Categories for Me
(memorandum)

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Contents

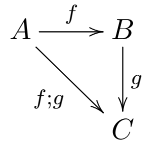
1 Categories

Definition 1 (Category)
A *category* $\mathbb{C}$ is given by a class of *objects* $obj(\mathbb{C})$ and, for each pair of objects A and B in $obj(\mathbb{C})$, a class of *morphisms* (or arrows) $\mathbb{C}(A, B)$ from A to B together with:

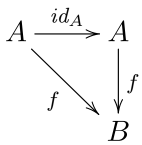
- *identities*: $id_A \in \mathbb{C}(A, A)$ for each object A:

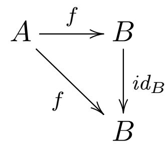
$$A \xrightarrow{id_A} A$$

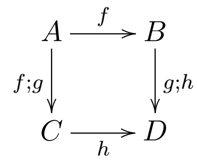
- *composition*: $\mathbb{C}(A,B) \times \mathbb{C}(B,C) \to \mathbb{C}(A,C)$, denoted by $(f,g) \mapsto f \,;\, g$:



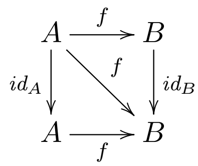
such that the following diagrams commute:

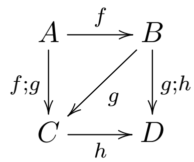






We can "summarize" these four diagrams into:





Example 1 (Category $\mathbb{S}\mathsf{et}$)
The *category of sets* $\mathbb{S}\mathsf{et}$ is given by:

- objects are sets
- morphisms are functions
- identities are identity functions
- composition is composition of functions

Definition 2 (Sub-Category)
A category $\mathbb{D}$ is a *sub-category* of the category $\mathbb{C}$ if its objects are objects of $\mathbb{C}$ ($obj(\mathbb{D}) \subseteq obj(\mathbb{C})$), its morphisms are morphisms of $\mathbb{C}$ ($\mathbb{D}(A,B) \subseteq \mathbb{C}(A,B)$), its identities are the identities of $\mathbb{C}$ ($id_A^{\mathbb{D}} = id_A^{\mathbb{C}}$) and its composition is the composition of $\mathbb{C}$ ($f \,;^{\mathbb{D}} g = f \,;^{\mathbb{C}} g$).
$\mathbb{D}$ is a *full sub-category* of $\mathbb{C}$ if, whenever A and B are objects of $\mathbb{D}$, $\mathbb{D}(A,B) = \mathbb{C}(A,B)$.
$\mathbb{D}$ is a *wide sub-category* of $\mathbb{C}$ if $obj(\mathbb{D}) = obj(\mathbb{C})$.

A full sub-category is characterized by its class of objects.

Example 2 (Full Wide Sub-Category)
The unique full wide sub-category of a category is itself.

1.1 Constructions

Definition 3 (Dual Category)
The *dual* (or *opposite*) $\mathbb{C}^{op}$ of a category $\mathbb{C}$ is the category with:

- objects of $\mathbb{C}^{op}$ are objects of $\mathbb{C}$
- morphisms of $\mathbb{C}^{op}$ from A to B are morphisms of $\mathbb{C}$ from B to A
- identities of $\mathbb{C}^{op}$ are identities of $\mathbb{C}$
- composition of f and g in $\mathbb{C}^{op}$ is $g\,;f$ in $\mathbb{C}$

Definition 4 (Unit Category)
The *unit category* $\mathbb{T}$ is given by:

- a unique object $\star$
- a unique morphism u from $\star$ to $\star$
- $id_\star = u$
- $u\,;u = u$

Definition 5 (Product Category)
The *product* $\mathbb{C} \times \mathbb{D}$ of two categories $\mathbb{C}$ and $\mathbb{D}$ is the category with:

- objects are pairs of objects of $\mathbb{C}$ and objects of $\mathbb{D}$
- morphisms from (A, A') to (B, B') are pairs of morphisms of $\mathbb{C}$ from A to B and morphisms of $\mathbb{D}$ from A' to B'
- identity on (A, A') is the pair $(id_A, id_{A'})$
- composition of (f, f') and (g, g') is $(f\,;g, f'\,;g')$

1.2 Morphisms

Definition 6 (Monomorphism)
A *monomorphism* f from the object A to the object B (denoted $f : A \hookrightarrow B$) is a morphism from A to B such that for any two morphisms g and h from some object C to A, we have:

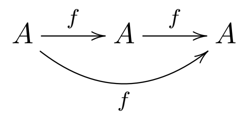
$$g\,;f = h\,;f \Longrightarrow g = h$$

Definition 7 (Epimorphism)
An *epimorphism* f from the object A to the object B (denoted $f : A \twoheadrightarrow B$) is a morphism from A to B such that for any two morphisms g and h from B to some object C, we have:

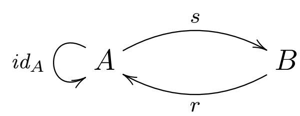
$$f\,;g = f\,;h \Longrightarrow g = h$$

It is thus a monomorphism in $\mathbb{C}^{op}$.

Definition 8 (Idempotent)
A morphism f from the object A to itself is an *idempotent* if $f\,;f = f$.
This can be written:



Definition 9 (Retract)
An object A is a *retract* of an object B (denoted $A \lhd B$) if there exist two morphisms $s \in \mathbb{C}(A, B)$ and $r \in \mathbb{C}(B, A)$ such that $s\,;r = id_A$.
This can be written:



s is then called a *section* of r, and r is called a *retraction* of s. (s, r) is called a *section-retraction pair*.

If (s, r) is a section-retraction pair, s is a monomorphism and r is an epimorphism. Such monomorphisms and epimorphisms coming from a section-retraction pair are called *split monomorphisms* and *split epimorphisms*. $r\,;s$ is an idempotent. Such idempotents coming from a section-retraction pair are called *split idempotents*.

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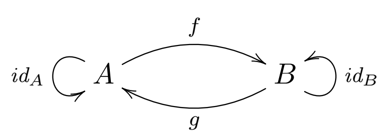
Definition 10 (Isomorphism)
An *isomorphism* f from the object A to the object B is a morphism from A to B such that there exists a morphism g from B to A (called the *inverse* of f) such that the following diagrams commute:

We can "summarize" these two diagrams into:



Property 1 (Retracts and Isomorphisms)
We have:

- *If there exists an isomorphism between A and B (denoted $A \simeq B$) then both $A \lhd B$ and $B \lhd A$.*
- *If $f \in \mathbb{C}(A, B)$ is both a section and a retraction then it is an isomorphism.*

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In particular an isomorphism is both a monomorphism and an epimorphism (the converse does not hold in general).

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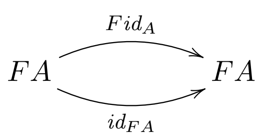
Definition 11 (Essentially Wide Sub-Category)
$\mathbb{D}$ is an *essentially wide sub-category* of $\mathbb{C}$ if it is a sub-category such that, for each object A of $\mathbb{C}$, there is an object A' of $\mathbb{D}$ such that $A' \simeq A$.

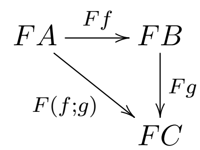
1.3 Functors

Definition 12 (Functor)
A *functor* F between two categories $\mathbb{C}$ and $\mathbb{D}$ is:

- a function from the objects of $\mathbb{C}$ to the objects of $\mathbb{D}$
- for each A and B, a function from $\mathbb{C}(A, B)$ to $\mathbb{D}(FA, FB)$

such that the following diagrams in $\mathbb{D}$ commute:





A functor from a category to itself is called an *endofunctor*.

Example 3 (Constant Functor)
If $\mathbb{C}$ and $\mathbb{D}$ are two categories and D is an object of $\mathbb{D}$, the *constant functor* C_D from $\mathbb{C}$ to $\mathbb{D}$ is defined by:

- for any $A \in obj(\mathbb{C})$, $C_D A = D$
- for any $f \in \mathbb{C}(A, B)$, $C_D f = id_D$

The constant functor $C_\star$ is the unique functor from any category $\mathbb{C}$ to $\mathbb{T}$.

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Example 4 (Inclusion Functor)
If $\mathbb{D}$ is a sub-category of $\mathbb{C}$, the *inclusion functor* I from $\mathbb{D}$ to $\mathbb{C}$ is defined by:

- for each $A \in obj(\mathbb{D})$, $IA = A$
- if A and B are in $obj(\mathbb{D})$ and $f \in \mathbb{D}(A, B)$, $If = f$

We denote by $Id_{\mathbb{C}}$ the *identity endofunctor* of $\mathbb{C}$ which is the inclusion functor of $\mathbb{C}$ into itself.

