On the Canonicity of Linear Logic Connectives

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Connectives of linear logic are sometimes classified as *canonical* or not. This refers to the possibility (or not) to characterise a connective (up to logical equivalence) by means of its introduction rules. The standard claim is that all linear-logic connectives, but the exponential ones, are canonical.

We want to stress that this claim strongly depends, not only on linear logic itself, but on the way rules are constructed. This means in particular that canonicity is *not* an intrinsic property of connectives but depends on how they are presented: *canonicity is not canonical...*

Canonicity of Linear Connectives in the Literature

Linear logic [Gir87] is known to be a logic with a lot of connectives. Propositional linear connectives are classified into three families: **multiplicative** ($\otimes$, $\parr$, $\multimap$, 1, $\bot$), **additive** ($\&$, $\oplus$, $\top$, 0) and **exponential** ($!$, $?$) connectives. The exponential ones are often presented as *less canonical* (discussion and formal definition provided later) than the others. Here are a few quotations from the linear logic literature and going in that direction:

> "As is well known, the sequent calculus rules for the exponentials, unlike those for the other connectives, do not imply their uniqueness modulo linear equivalence: if we introduce a unary connective $!'$, with the *same* rules as the exponential $!$, then neither $!A \multimap !'A$, nor $!'A \multimap !A$ are derivable"
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> [DJS93, page 168]

> "Girard has pointed out that the exponential connectives or modalities, $!$ and $?$, unlike the other connectives, are not determined by the axioms of linear logic. More precisely, if one added to linear logic a second pair of modalities, say $!'$ and $?'$, subject to the same rules of inference as the original pair, then one could not deduce that the new modalities are equivalent to the old." [Bla95, page 80]

> "Even if we fix the inference rules for the exponentials, as in standard linear logic, the rules do not describe unique exponentials. If one gives a red tensor and a blue tensor the same inference rules, then one can prove that these two tensors are, in fact, equivalent. All of linear logic connectives except the exponentials yield similar theorems." [BM07, page 93]

> "More precisely, if we add to the language of linear logic two more operators, $!'$ and $?'$, and postulate of them the same rules as for $!$ and $?$, we cannot prove that $!A \multimapboth !'A$ and $?A \multimapboth ?'A$. In contrast, if we introduce $\&'$, $\bot'$, etc, we can prove that the new operators are equivalent to the old ones."
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> nLab wiki

> "One interesting aspect of the exponentials is that they are not *canonical*: if we introduce copies of the exponentials, say $!'$, $?'$ satisfying the same respective laws as $!$, $?$, we cannot prove that necessarily $! = !'$ and $? = ?'$ on this basis alone (see Section 6.1 of [DCM23]). This is in stark contrast with the multiplicative and additive connectives, which are indeed canonical in this way."
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> Proof Theory blog

They are mostly correct because they (sometimes implicitly) rely on Girard's sequent calculus rules for linear logic. Nevertheless such statements often lead to a common belief that this (non-) canonicity properties are an intrinsic property of linear logic connectives. We will discuss what happens with different presentations of the logic leading to the conclusion that canonicity depends on the presentation, not only on the logic itself.

Defining Canonicity of Connectives

Following the quotations above, here is a proposal definition for canonicity of connectives:

Definition (Canonical Connective)
*A connective $\star$ is **canonical** if, whenever we consider two copies $\color{blue}\star$ and $\color{red}\star$ with the same rules as $\star$, we can prove $\color{blue}\star$ and $\color{red}\star$ to be logically equivalent.*

It is important to notice that, by relying on what are the rules, this notion has no reason to be invariant when moving from a proof calculus to another.

We will focus on intuitionistic linear Logic [GL87] restricted to the connectives $\otimes$, 1, $\multimap$, $\&$, $\top$, $!$ as it is sufficient for our discussion and slightly simpler for the presentation of some systems. Things are very similar in classical linear logic.

Traditional Sequents

We consider here Girard-Lafont's sequent calculus [GL87] for intuitionistic linear logic based on sequents $\Gamma \vdash A$ with Γ a list of formulas and A a formula:

$$\frac{}{A \vdash A}\,ax \qquad \frac{\Gamma \vdash A \quad \Delta, A \vdash C}{\Gamma, \Delta \vdash C}\,cut \qquad \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\,ex$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\,\otimes\mathrm{R} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\,\otimes\mathrm{L} \qquad \frac{}{\vdash 1}\,1\mathrm{R} \qquad \frac{\Gamma \vdash C}{\Gamma, 1 \vdash C}\,1\mathrm{L}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\,\multimap\mathrm{R} \qquad \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\,\multimap\mathrm{L}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B}\,\&\mathrm{R} \qquad \frac{\Gamma, A \vdash C}{\Gamma, A \& B \vdash C}\,\&_1\mathrm{L} \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \& B \vdash C}\,\&_2\mathrm{L} \qquad \frac{}{\Gamma \vdash \top}\,\top\mathrm{R}$$

