash^t \Pi^+}\;cut$$

and

$$\dfrac{\dfrac{!\neg\Gamma^+ \vdash^t \neg F^-}{!\neg\Gamma^+ \vdash^t !\neg F^-}\;!R \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{F^- \vdash^t F^-}\;ax}{\neg F^-, F^- \vdash^t}\;\neg L}{!\neg F^-, F^- \vdash^t}\;!L}{!\neg F^- \vdash^t \neg F^-}\;\neg R}{!\neg F^- \vdash^t !\neg F^-}\;!R}{!\neg\Gamma^+ \vdash^t !\neg F^-}\;cut$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{!\neg\Gamma^+ \vdash^t \neg F^- \qquad \dfrac{\dfrac{}{F^- \vdash^t F^-}\;ax}{\neg F^-, F^- \vdash^t}\;\neg L}{!\neg\Gamma^+, F^- \vdash^t}\;cut}{!\neg\Gamma^+ \vdash^t \neg F^-}\;\neg R}{!\neg\Gamma^+ \vdash^t !\neg F^-}\;!R$$

$$\mapsto \quad \dfrac{\dfrac{!\neg\Gamma^+ \vdash^t \neg F^- \qquad \dfrac{\dfrac{}{F^- \vdash^t F^-}\;ax}{\neg F^-, F^- \vdash^t}\;\neg L}{!\neg\Gamma^+, F^- \vdash^t}\;cut}{!\neg\Gamma^+ \vdash^t !\neg F^-}\;!R'$$

Corollary 4.6. If $\vdash \Gamma$ is provable in LL then $\vdash^f \Gamma \mid$ is provable in LL_{foc}.

Proof. If $\vdash \Gamma$ is provable in LL, by Theorem 4.3, we have $\Gamma^- \vdash^t$ in TL. Thus $\Gamma^- \vdash^t$ in TL′ by Lemma 4.4. Finally Theorem 4.5 gives us $\vdash^f \Gamma \mid$ in LL_{foc}. □

Section 5

Lemma 5.1. For all F, G and X, if $\mathsf{R}[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}] = \mathsf{R}$ then $F^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}], \neg_{\mathsf{R}}(F[{}^{G}/_{X}])^\bullet \vdash^i \mathsf{R}$ in ILL.

Proof. By induction on F (let Y denote a propositional variable different from X):

$$\dfrac{\dfrac{}{\neg_{\mathsf{R}} G^\bullet \vdash^i \neg_{\mathsf{R}} G^\bullet}\,ax}{\neg_{\mathsf{R}}\neg_{\mathsf{R}} G^\bullet, \neg_{\mathsf{R}} G^\bullet \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}} L \qquad \dfrac{\dfrac{}{\neg_{\mathsf{R}} Y \vdash^i \neg_{\mathsf{R}} Y}\,ax}{\neg_{\mathsf{R}} Y, \neg_{\mathsf{R}}\neg_{\mathsf{R}} Y \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}} L \qquad \begin{array}{c}\text{Lemma 2.3}\\ \neg_{\mathsf{R}} G^\bullet, \neg_{\mathsf{R}}(G^\perp)^\bullet \vdash^i \mathsf{R}\end{array} \qquad \dfrac{\dfrac{}{Y \vdash^i Y}\,ax}{Y, \neg_{\mathsf{R}} Y \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}} L$$