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Example 5 (Category $\mathbb{C}\mathsf{at}$)
The *category of categories* $\mathbb{C}\mathsf{at}$ is given by:

- objects are (small) categories
- morphisms are functors
- identities are identity functors

- composition is composition of functors: if F is a functor from $\mathbb{C}$ to $\mathbb{D}$ and G is a functor from $\mathbb{D}$ to $\mathbb{E}$, their composition $F\,;G$ (or GF) is the functor from $\mathbb{C}$ to $\mathbb{E}$ which maps the object A to $G(FA)$ and the morphism f to $G(Ff)$.

If F is an endofunctor of a category $\mathbb{C}$, we use the notations F^2 for $F\,;F = FF$, F^3 for $F\,;F\,;F = FFF$, ...

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Property 2 (Preservation of Retracts)
Functors preserve retracts and isomorphisms: if F is a functor,

- $A \lhd B \Longrightarrow FA \lhd FB$
- $A \simeq B \Longrightarrow FA \simeq FB$

Definition 13 (Bi-Functor)
A *bi-functor* from two categories $\mathbb{C}$ and $\mathbb{D}$ to a category $\mathbb{E}$ is a functor from $\mathbb{C} \times \mathbb{D}$ to $\mathbb{E}$.
More concretely, it is given by:

- a function from $obj(\mathbb{C}) \times obj(\mathbb{D})$ to $obj(\mathbb{E})$
- for each A and B in $obj(\mathbb{C})$ and A' and B' in $obj(\mathbb{D})$, a function from $\mathbb{C}(A,B) \times \mathbb{D}(A',B')$ to $\mathbb{E}(FAA', FBB')$

such that the following diagrams in $\mathbb{E}$ commute:

One often uses the notations FAf for Fid_Af and FfA for $Ffid_A$, if A is an object.

Example 6 (Homset Functor)
The *homset functor* $\mathbb{C}(\_,\_)$ of a category $\mathbb{C}$ is the bi-functor from $\mathbb{C}^{op}$ and $\mathbb{C}$ to $\mathbb{S}$et given by:

- $\mathbb{C}(\_,\_)(A,B) = \mathbb{C}(A,B)$
- $\mathbb{C}(\_,\_)(f,g)h = f\,;h\,;g$ (for $f \in \mathbb{C}(A',A)$, $g \in \mathbb{C}(B,B')$ and $h \in \mathbb{C}(A,B)$)

Example 7 (Fixed Component Bi-Functor)
If F is a bi-functor from $\mathbb{C}$ and $\mathbb{D}$ to $\mathbb{E}$ and if A is an object of C, we can define a functor F_A from $\mathbb{D}$ to $\mathbb{E}$ by:

- for any object B of $\mathbb{D}$, $F_AB = FAB$
- for any morphism $g \in \mathbb{D}(B,B')$, $F_Ag = Fid^{\mathbb{C}}_A g$

Definition 14 (Full and Faithful Functors)
A functor F between two categories $\mathbb{C}$ and $\mathbb{D}$ is *full* if, for any pair (A,B) of objects of $\mathbb{C}$, F is surjective from $\mathbb{C}(A,B)$ to $\mathbb{D}(FA,FB)$.
A functor F between two categories $\mathbb{C}$ and $\mathbb{D}$ is *faithful* if, for any pair (A,B) of objects of $\mathbb{C}$, F is injective from $\mathbb{C}(A,B)$ to $\mathbb{D}(FA,FB)$.

Definition 15 (Essentially Surjective Functor)
A functor F between two categories $\mathbb{C}$ and $\mathbb{D}$ is *essentially surjective* if, for each object A' of $\mathbb{D}$, there exists an object A of $\mathbb{C}$ such that A' is isomorphic to FA.

Example 8 (Inclusion Functor (bis))
If $\mathbb{D}$ is a sub-category of $\mathbb{C}$, the inclusion functor is faithful. It is full if and only if $\mathbb{D}$ is a full sub-category of $\mathbb{C}$. It is essentially surjective if and only if $\mathbb{D}$ is an essentially wide sub-category of $\mathbb{C}$.

Example 9 (Projection Functor)
Let $\mathbb{C}$ and $\mathbb{D}$ be two categories, the *projection functor* P from $\mathbb{C} \times \mathbb{D}$ to $\mathbb{C}$ is defined by:

- for each $(A, B) \in obj(\mathbb{C} \times \mathbb{D})$, $P(A, B) = A \in obj(\mathbb{C})$
- if A and A' are objects in $\mathbb{C}$, B and B' are objects in $\mathbb{D}$, and $(f, g) \in \mathbb{C} \times \mathbb{D}((A, B), (A', B'))$, $P(f, g) = f \in \mathbb{C}(A, A')$

It is a full functor if $\mathbb{D}$ has at least one morphism between any two objects.

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Definition 16 (Algebra)
An *algebra* for the endofunctor F is a pair (A, h_A) where:

- A is an object
- h_A is a morphism from FA to A

Definition 17 (Algebra Morphism)
An *algebra morphism* f from (A, h_A) to (B, h_B) is a morphism from A to B such that the following diagram commutes:

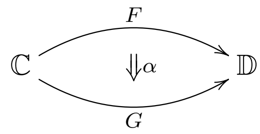
$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle h_A}\downarrow & & \downarrow{\scriptstyle h_B} \\ A & \xrightarrow[f]{} & B \end{array}$$

If F is a functor, its *category of algebras* $\mathbb{A}\mathsf{lg}(F)$ has objects the algebras of F and morphisms the algebra morphisms between them.

Definition 18 (Natural Transformation)
A *transformation* α between two functions F and G from the objects of a category $\mathbb{C}$ to the objects of a category $\mathbb{D}$ (in particular between two functors from $\mathbb{C}$ to $\mathbb{D}$) is a family $(\alpha_A)_{A \in obj(\mathbb{C})}$ of morphisms from FA to GA.
A transformation α between two functors F and G is *natural* if the following diagram in $\mathbb{D}$ commutes for all $f \in \mathbb{C}(A, B)$:

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_B} \\ GA & \xrightarrow[Gf]{} & GB \end{array}$$

It is represented:



A *natural isomorphism* is a natural transformation such that each element is an isomorphism.

Example 10 (Identity Natural Transformation)
If F is a functor between the categories $\mathbb{C}$ and $\mathbb{D}$, $(id_{FA})_{A \in obj(\mathbb{C})}$ is a natural isomorphism from F to itself.

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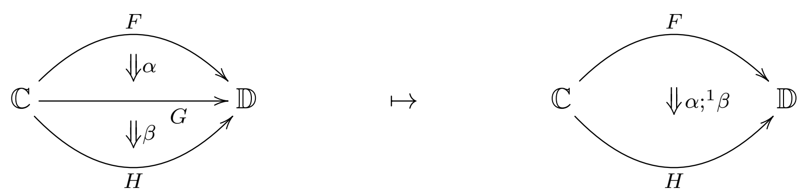
Definition 19 (Vertical Composition)
Let F, G and H be three functors between the same two categories $\mathbb{C}$ and $\mathbb{D}$, if α is a natural transformation for F to G and β is a natural transformation from G to H, the *vertical composition* $\alpha \,;^1 \beta$ is the natural transformation from F to H defined by $(\alpha \,;^1 \beta)_A = \alpha_A \,; \beta_A$.



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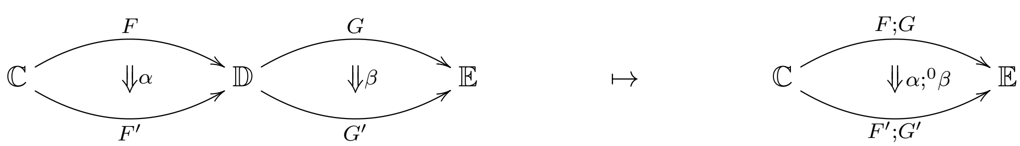
Definition 20 (Horizontal Composition)
Let $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{E}$ be three categories, F and F' be two functors from $\mathbb{C}$ to $\mathbb{D}$ and G and G' be two functors from $\mathbb{D}$ to $\mathbb{E}$, if α is a natural transformation for F to F' and β is a natural transformation from G to G', the *horizontal composition* $\alpha \,;^0 \beta$ is the natural transformation from $F \,; G$ to $F' \,; G'$ defined by $(\alpha \,;^0 \beta)_A = G\alpha_A \,; \beta_{F'A} = \beta_{FA} \,; G'\alpha_A$.



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Example 11 (Category of Functors)
Let $\mathbb{C}$ and $\mathbb{D}$ be two categories, the *category of functors* $\mathsf{Func}(\mathbb{C}, \mathbb{D})$ is given by:

- objects are functors between $\mathbb{C}$ and $\mathbb{D}$
- morphisms are natural transformations
- identities are the identity natural transformations
- composition is the vertical composition of natural transformations

1.4 Objects

Definition 21 (Terminal Object)
A *terminal object* in a category $\mathbb{C}$ is an object $\top$ such that, for any object A of $\mathbb{C}$, there exists a unique morphism t_A from A to $\top$.