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\,!\mathrm{R} \qquad \frac{\Gamma, A \vdash C}{\Gamma, !A \vdash C}\,!\mathrm{L} \qquad \frac{\Gamma, !A, !A \vdash C}{\Gamma, !A \vdash C}\,!c \qquad \frac{\Gamma \vdash C}{\Gamma, !A \vdash C}\,!w$$

Assuming now we have two different copies of each connective, equivalences come from decorated axiom expansions:

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,ax \quad \dfrac{}{B \vdash B}\,ax}{A, B \vdash A \mathbin{\color{blue}\otimes} B}\,{\color{blue}\otimes}\mathrm{R}}{A \mathbin{\color{red}\otimes} B \vdash A \mathbin{\color{blue}\otimes} B}\,{\color{red}\otimes}\mathrm{L} \qquad \dfrac{\dfrac{}{\vdash {\color{blue}1}}\,{\color{blue}1}\mathrm{R}}{{\color{red}1} \vdash {\color{blue}1}}\,{\color{red}1}\mathrm{L} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,ax \quad \dfrac{}{B \vdash B}\,ax}{A, A \mathbin{\color{red}\multimap} B \vdash B}\,{\color{red}\multimap}\mathrm{L}}{A \mathbin{\color{red}\multimap} B, A \vdash B}\,ex}{A \mathbin{\color{red}\multimap} B \vdash A \mathbin{\color{blue}\multimap} B}\,{\color{blue}\multimap}\mathrm{R}$$

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,ax}{A \mathbin{\color{red}\&} B \vdash A}\,{\color{red}\&_1}\mathrm{L} \quad \dfrac{\dfrac{}{B \vdash B}\,ax}{A \mathbin{\color{red}\&} B \vdash B}\,{\color{red}\&_2}\mathrm{L}}{A \mathbin{\color{red}\&} B \vdash A \mathbin{\color{blue}\&} B}\,{\color{blue}\&}\mathrm{R} \qquad \dfrac{}{{\color{red}\top} \vdash {\color{blue}\top}}\,{\color{blue}\top}\mathrm{R}$$

By also considering the dual proofs which swap red and blue, we get ${\color{red}\otimes} \dashv\vdash {\color{blue}\otimes}$, ${\color{red}\&} \dashv\vdash {\color{blue}\&}$, etc. So that multiplicative and additive connectives are proved to be canonical. However, because of the contextual constraint in the ($!\mathrm{R}$) rule, it is not possible to prove ${\color{red}!} \dashv\vdash {\color{blue}!}$:

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,ax}{{\color{red}!}A \vdash A}\,{\color{red}!}\mathrm{L}}{{\color{red}!}A \vdash {\color{blue}!}A}\,{\color{blue}!}\mathrm{R}$$

since the dashed rule application is not valid as it violates the condition on the context required to contain formulas starting with ${\color{blue}!}$ only.

One can interpret the situation by saying that, in Girard-Lafont's sequent calculus, the multiplicative connectives are canonical because there is a *single* (black) "," representing them at the sequent level. Exponential connectives are not canonical because they rely on the *contextual* (!R) rule fixing the copy of ! under consideration.

This means canonicity is partly driven by how connectives are dissolved into the sequent structure when reading rules bottom-up. From this remark, one can see that everything changes if we modify the structure of sequents as it happens with Andreoli's dyadic sequents or with bunches.

Dyadic Sequents

In the context of proof-search, Andreoli [And92] introduced alternative sequent calculi for linear logic. The status of canonical connectives happens to be different there. We consider only the dyadic case as it is sufficient for our purpose, and adapt it to intuitionistic linear logic. **Dyadic intuitionistic sequents** have the shape: $\Gamma \mid \Delta \vdash A$, with the property that such a sequent is provable if and only if $!\Gamma, \Delta \vdash A$ is provable in Girard-Lafont's system.

Let us focus on the exponential connectives. The dyadic axiom and exponential rules are:

$$\frac{}{\Gamma \mid A \vdash A}\ ax \qquad \frac{\Gamma, A \mid A, \Delta \vdash C}{\Gamma, A \mid \Delta \vdash C}\ sel \qquad \frac{\Gamma \mid \ \vdash A}{\Gamma \mid \ \vdash !A}\ !\mathrm{R} \qquad \frac{\Gamma, A \mid \Delta \vdash C}{\Gamma \mid !A, \Delta \vdash C}\ !\mathrm{L}$$

Now by considering two copies of !, we get:

$$\cfrac{\cfrac{\cfrac{\cfrac{}{A \mid A \vdash A}\ ax}{A \mid \ \vdash A}\ sel}{A \mid \ \vdash {\color{blue}!}A}\ {\color{blue}!}\mathrm{R}}{\mid {\color{red}!}A \vdash {\color{blue}!}A}\ {\color{red}!}\mathrm{L}$$

so that ${\color{red}!} \dashv\vdash {\color{blue}!}$, *i.e.* the exponential connective ! is canonical. The reason for that is that we have only *one* (black) "|" and the (!R) rule does not rely any more on the explicit mention of other occurrences of connectives in the context.