$$\dfrac{\dfrac{}{\neg_{\mathsf{R}} 1 \vdash^i \neg_{\mathsf{R}} 1}\,ax}{\neg_{\mathsf{R}} 1, \neg_{\mathsf{R}}\neg_{\mathsf{R}} 1 \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}} L \qquad \dfrac{\dfrac{}{1 \vdash^i 1}\,ax}{1, \neg_{\mathsf{R}} 1 \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}} L$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{IH}\\ F_1^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}], \neg_{\mathsf{R}}(F_1[{}^{G}/_{X}])^\bullet \vdash^i \mathsf{R}\end{array}}{\neg_{\mathsf{R}}(F_1[{}^{G}/_{X}])^\bullet \vdash^i \neg_{\mathsf{R}} F_1^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}]}\,\neg_{\mathsf{R}} R}{\neg_{\mathsf{R}}\neg_{\mathsf{R}} F_1^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}], \neg_{\mathsf{R}}(F_1[{}^{G}/_{X}])^\bullet \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}} L \quad \dfrac{\dfrac{\begin{array}{c}\text{IH}\\ F_2^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}], \neg_{\mathsf{R}}(F_2[{}^{G}/_{X}])^\bullet \vdash^i \mathsf{R}\end{array}}{\neg_{\mathsf{R}}(F_2[{}^{G}/_{X}])^\bullet \vdash^i \neg_{\mathsf{R}} F_2^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}]}\,\neg_{\mathsf{R}} R}{\neg_{\mathsf{R}}\neg_{\mathsf{R}} F_2^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}], \neg_{\mathsf{R}}(F_2[{}^{G}/_{X}])^\bullet \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}} L}{\neg_{\mathsf{R}}(\neg_{\mathsf{R}} F_1^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}] \otimes \neg_{\mathsf{R}} F_2^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}]), \neg_{\mathsf{R}}(F_1[{}^{G}/_{X}])^\bullet \otimes \neg_{\mathsf{R}}(F_2[{}^{G}/_{X}])^\bullet \vdash^i \mathsf{R}}\,\text{Lemma 2.1 } (i)}{\neg_{\mathsf{R}}(\neg_{\mathsf{R}} F_1^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}] \otimes \neg_{\mathsf{R}} F_2^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}]) \vdash^i \neg_{\mathsf{R}}(\neg_{\mathsf{R}}(F_1[{}^{G}/_{X}])^\bullet \otimes \neg_{\mathsf{R}}(F_2[{}^{G}/_{X}])^\bullet)}\,\neg_{\mathsf{R}} R}{\neg_{\mathsf{R}}(\neg_{\mathsf{R}} F_1^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}] \otimes \neg_{\mathsf{R}} F_2^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}]), \neg_{\mathsf{R}}\neg_{\mathsf{R}}(\neg_{\mathsf{R}}(F_1[{}^{G}/_{X}])^\bullet \otimes \neg_{\mathsf{R}}(F_2[{}^{G}/_{X}])^\bullet) \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}} L$$

$$\dfrac{\begin{array}{c}\text{IH}\\ F_1^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}], \neg_{\mathsf{R}}(F_1[{}^{G}/_{X}])^\bullet \vdash^i \mathsf{R}\end{array} \qquad \begin{array}{c}\text{IH}\\ F_2^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}], \neg_{\mathsf{R}}(F_2[{}^{G}/_{X}])^\bullet \vdash^i \mathsf{R}\end{array}}{F_1^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}] \otimes F_2^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}], \neg_{\mathsf{R}}((F_1[{}^{G}/_{X}])^\bullet \otimes (F_2[{}^{G}/_{X}])^\bullet) \vdash^i \mathsf{R}}\,\text{Lemma 2.1 } (i)$$