If $\mathbb{C}$ is a category with a terminal object $\top$, a *point* of an object A of $\mathbb{C}$ is a morphism from $\top$ to A.

Definition 22 (Initial Object)
An *initial object* in a category $\mathbb{C}$ is an object $\bot$ such that, for any object A of $\mathbb{C}$, there exists a unique morphism i_A from $\bot$ to A.
It is thus a terminal object in $\mathbb{C}^{op}$.

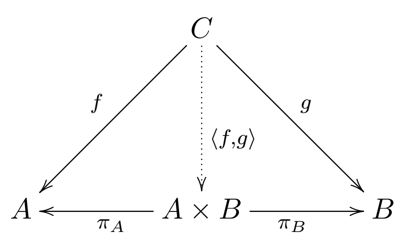
A *zero object* is an object 0 which is both initial and terminal. If 0 is a zero object in the category $\mathbb{C}$ and A and B are two objects of $\mathbb{C}$, the *zero morphism* $z_{A,B}$ is:

$$A \xrightarrow{t_A} 0 \xrightarrow{i_A} B$$

Definition 23 (Product)
A *product* of two objects A and B in a category $\mathbb{C}$ is a triple $(A \times B, \pi_A, \pi_B)$ where:

- $A \times B$ is an object of $\mathbb{C}$
- π_A is a morphism from $A \times B$ to A
- π_B is a morphism from $A \times B$ to B

such that, for any triple (C, f, g), where C is an object of $\mathbb{C}$, f is a morphism from C to A and g is a morphism from C to B, there exists a unique morphism $\langle f, g\rangle$ from C to $A \times B$ such that $\langle f, g\rangle \,;\pi_A = f$ and $\langle f, g\rangle \,;\pi_B = g$.
This can be written:



If $(A \times A, \pi^l_A, \pi^r_A)$ is a product of A and A in $\mathbb{C}$, the *diagonal morphism* Δ_A is $\langle id_A, id_A\rangle$ from A to $A \times A$. It a section of both projections π^l_A and π^r_A.
A category equipped with a product for each pair of objects and which has a terminal object is called a *cartesian category*. In such a category, one can form all products of finite families of objects. If $\mathbb{C}$ is a cartesian category, $\times$ defines a bi-functor from $\mathbb{C}$ and $\mathbb{C}$ to $\mathbb{C}$, and Δ is a natural transformation from $Id_{\mathbb{C}}$ to $\_ \times \_$.

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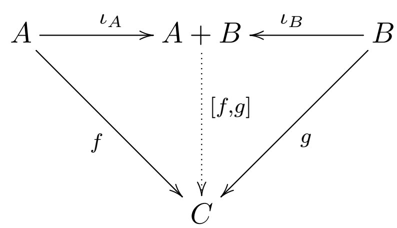
Definition 24 (Co-Product)
A *co-product* of two objects A and B in a category $\mathbb{C}$ is a triple $(A + B, \iota_A, \iota_B)$ where:

- $A + B$ is an object of $\mathbb{C}$
- ι_A is a morphism from A to $A + B$
- ι_B is a morphism from B to $A + B$

such that, for any triple (C, f, g), where C is an object of $\mathbb{C}$, f is a morphism from A to C and g is a morphism from B to C, there exists a unique morphism $[f, g]$ from $A + B$ to C such that $\iota_A \,;\, [f, g] = f$ and $\iota_B \,;\, [f, g] = g$.



It is thus a product in $\mathbb{C}^{op}$.

If $(A + A, \iota^l_A, \iota^r_A)$ is a co-product of A and A in $\mathbb{C}$, the *co-diagonal morphism* ∇_A is $[id_A, id_A]$ from $A + A$ to A.

Example 12 (Products and Co-Products in $\mathbb{S}$et)
If A and B are two sets, the cartesian product $A \times B$ (with the projection functions) defines a product of A and B in $\mathbb{S}$et. The singleton set $\{\star\}$ is terminal in $\mathbb{S}$et. With this structure, $\mathbb{S}$et is a cartesian category.
The disjoint union $A \uplus B$ (with the injection functions) is a co-product in $\mathbb{S}$et. The empty set $\emptyset$ is an initial object in $\mathbb{S}$et.

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Example 13 (Products in $\mathbb{C}$at)
If $\mathbb{C}$ and $\mathbb{D}$ are two categories, the product category $\mathbb{C} \times \mathbb{D}$ (with the projection functors) defines a product of $\mathbb{C}$ and $\mathbb{D}$ in $\mathbb{C}$at. The unit category $\mathbb{T}$ is terminal in $\mathbb{C}$at. With this structure, $\mathbb{C}$at is a cartesian category.

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Example 14 (Co-Products in $\mathbb{C}$at)
If $\mathbb{C}$ and $\mathbb{D}$ are two categories, the category $\mathbb{C} + \mathbb{D}$ is given by:

- objects are in the disjoint union $obj(\mathbb{C}) \uplus obj(\mathbb{D})$
- morphisms from $(0, A)$ to $(0, B)$ are $\mathbb{C}(A, B)$, morphisms from $(1, A')$ to $(1, B')$ are $\mathbb{D}(A', B')$ (and there is no morphism from (i, A) to (j, B') if $i \neq j$)
- composition and identities come from those of $\mathbb{C}$ and $\mathbb{D}$

Up to the identification of $obj(\mathbb{C})$ and $obj(\mathbb{D})$ with their disjoint copies in $obj(\mathbb{C}) \uplus obj(\mathbb{D})$, one can consider the inclusion functors as functors from $\mathbb{C}$ to $\mathbb{C} + \mathbb{D}$ and from $\mathbb{D}$ to $\mathbb{C} + \mathbb{D}$. The category $\mathbb{C} + \mathbb{D}$ with these two functors defines a co-product of $\mathbb{C}$ and $\mathbb{D}$ in $\mathbb{C}$at.
The *empty category* $\perp\!\!\!\perp$ with no object and no morphism is initial in $\mathbb{C}$at.

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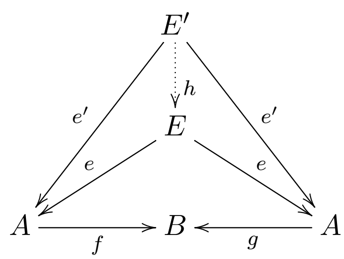
Definition 25 (Bi-Product)
Let $\mathbb{C}$ be a category with a zero object 0 and A and B two objects of $\mathbb{C}$, a *bi-product* of A and B is a 5-tuple $(A \oplus B, \iota_A, \iota_B, \pi_A, \pi_B)$ where:

- $(A \oplus B, \pi_A, \pi_B)$ is a product of A and B in $\mathbb{C}$
- $(A \oplus B, \iota_A, \iota_B)$ is a co-product of A and B in $\mathbb{C}$

and such that:

$$\iota_A \,;\, \pi_A = id_A$$
$$\iota_B \,;\, \pi_B = id_B$$
$$\iota_A \,;\, \pi_B = z_{A,B}$$
$$\iota_B \,;\, \pi_A = z_{B,A}$$

Definition 26 (Equalizer)
An *equalizer* of two morphisms f and g between the same two objects A and B in a category $\mathbb{C}$ is a pair (E, e) where E is an object of $\mathbb{C}$ and e is a morphism from E to A such that $e \,;\, f = e \,;\, g$ and, for any pair (E', e'), where E' is an object of $\mathbb{C}$ and e' is a morphism from E' to A such that $e' \,;\, f = e' \,;\, g$, there exists a unique morphism h from E' to E such that $e' = h \,;\, e$.
This can be written:



If (E, e) is an equalizer, e is a monomorphism. Such monomorphisms coming from an equalizer are called *regular monomorphisms*. A split monomorphism is a regular monomorphism.
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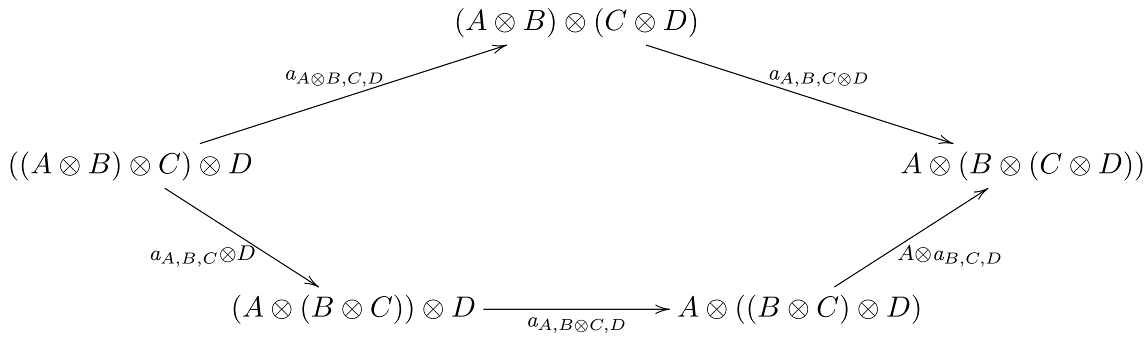
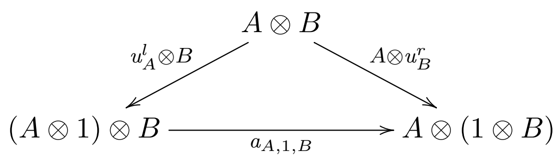
2 Monoidal Categories