Bunched Sequents

Defining contexts in sequents as lists was imposing a single notion of ",". If we follow the theory of bunched implication logic [OP99, Pym02], we can generalise lists to binary trees. We consider here the multiplicative fragment of bunched implication logic only.

A **bunch** is a binary tree with leaves which are either formulas or an empty bunch {}:

$$\Xi ::= A \mid \{\} \mid (\Xi, \Xi)$$

They are equipped with the congruence ≡ generated by the structure of a commutative monoid:

$$(\Xi_1, (\Xi_2, \Xi_3)) \equiv ((\Xi_1, \Xi_2), \Xi_3) \qquad (\Xi, \{\}) \equiv \Xi \qquad (\Xi_1, \Xi_2) \equiv (\Xi_2, \Xi_1)$$

If Θ is a bunch context (*i.e.* a bunch with a hole amongst its leaves), we use the notation $\Theta[\Xi]$ for the bunch obtained by filing the hole of Θ with the bunch Ξ. The relation ≡ being a congruence, we can deduce $\Theta[\Xi_1] \equiv \Theta[\Xi_2]$ from $\Xi_1 \equiv \Xi_2$.

In this bunch-style presentation of intuitionistic linear logic, sequents have the shape $\Xi \vdash A$ where Ξ is a bunch and A is a formula:

$$\frac{}{A \vdash A}\ ax \qquad \frac{\Xi \vdash A \quad \Theta[A] \vdash C}{\Theta[\Xi] \vdash C}\ cut \qquad \frac{\Xi \vdash C \quad \Xi \equiv \Xi'}{\Xi' \vdash C}\ \equiv$$

$$\frac{\Xi_1 \vdash A \quad \Xi_2 \vdash B}{(\Xi_1, \Xi_2) \vdash A \otimes B}\ \otimes\mathrm{R} \qquad \frac{\Theta[(A, B)] \vdash C}{\Theta[A \otimes B] \vdash C}\ \otimes\mathrm{L} \qquad \frac{}{\{\} \vdash 1}\ 1\mathrm{R} \qquad \frac{\Theta[\{\}] \vdash C}{\Theta[1] \vdash C}\ 1\mathrm{L}$$

$$\frac{(\Xi, A) \vdash B}{\Xi \vdash A \multimap B}\ \multimap\mathrm{R} \qquad \frac{\Xi \vdash A \quad \Theta[B] \vdash C}{\Theta[(\Xi, A \multimap B)] \vdash C}\ \multimap\mathrm{L}$$

Assume now, we consider two copies of ⊗, but also two associated copies of "," (and {}) in the construction of bunches:

$$\Xi ::= A \mid \{\} \mid (\Xi,\Xi) \mid \{\} \mid (\Xi,\Xi)$$

then it is not possible any more to prove $\otimes \vdash \otimes$:

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,ax \quad \dfrac{}{B \vdash B}\,ax}{(A,B) \vdash A \otimes B}\,\otimes\text{R}}{A \otimes B \vdash A \otimes B}\,\otimes\text{L}$$

since the dashed rule application is not valid as it violates the condition on the context required to be of the shape (Ξ_1,Ξ_2).

The situation is similar in, for example, calculus of structure presentations of linear logic [Str03].

Concluding with some Semantics

Moving to more semantic arguments, it is often used in the literature that, in a given denotational model of the multiplicative-additive fragment of linear logic, it may be possible to define multiple interpretations of the exponential connectives. This is in particular the case in coherent spaces [Gir87] which have both a set-based exponential construction (the finitary one) and a multiset-based exponential construction (the free one). This is arguing for the *non*-canonicity of exponential connectives.

But already at the level of the multiplicative structure, there is no uniqueness. Given a category, there may be multiple monoidal[1] or even ⋆-autonomous[2] structures on it.

The situation is however different for the additive connectives which correspond to product/co-product constructions thus, because they satisfy a universal property, are unique up to unique isomorphism.

Navigating through various ways of presenting the connectives of linear logic, we have seen that only the additive connectives really appear to be canonical by remaining canonical even when the rules are presented differently.

If rules are fixed, that is for a given presentation of the logic, more connectives may become canonical. This is what happens in Girard's sequent calculus for classical linear logic and for Girard-Lafont's sequent calculus for intuitionistic linear logic. Together with the additive connectives, the multiplicative ones become canonical in this particular system, while exponential connectives are not.

To sum up, if canonicity is considered globally with respect to the logic (*i.e.* independently from its presentations), only the additive connective should really be considered canonical. In the more local framework of a fixed particular proof system, more connectives may appear canonical (as it is the case for the multiplicative ones in the standard sequent calculus).

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[1] https://ncatlab.org/nlab/show/bunched+logic#categorical_semantics

[2] https://mathoverflow.net/questions/309605/

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