$$\dfrac{\dfrac{}{\neg_{\mathsf{R}} 0 \vdash^i \neg_{\mathsf{R}} 0}\,ax}{\neg_{\mathsf{R}} 0, \neg_{\mathsf{R}}\neg_{\mathsf{R}} 0 \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}} L \qquad \dfrac{\dfrac{}{0 \vdash^i 0}\,ax}{0, \neg_{\mathsf{R}} 0 \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}} L$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{IH}\\ F_1^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}], \neg_{\mathsf{R}}(F_1[{}^{G}/_{X}])^\bullet \vdash^i \mathsf{R}\end{array}}{\neg_{\mathsf{R}}(F_1[{}^{G}/_{X}])^\bullet \vdash^i \neg_{\mathsf{R}} F_1^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}]}\,\neg_{\mathsf{R}} R}{\neg_{\mathsf{R}}\neg_{\mathsf{R}} F_1^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}], \neg_{\mathsf{R}}(F_1[{}^{G}/_{X}])^\bullet \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}} L \quad \dfrac{\dfrac{\begin{array}{c}\text{IH}\\ F_2^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}], \neg_{\mathsf{R}}(F_2[{}^{G}/_{X}])^\bullet \vdash^i \mathsf{R}\end{array}}{\neg_{\mathsf{R}}(F_2[{}^{G}/_{X}])^\bullet \vdash^i \neg_{\mathsf{R}} F_2^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}]}\,\neg_{\mathsf{R}} R}{\neg_{\mathsf{R}}\neg_{\mathsf{R}} F_2^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}], \neg_{\mathsf{R}}(F_2[{}^{G}/_{X}])^\bullet \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}} L}{\neg_{\mathsf{R}}(\neg_{\mathsf{R}} F_1^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}] \oplus \neg_{\mathsf{R}} F_2^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}]), \neg_{\mathsf{R}}(F_1[{}^{G}/_{X}])^\bullet \oplus \neg_{\mathsf{R}}(F_2[{}^{G}/_{X}])^\bullet \vdash^i \mathsf{R}}\,\text{Lemma 2.1 } (ii)}{\neg_{\mathsf{R}}(\neg_{\mathsf{R}} F_1^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}] \oplus \neg_{\mathsf{R}} F_2^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}]) \vdash^i \neg_{\mathsf{R}}(\neg_{\mathsf{R}}(F_1[{}^{G}/_{X}])^\bullet \oplus \neg_{\mathsf{R}}(F_2[{}^{G}/_{X}])^\bullet)}\,\neg_{\mathsf{R}} R}{\neg_{\mathsf{R}}(\neg_{\mathsf{R}} F_1^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}] \oplus \neg_{\mathsf{R}} F_2^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}]), \neg_{\mathsf{R}}\neg_{\mathsf{R}}(\neg_{\mathsf{R}}(F_1[{}^{G}/_{X}])^\bullet \oplus \neg_{\mathsf{R}}(F_2[{}^{G}/_{X}])^\bullet) \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}} L$$