Definition 27 (Monoidal Category)
A *monoidal category* is a 6-tuple $(\mathbb{C}, \otimes, 1, a, u^l, u^r)$ where:

- $\otimes$ is a bi-functor from $\mathbb{C}$ and $\mathbb{C}$ to $\mathbb{C}$
- 1 is an object of $\mathbb{C}$
- a is a natural isomorphism from $(\_ \otimes \_') \otimes \_''$ to $\_ \otimes (\_' \otimes \_'')$
- u^l is a natural isomorphism from $Id_{\mathbb{C}}$ to $\_ \otimes 1$
- u^r is a natural isomorphism from $Id_{\mathbb{C}}$ to $1 \otimes \_$

such that the following diagrams commute:



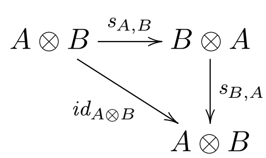


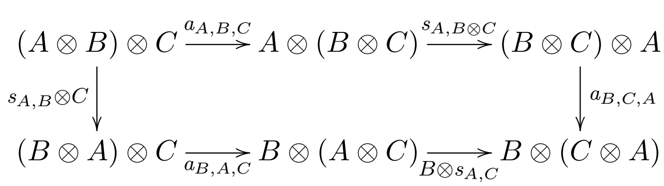
A monoidal category is *strict* if the natural isomorphisms a, u^l and u^r are the identity natural isomorphism.

A *symmetric monoidal category* is a 7-tuple $(\mathbb{C}, \otimes, 1, a, u^l, u^r, s)$ where:

- $(\mathbb{C}, \otimes, 1, a, u^l, u^r)$ is a monoidal category
- s is a natural isomorphism from $\_ \otimes \_'$ to $\_' \otimes \_$

such that the following diagrams commute:





From this definition, it is possible to deduce that, in any monoidal category, $u^r_1 = u^l_1$.

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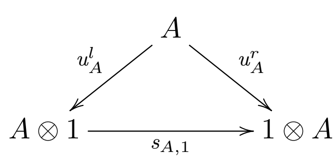
From this definition, it is possible to deduce that, in any symmetric monoidal category:



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If $(\mathbb{C}, \otimes, 1, a, u^l, u^r)$ is a monoidal category (resp. a symmetric monoidal category) then $(\mathbb{C}^{op}, \otimes, 1, a^{-1}, {u^l}^{-1}, {u^r}^{-1})$ as well.

Example 15 (Cartesian Category)
A cartesian category $\mathbb{C}$ is a symmetric monoidal category $(\mathbb{C}, \times, \top)$ with the natural isomorphisms:

- $a_{A,B,C} = \langle \pi_{A\times B} \,;\, \pi_A, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C \rangle\rangle$
- $u^l_A = \langle id_A, t_A \rangle$
- $u^r_A = \langle t_A, id_A \rangle$
- $s_{A,B} = \langle \pi_B, \pi_A \rangle$

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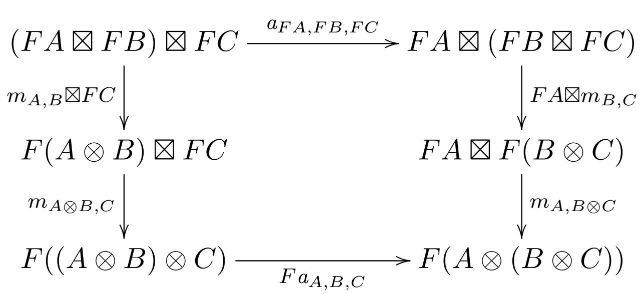
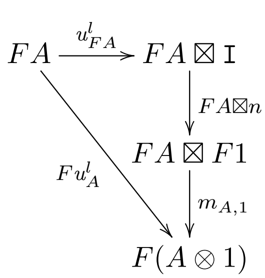
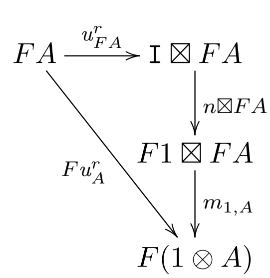
Definition 28 (Monoidal Functor)
A *monoidal functor* between two monoidal categories $(\mathbb{C}, \otimes, 1)$ and $(\mathbb{D}, \boxtimes, \mathtt{I})$ is a triple (F, m, n) where:

- F is a functor from $\mathbb{C}$ to $\mathbb{D}$
- m is a natural transformation from $F\_ \boxtimes F\_'$ to $F(\_ \otimes \_')$
- n is a morphism from $\mathtt{I}$ to $F1$

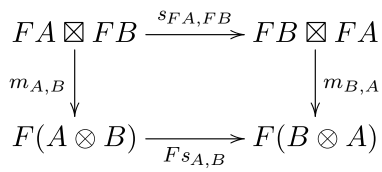
such that the following diagrams in $\mathbb{D}$ commute:





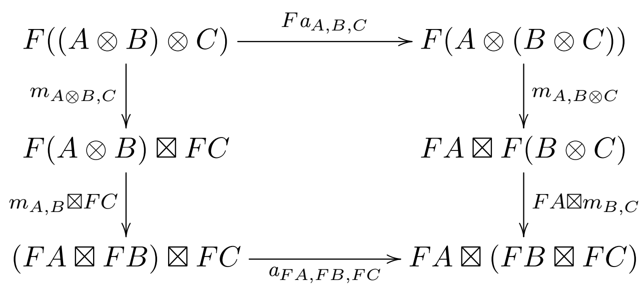


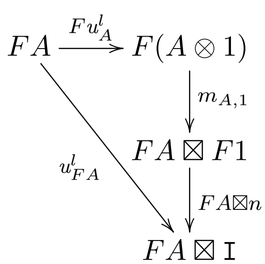
If $\mathbb{C}$ and $\mathbb{D}$ are symmetric monoidal, a *symmetric monoidal functor* is a monoidal functor such that the following diagram in $\mathbb{D}$ commutes:

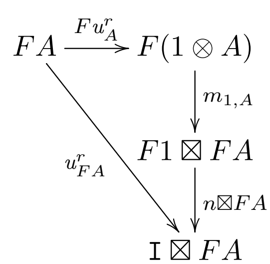


Let (F, m, n) be a monoidal functor, F is *strong* if $m_{A,B}$ and n are isomorphisms and F is *strict* if they are equalities.

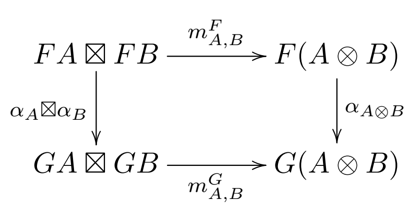
Definition 29 (Co-Monoidal Functor)
A *co-monoidal functor* between two monoidal categories $(\mathbb{C}, \otimes, 1)$ and $(\mathbb{D}, \boxtimes, \mathtt{I})$ is a triple (F, m, n) which is a monoidal functor between $(\mathbb{C}^{op}, \otimes, 1)$ and $(\mathbb{D}^{op}, \boxtimes, \mathtt{I})$, thus: m natural transformation from $F(\_ \otimes \_')$ to $F\_ \boxtimes F\_'$ and n morphism from $F1$ to $\mathtt{I}$.
We thus have the following commutative diagrams:

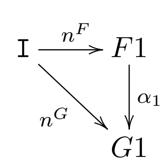






Definition 30 (Monoidal Natural Transformation)
A *monoidal natural transformation* α between two monoidal functors F and G between the same two monoidal categories $(\mathbb{C}, \otimes, 1)$ and $(\mathbb{D}, \boxtimes, \mathtt{I})$ is a natural transformation such that the following diagrams in $\mathbb{D}$ commute:





2.1 Monoids

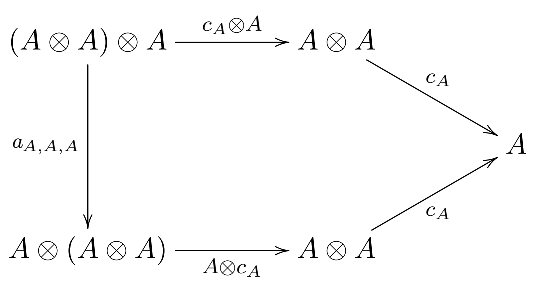
Definition 31 (Monoid)
A *monoid* in a monoidal category $(\mathbb{C}, \otimes, 1)$ is a triple (A, c_A, w_A) where:

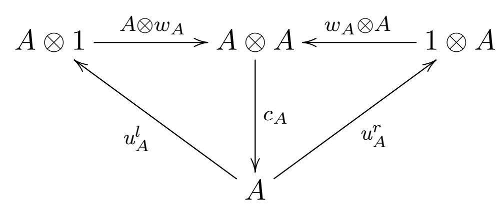
- A is an object
- c_A is a morphism from $A \otimes A$ to A
- w_A is a morphism from 1 to A

that is:

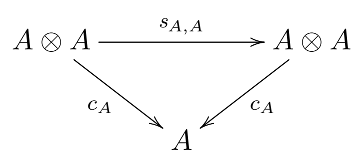
$$A \otimes A \xrightarrow{c_A} A \xleftarrow{w_A} 1$$

such that the following diagrams commute:

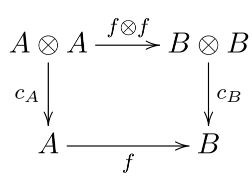


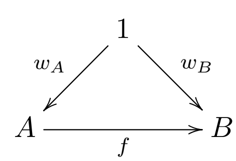


If $\mathbb{C}$ is symmetric monoidal, a monoid is *symmetric* if the following diagram commutes:



Definition 32 (Monoidal Morphism)
A *monoidal morphism* f between two monoids (A, c_A, w_A) and (B, c_B, w_B) in a monoidal category is a morphism from A to B such that the following diagrams commute:

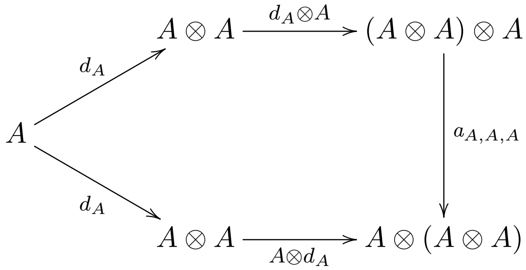


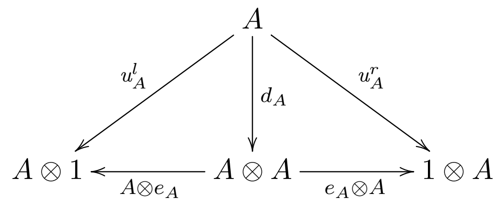


Monoids of a monoidal category $(\mathbb{C}, \otimes, 1)$ and monoidal morphisms between them define a category $\mathbb{Mon}(\mathbb{C})$ called the *category of monoids* of $\mathbb{C}$.

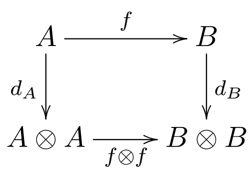
Definition 33 (Co-Monoid)
A *co-monoid* in $\mathbb{C}$ is a monoid in $\mathbb{C}^{op}$. It is thus a triple (A, d_A, e_A) with d_A morphism from A to

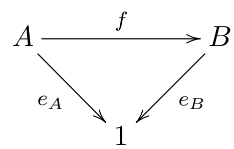
$A \otimes A$ and e_A morphism from A to 1 such that:





Definition 34 (Co-Monoidal Morphism)
A *co-monoidal morphism* f between two co-monoids (A, d_A, e_A) and (B, d_B, e_B) in a monoidal category is a morphism from A to B such that the following diagrams commute:

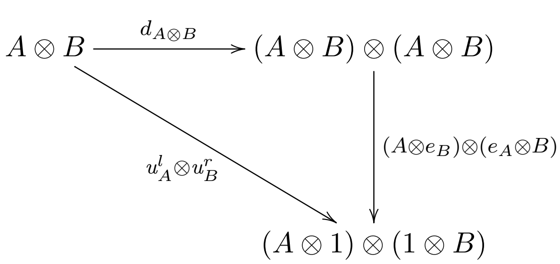




Co-monoids of a monoidal category $(\mathbb{C}, \otimes, 1)$ and co-monoidal morphisms between them define a category $\mathsf{coMon}(\mathbb{C})$ called the *category of co-monoids* of $\mathbb{C}$.

Example 16 (Co-Monoids and Cartesian Categories)
In a cartesian category $\mathbb{C}$, each object A comes with a canonical structure of symmetric co-monoid (A, Δ_A, t_A). Since any morphism of $\mathbb{C}$ is co-monoidal for these co-monoid structures, one can see $\mathbb{C}$ as a full sub-category of $\mathsf{coMon}(\mathbb{C})$.
Conversely, let $\mathbb{C}$ be a monoidal category and $\mathbb{M}$ be a sub-category of $\mathsf{coMon}(\mathbb{C})$ such that:

- the forgetful functor U from $\mathbb{M}$ to $\mathbb{C}$ which maps triples (A, d_A, e_A) to A is full and injective on objects
- if A and B are in the image of U then $A \otimes B$ as well
- 1 is in the image of U
- the following diagram commutes:



- $e_1 = id_1$

then $U\mathbb{M}$ is a cartesian category with $\otimes$ as product and 1 as terminal object.

Property 3 (Preservation of Monoids)
If (F, m, n) is a monoidal functor from $(\mathbb{C}, \otimes, 1)$ to $(\mathbb{D}, \boxtimes, \mathtt{I})$ and (A, c_A, w_A) is a monoid in $(\mathbb{C}, \otimes, 1)$, then $(FA, m_{A,A}\,;Fc_A, n\,;Fw_A)$ is a monoid in $(\mathbb{D}, \boxtimes, \mathtt{I})$. We say that monoidal functors preserve monoids.

$$FA \boxtimes FA \xrightarrow{m_{A,A}} F(A \otimes A) \xrightarrow{Fc_A} FA \xleftarrow{Fw_A} F1 \xleftarrow{n} \mathtt{I}$$

Similarly, symmetric monoidal functors preserve symmetric monoids, and co-monoidal functors preserve co-monoids.

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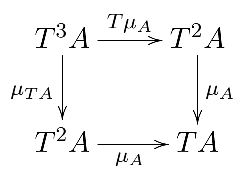
3 Monads

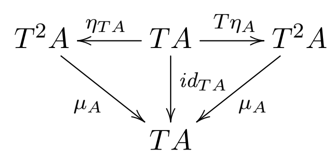
Definition 35 (Monad)
A *monad* on a category $\mathbb{C}$ is a triple (T, η, μ) where:

- T is an endofunctor of $\mathbb{C}$
- η is a natural transformation from $Id_{\mathbb{C}}$ to T
- μ is a natural transformation from T^2 to T

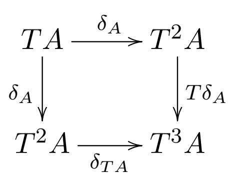
$$T^2 \xrightarrow{\mu} T \xleftarrow{\eta} Id_{\mathbb{C}}$$

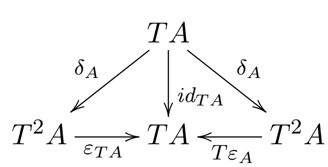
such that the following diagrams commute:





A *co-monad* on $\mathbb{C}$ is a monad on $\mathbb{C}^{op}$, that is a triple (T, ε, δ) (T endofunctor of $\mathbb{C}$, ε natural transformation from T to $Id_{\mathbb{C}}$ and δ natural transformation from T to T^2) such that:

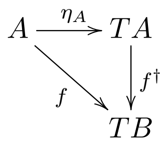


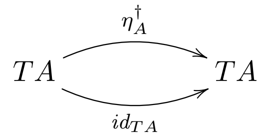


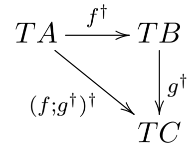
Definition 36 (Kleisli Triple)
A *Kleisli triple* on a category $\mathbb{C}$ is a triple $(T, \eta, (\_)^\dagger)$ where:

- T is a function from $obj(\mathbb{C})$ to $obj(\mathbb{C})$
- η is a transformation from $Id_{\mathbb{C}}$ to T
- $(\_)^\dagger$ is a function from $\mathbb{C}(A, TB)$ to $\mathbb{C}(TA, TB)$

such that the following diagrams commute:







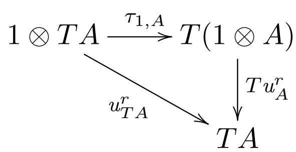
The notions of monad and Kleisli triple are equivalent through:

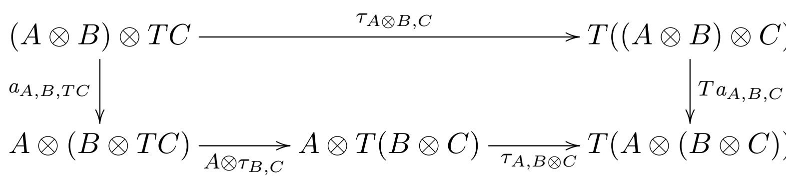
$$(T, \eta, \mu) \mapsto (T, \eta, T_- \,;\, \mu)$$
$$(T, \eta, (\_)^\dagger) \mapsto (T, \eta, id^\dagger_{T\_})$$

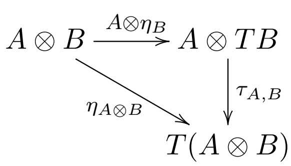
Definition 37 (Strong Monad)
A *strong monad* on a monoidal category $\mathbb{C}$ is a monad equipped with τ where:

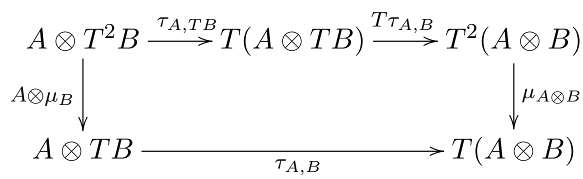
- τ is a natural transformation from $\_ \otimes T\_'$ to $T(\_ \otimes \_')$

such that the following diagrams commute:





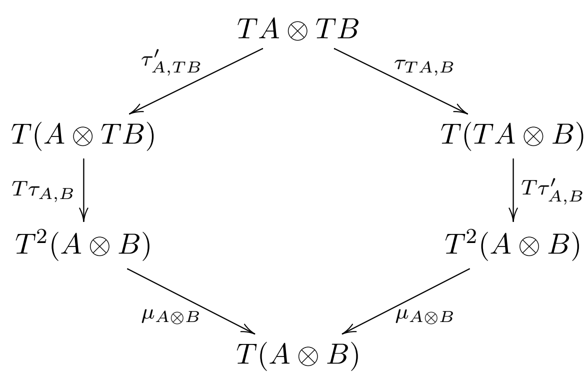




Definition 38 (Commutative Monad)
A *commutative monad* on a symmetric monoidal category $\mathbb{C}$ is a strong monad such that, if:

$$\tau'_{A,B} = TA \otimes B \xrightarrow{s_{TA,B}} B \otimes TA \xrightarrow{\tau_{B,A}} T(B \otimes A) \xrightarrow{Ts_{B,A}} T(A \otimes B)$$

then the following diagram commutes:



Definition 39 (Monoidal Monad)
A monad (T, η, μ) on a monoidal category $\mathbb{C}$ is *monoidal* if T is a monoidal functor, and η and μ are monoidal natural transformations.
If $\mathbb{C}$ is symmetric monoidal, the monad is *symmetric monoidal* if, moreover, T is a symmetric monoidal functor.

Property 4 (Monoidal and Commutative Monads)
Let $\mathbb{C}$ be a symmetric monoidal category and T be a strong monad on $\mathbb{C}$:

- *T equipped with either:*

$$TA \otimes TB \xrightarrow{\tau_{TA,B}} T(TA \otimes B) \xrightarrow{T\tau'_{A,B}} T^2(A \otimes B) \xrightarrow{\mu_{A\otimes B}} T(A \otimes B)$$

  or

$$TA \otimes TB \xrightarrow{\tau'_{A,TB}} T(A \otimes TB) \xrightarrow{T\tau_{A,B}} T^2(A \otimes B) \xrightarrow{\mu_{A\otimes B}} T(A \otimes B)$$

  and $\eta_1 : 1 \to T1$ is a monoidal functor

- *in both cases, η and μ are monoidal natural transformations*

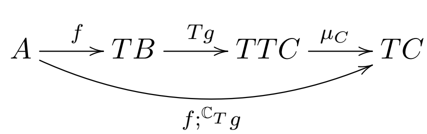
- *T is a symmetric monoidal functor $\iff$ T is a commutative monad*

Definition 40 (Algebra)
An *algebra* for the monad T is a pair (A, h_A) which is an algebra for the functor T such that the following diagrams commute:

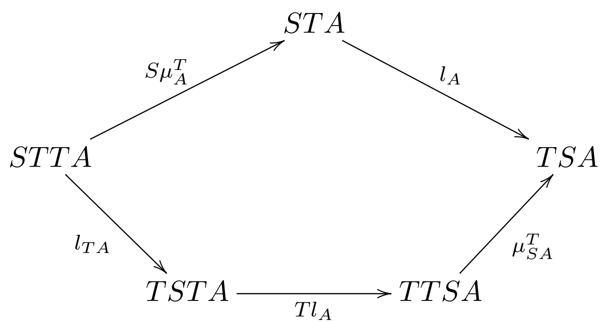
Example 17 (Free Algebra)
For any object A, (TA, μ_A) is an algebra called the *free algebra* generated by A.

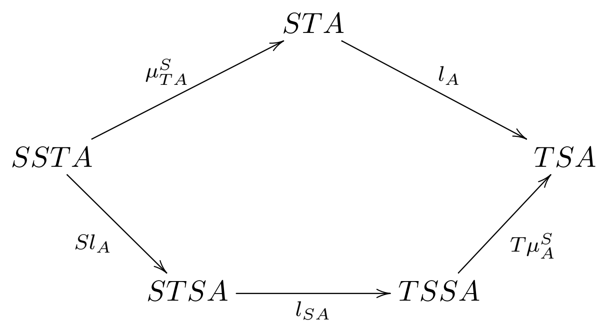
Definition 41 (Eilenberg-Moore Category)
If T is a monad on the category $\mathbb{C}$, its *category of algebras* is the full sub-category of the category of algebras of the functor T whose objects are the algebras of the monad T. It is also called the *Eilenberg-Moore category* of T and denoted $\mathbb{C}^T$.

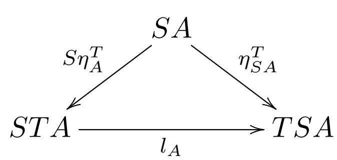
Definition 42 (Kleisli Category)
If T is a monad on the category $\mathbb{C}$, the *Kleisli category* $\mathbb{C}_T$ has objects the objects of $\mathbb{C}$ and for morphisms: $\mathbb{C}_{\mathbb{T}}(A, B) = \mathbb{C}(A, TB)$. The identities are $\eta_A \in \mathbb{C}(A, TA)$, and the composition of $f \in \mathbb{C}(A, TB)$ and $g \in \mathbb{C}(B, TC)$ is $f \,;Tg\,; \mu_C \in \mathbb{C}(A, TC)$.

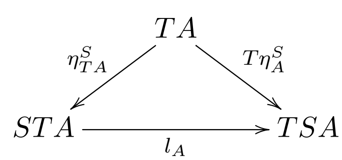


Definition 43 (Distributive Law)
If (T, η^T, μ^T) and (S, η^S, μ^S) are two monads on the category $\mathbb{C}$, a *distributive law* of T over S is a natural transformation l from ST to TS such that the following diagrams commute:









Example 18 (Composition of Monads)
Let (T, η^T, μ^T) and (S, η^S, μ^S) be two monads on the category $\mathbb{C}$, and l be a distributive law of T over S, TS equipped with

$$A \xrightarrow{\eta^S_A} SA \xrightarrow{\eta^T_{SA}} TSA \qquad \text{and} \qquad TSTSA \xrightarrow{Tl_{SA}} TTSSA \xrightarrow{\mu^T_{SSA}} TSSA \xrightarrow{T\mu^S_A} TSA$$

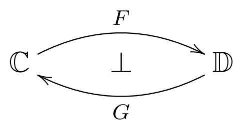
is a monad on $\mathbb{C}$.

4 Adjunctions

Definition 44 (Adjunction)
An *adjunction $F \dashv G$* between two categories $\mathbb{C}$ and $\mathbb{D}$ is a triple (F, G, φ) where:

- F is a functor from $\mathbb{C}$ to $\mathbb{D}$
- G is a functor from $\mathbb{D}$ to $\mathbb{C}$

- φ is a natural isomorphism from the functor $\mathbb{D}(F\_, \_')$ to the functor $\mathbb{C}(\_, G\_')$ (both from $\mathbb{C}^{op} \times \mathbb{D}$ to $\mathbb{S}\mathsf{et}$).



$$\frac{FA \longrightarrow B'}{A \longrightarrow GB'}\ \varphi$$

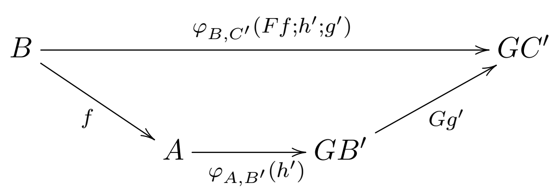
Equivalently, an *adjunction* $F \dashv G$ between two categories $\mathbb{C}$ and $\mathbb{D}$ is a quadruple $(F, G, \eta, \varepsilon)$ where:

- F is a functor from $\mathbb{C}$ to $\mathbb{D}$
- G is a functor from $\mathbb{D}$ to $\mathbb{C}$
- η is a natural transformation from $Id_{\mathbb{C}}$ to GF
- ε is a natural transformation from FG to $Id_{\mathbb{D}}$

such that the following diagrams commute:

If $F \dashv G$ is an adjunction, F is called a *left adjoint* and G is called a *right adjoint*.