$$\dfrac{\begin{array}{c}\text{IH}\\ F_1^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}], \neg_{\mathsf{R}}(F_1[{}^{G}/_{X}])^\bullet \vdash^i \mathsf{R}\end{array} \qquad \begin{array}{c}\text{IH}\\ F_2^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}], \neg_{\mathsf{R}}(F_2[{}^{G}/_{X}])^\bullet \vdash^i \mathsf{R}\end{array}}{F_1^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}] \oplus F_2^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}], \neg_{\mathsf{R}}((F_1[{}^{G}/_{X}])^\bullet \oplus (F_2[{}^{G}/_{X}])^\bullet) \vdash^i \mathsf{R}}\,\text{Lemma 2.1 } (ii)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{IH}\\ F^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}], \neg_{\mathsf{R}}(F[{}^{G}/_{X}])^\bullet \vdash^i \mathsf{R}\end{array}}{\neg_{\mathsf{R}}(F[{}^{G}/_{X}])^\bullet \vdash^i \neg_{\mathsf{R}} F^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}]}\,\neg_{\mathsf{R}} R}{!\neg_{\mathsf{R}}(F[{}^{G}/_{X}])^\bullet \vdash^i \neg_{\mathsf{R}} F^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}]}\,!L}{!\neg_{\mathsf{R}}(F[{}^{G}/_{X}])^\bullet \vdash^i !\neg_{\mathsf{R}} F^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}]}\,!R}{\neg_{\mathsf{R}}!\neg_{\mathsf{R}} F^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}], !\neg_{\mathsf{R}}(F[{}^{G}/_{X}])^\bullet \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}} L}{\neg_{\mathsf{R}}!\neg_{\mathsf{R}} F^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}] \vdash^i \neg_{\mathsf{R}}!\neg_{\mathsf{R}}(F[{}^{G}/_{X}])^\bullet}\,\neg_{\mathsf{R}} R}{\neg_{\mathsf{R}}!\neg_{\mathsf{R}} F^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}], \neg_{\mathsf{R}}\neg_{\mathsf{R}}!\neg_{\mathsf{R}}(F[{}^{G}/_{X}])^\bullet \vdash^i \mathsf{R}}\,\neg_{\mathsf{R}} L$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{IH}\\ F^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}], \neg_{\mathsf{R}}(F[{}^{G}/_{X}])^\bullet \vdash^i \mathsf{R}\end{array}}{\neg_{\mathsf{R}}(F[{}^{G}/_{X}])^\bullet \vdash^i \neg_{\mathsf{R}} F^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}]}\,\neg R}{\neg_{\mathsf{R}}\neg_{\mathsf{R}} F^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}], \neg_{\mathsf{R}}(F[{}^{G}/_{X}])^\bullet \vdash^i \mathsf{R}}\,\neg L}{\neg_{\mathsf{R}}\neg_{\mathsf{R}} F^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}] \vdash^i \neg_{\mathsf{R}}\neg_{\mathsf{R}}(F[{}^{G}/_{X}])^\bullet}\,\neg R}{!\neg_{\mathsf{R}}\neg_{\mathsf{R}} F^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}] \vdash^i \neg_{\mathsf{R}}\neg_{\mathsf{R}}(F[{}^{G}/_{X}])^\bullet}\,!L}{!\neg_{\mathsf{R}}\neg_{\mathsf{R}} F^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}] \vdash^i !\neg_{\mathsf{R}}\neg_{\mathsf{R}}(F[{}^{G}/_{X}])^\bullet}\,!R}{!\neg_{\mathsf{R}}\neg_{\mathsf{R}} F^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_{X}], \neg_{\mathsf{R}}!\neg_{\mathsf{R}}\neg_{\mathsf{R}}(F[{}^{G}/_{X}])^\bullet \vdash^i \mathsf{R}}\,\neg L$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{IH}}{F^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_X], \neg_{\mathsf{R}}(F[{}^G/_X])^\bullet \vdash^i \mathsf{R}}}{\neg_{\mathsf{R}}(F[{}^G/_X])^\bullet \vdash^i \neg_{\mathsf{R}} F^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_X]}\ \neg_{\mathsf{R}} R}{\neg_{\mathsf{R}}(F[{}^G/_X])^\bullet \vdash^i \exists Y.\neg_{\mathsf{R}} F^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_X]}\ \exists R}{\exists Y.\neg_{\mathsf{R}}(F[{}^G/_X])^\bullet \vdash^i \exists Y.\neg_{\mathsf{R}} F^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_X]}\ \exists L}{\neg_{\mathsf{R}}\exists Y.\neg_{\mathsf{R}} F^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_X], \exists Y.\neg_{\mathsf{R}}(F[{}^G/_X])^\bullet \vdash^i \mathsf{R}}\ \neg L}{\neg_{\mathsf{R}}\exists Y.\neg_{\mathsf{R}} F^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_X] \vdash^i \neg_{\mathsf{R}}\exists Y.\neg_{\mathsf{R}}(F[{}^G/_X])^\bullet}\ \neg R}{\neg_{\mathsf{R}}\exists Y.\neg_{\mathsf{R}} F^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_X], \neg_{\mathsf{R}}\neg_{\mathsf{R}}\exists Y.\neg_{\mathsf{R}}(F[{}^G/_X])^\bullet \vdash^i \mathsf{R}}\ \neg L
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(F[{}^G/_X])^\bullet \vdash^i (F[{}^G/_X])^\bullet}\ ax}{(F[{}^G/_X])^\bullet \vdash^i \exists Y.(F[{}^G/_X])^\bullet}\ \exists R}{(F[{}^G/_X])^\bullet, \neg_{\mathsf{R}}\exists Y.(F[{}^G/_X])^\bullet \vdash^i \mathsf{R}}\ \neg L}{\neg_{\mathsf{R}}\exists Y.(F[{}^G/_X])^\bullet \vdash^i \neg_{\mathsf{R}}(F[{}^G/_X])^\bullet}\ \neg R \qquad \dfrac{\text{IH}}{F^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_X], \neg_{\mathsf{R}}(F[{}^G/_X])^\bullet \vdash^i \mathsf{R}}}{F^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_X], \neg_{\mathsf{R}}\exists Y.(F[{}^G/_X])^\bullet \vdash^i \mathsf{R}}\ cut}{\exists Y.F^\bullet[{}^{\neg_{\mathsf{R}} G^\bullet}/_X], \neg_{\mathsf{R}}\exists Y.(F[{}^G/_X])^\bullet \vdash^i \mathsf{R}}\ \exists L
$$

□