The diagram underlying the naturality of φ is, in $\mathbb{C}$:



The equivalence between the two definitions is given by:

$$\varphi_{A,A'}(f) = A \xrightarrow{\eta_A} GFA \xrightarrow{Gf} GA'$$

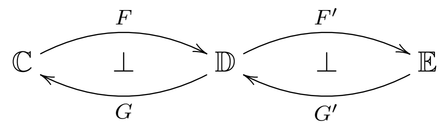
$$\eta_A = A \xrightarrow{\varphi_{A,FA}(id_{FA})} GFA$$

$$\varepsilon_{A'} = FGA' \xrightarrow{\varphi^{-1}_{GA',A'}(id_{GA'})} A'$$

Example 19 (Category of Adjunctions)
The *category of adjunctions* $\mathbb{A}\mathsf{dj}$ is given by:

- objects are (small) categories
- morphisms in $\mathbb{A}\mathsf{dj}(\mathbb{C}, \mathbb{D})$ are adjunctions between $\mathbb{C}$ and $\mathbb{D}$
- identities are identity adjunctions (Id, Id, id)

- composition is composition of adjunctions: if (F, G, φ) is an adjunction between $\mathbb{C}$ and $\mathbb{D}$ and (F', G', φ') is an adjunction between $\mathbb{D}$ and $\mathbb{E}$ then $(F \mathbin{;} F', G' \mathbin{;} G, \varphi'_{F\_,\_'} \mathbin{;} \varphi_{\_,G'\_'})$ is an adjunction between $\mathbb{C}$ and $\mathbb{E}$.

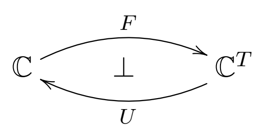


Definition 45 (Monoidal Adjunction)
An adjunction $(F, G, \eta, \varepsilon)$ between two monoidal categories $\mathbb{C}$ and $\mathbb{D}$ is *monoidal* if F and G are monoidal functors and η and ε are monoidal natural transformations.
If $\mathbb{C}$ and $\mathbb{D}$ are symmetric monoidal, the adjunction is *symmetric monoidal* if, moreover, F and G are symmetric monoidal functors.

In a monoidal adjunction, F is strong.

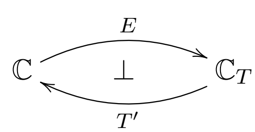
Property 5 (Monad of an Adjunction)
If $(F, G, \eta, \varepsilon)$ is an adjunction, $(GF, \eta, G\varepsilon_{F\_})$ is a monad called the monad of the adjunction.
Similarly, $(FG, \varepsilon, F\eta_{G\_})$ is a co-monad.
If the adjunction is monoidal, the monad is monoidal. If the adjunction is symmetric monoidal, the monad is symmetric monoidal.

Example 20 (Eilenberg-Moore Adjunction)
Let T be a monad on $\mathbb{C}$, let F be the free-algebra functor from $\mathbb{C}$ to $\mathbb{C}^T$ associating (TA, μ_A) with A, and associating $Tf \in \mathbb{C}^{\mathbb{T}}((TA, \mu_A), (TB, \mu_B))$ with $f \in \mathbb{C}(A, B)$.
Let U be the forgetful functor from $\mathbb{C}^T$ to $\mathbb{C}$ associating A with the algebra (A, h_A) and such that $Uf = f$.



F is a left adjoint to U and the monad associated with this adjunction is T.

Example 21 (Kleisli Adjunction)
Let T be a monad on $\mathbb{C}$, let E be the embedding functor from $\mathbb{C}$ to $\mathbb{C}_T$ associating A with A ($EA = A$), and associating $f \mathbin{;} \eta_A \in \mathbb{C}_{\mathbb{T}}(A, B)$ with $f \in \mathbb{C}(A, B)$.
Let T' be the functor from $\mathbb{C}_T$ to $\mathbb{C}$ defined by $T'A = TA$ and $T'f = Tf \mathbin{;} \mu_B$ for $f \in \mathbb{C}_{\mathbb{T}}(A, B)$.

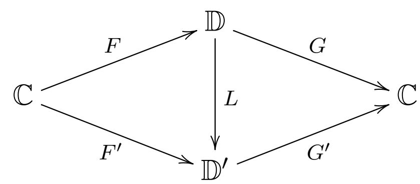


E is a left adjoint to T' and the monad associated with this adjunction is T.

Example 22 (Category of Adjunctions of a Monad)
Let T be a monad on a category $\mathbb{C}$, the category T-$\mathbb{A}\mathsf{dj}$ of adjunctions of the monad T is given by:

- objects are tuples $(\mathbb{D}, F, G, \eta, \varepsilon)$ where $(F, G, \eta, \varepsilon)$ is an adjunction between $\mathbb{C}$ and $\mathbb{D}$ which induces the monad T on $\mathbb{C}$ (Property 5)

- morphisms between $(\mathbb{D}, F, G, \eta, \varepsilon)$ and $(\mathbb{D}', F', G', \eta', \varepsilon')$ are functors L from $\mathbb{D}$ to $\mathbb{D}'$ such that the following diagram commutes:



  and $L\varepsilon = \varepsilon'_L$.

The Kleisli adjunction is the initial object of T-$\mathbb{A}$dj.
The Eilenberg-Moore adjunction is the terminal object of T-$\mathbb{A}$dj.

Definition 46 (Equivalence of Categories)
A functor F between two categories $\mathbb{C}$ and $\mathbb{D}$ is an *equivalence of categories* if one of the two following equivalent properties is true:

- There exists an adjunction $(G, F, \eta, \varepsilon)$ between $\mathbb{D}$ and $\mathbb{C}$ such that η and ε are natural isomorphisms.
- F is full, faithful and essentially surjective.

Property 6 (Strict Monoidal Categories)
Every monoidal category is equivalent to a strict monoidal category.

Property 7 (Kleisli Category and Free Algebras)
If T is a monad on the category $\mathbb{C}$, the category $\mathbb{C}_T$ is equivalent to the full-subcategory of $\mathbb{C}^T$ consisting of free algebras.

5 Closed Categories

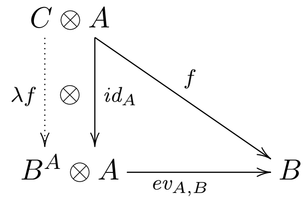
Definition 47 (Symmetric Monoidal Closed Category)
A symmetric monoidal category $(\mathbb{C}, \otimes, 1, a, u^l, u^r, s)$ is *closed* if, for any object A of $\mathbb{C}$, the functor $\_ \otimes A$ has a right adjoint (noted $A \multimap \_$).

$$\frac{C \otimes A \longrightarrow B}{C \longrightarrow A \multimap B}\ \mathit{curry}$$

In a symmetric monoidal closed category, if f is a morphism from $C \otimes A$ to B, we denote by $\mathit{curry}(f)$ the induced morphism from C to $A \multimap B$. We define $ev_{A,B}$ as $\mathit{curry}^{-1}(id_{A \multimap B}) \in \mathbb{C}((A \multimap B) \otimes A, B)$.

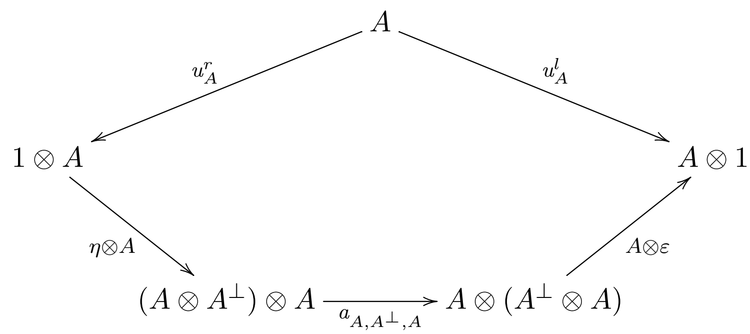
Definition 48 (Exponential Object)
If A and B are two objects of a symmetric monoidal category $\mathbb{C}$, an *exponential object* of A and B is a pair $(B^A, ev_{A,B})$ where B^A is an object of $\mathbb{C}$ and $ev_{A,B} \in \mathbb{C}(B^A \otimes A, B)$ such that, for any morphism $f \in \mathbb{C}(C \otimes A, B)$, there exists a unique morphism $\lambda f \in \mathbb{C}(C, B^A)$ such that $f = (\lambda f \otimes id_A)\,;\, ev_{A,B}$.

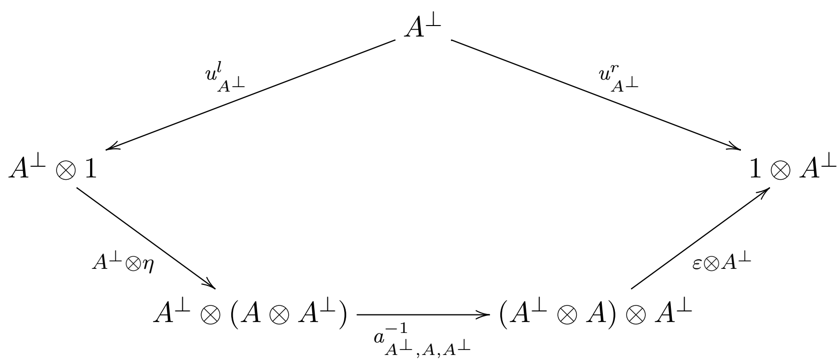
This can be written:



The notions of symmetric monoidal closed category and exponential object are related by the fact that a symmetric monoidal category is closed if and only if each pair of objects has an associated exponential object.

Definition 49 (Dual Object)
In a symmetric monoidal category $(\mathbb{C}, \otimes, 1, a, u^l, u^r, s)$, a *dual* of an object A is an object $A^\perp$ with two morphisms $\eta \in \mathbb{C}(1, A \otimes A^\perp)$ and $\varepsilon \in \mathbb{C}(A^\perp \otimes A, 1)$ such that the following diagrams commute:





Definition 50 (Compact Closed Category)
A symmetric monoidal category is *compact closed* if each object has a dual object.

Example 23 (Closure of Compact Closed Categories)
A compact closed category is a symmetric monoidal closed category with $A \multimap \_ = A^\perp \otimes \_$.

Remember (Example 15) that a cartesian category has a canonical symmetric monoidal structure.

Definition 51 (Cartesian Closed Category)
A cartesian category is *cartesian closed* if, as a symmetric monoidal category, it is closed.

Definition 52 (∗-Autonomous Category)
A symmetric monoidal closed category $\mathbb{C}$ is *∗-autonomous* if it contains a *dualizing object*, that is an object $\bot$ such that, for each object A of $\mathbb{C}$, the following morphism is an isomorphism between A and $(A \multimap \bot) \multimap \bot$:

$$curry\left(A \otimes (A \multimap \bot) \xrightarrow{s_{A,A\multimap\bot}} (A \multimap \bot) \otimes A \xrightarrow{ev_{A,\bot}} \bot \right)$$

Example 24 (Compact Closed and ∗-Autonomous Categories)
Any compact closed category is ∗-autonomous with $1^\bot$ as dualizing object.
Any ∗-autonomous category such that $(A \otimes B) \multimap \bot \simeq (B \multimap \bot) \otimes (A \multimap \bot)$ is compact closed with $A \multimap \bot$ as dual of A.

6 2-Categories

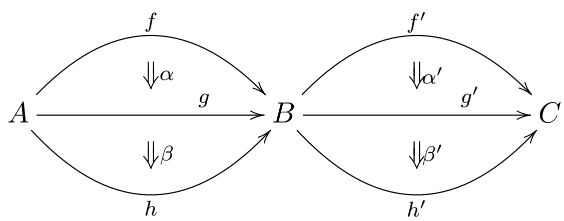
Definition 53 (2-Category)
A *2-category* $\mathbb{C}$ is given by:

- a class of objects $obj(\mathbb{C})$
- for any two objects A and B, a class of *1-morphisms* $\mathbb{C}(A, B)$
- for any two object A and B and any two morphisms f and g in $\mathbb{C}(A, B)$, a class of *2-morphisms* (or 2-cells) $\mathbb{C}^2(f, g)$
- for any object A, a *1-identity* morphism id_A in $\mathbb{C}(A, A)$
- for any 1-morphism f, a *2-identity* morphism id^1_f in $\mathbb{C}^2(f, f)$
- for any two morphisms $f \in \mathbb{C}(A, B)$ and $g \in \mathbb{C}(B, C)$, a composition $f\,;g \in \mathbb{C}(A, C)$
- for any two 2-morphisms $\alpha \in \mathbb{C}^2(f, g)$ and $\beta \in \mathbb{C}^2(g, h)$, a *vertical composition* $\alpha;^1\beta \in \mathbb{C}^2(f, h)$
- for any two 2-morphisms $\alpha \in \mathbb{C}^2(f, g)$ and $\beta \in \mathbb{C}^2(f', g')$ with f and g in $\mathbb{C}(A, B)$ and f' and g' in $\mathbb{C}(B, C)$, an *horizontal composition* $\alpha\,;^0\beta \in \mathbb{C}^2(f\,; f', g\,; g')$

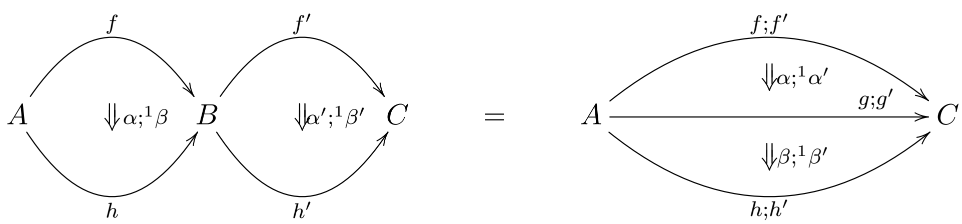
such that:

- $obj(\mathbb{C})$ with 1-morphisms, 1-identities, and composition is a category
- for any two objects A and B, $\mathbb{C}(A, B)$ with $\mathbb{C}^2(A, B)$ $(= \bigcup_{f,g\in\mathbb{C}(A,B)} \mathbb{C}^2(f, g))$ for morphisms, 2-identities between morphisms of $\mathbb{C}(A, B)$ for identities, and vertical composition for composition is a category
- $obj(\mathbb{C})$ with 2-morphisms for morphisms, 2-identities between 1-identities as identities, and horizontal composition for composition is a category

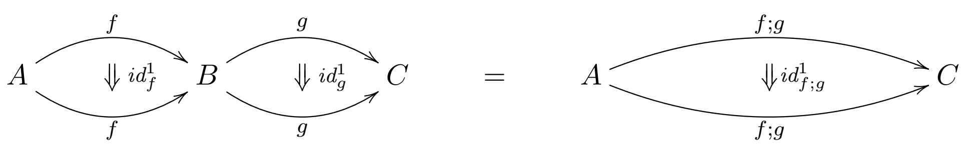
and given any four 2-morphisms of the following shape:



we have:



and we also have:



Example 25 (2-Category $\mathbb{C}$at)
(Small) Categories with functors for 1-morphisms, natural transformations for 2-morphisms, identity functors for 1-identities, identity natural transformations for 2-identities, composition of functors for composition, vertical composition of natural transformations for vertical composition, and horizontal composition of natural transformations for horizontal composition is a 2-category.

Example 26 (Monoidal Categories)
A 2-category with one object is the same thing as a strict monoidal category.

Property 8 (Monoidal Structures in 2-Categories)
Each object A of a 2-category $\mathbb{C}$ defines a strict monoidal category:

- *objects are 1-morphisms in $\mathbb{C}(A, A)$*
- *morphisms are 2-morphisms between them*
- *identities are id^1*
- *composition is vertical composition*

- *tensor product on objects is composition of 1-morphisms*
- *tensor product on morphisms is horizontal composition of 2-morphisms*
- *unit of the tensor is* id_A

Example 27 (Monads as Monoids)
Let $\mathbb{C}$ be a category, since it is an object in the 2-category $\mathbb{C}\mathsf{at}$, $\mathbb{F}\mathsf{unc}(\mathbb{C}, \mathbb{C})$ has a strict monoidal category structure given by Property 8. A monad is exactly a monoid in this monoidal category.

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Additional Properties

Cartesian Product

We consider a category $\mathbb{C}$, two objects A and B of $\mathbb{C}$ and a product $(A \times B, \pi_A, \pi_B)$ of A and B in $\mathbb{C}$.

Fact 1 (Pair of Projections)
$\langle \pi_A, \pi_B \rangle = id_{A\times B}$.

PROOF: $\langle \pi_A, \pi_B \rangle \,;\pi_A = \pi_A = id_{A\times B}\,;\pi_A$ and $\langle \pi_A, \pi_B \rangle \,;\pi_B = \pi_B = id_{A\times B}\,;\pi_B$ thus, by uniqueness of the pair, we have $\langle \pi_A, \pi_B \rangle = id_{A\times B}$. □

Fact 2 (Composition with Pair)
Let C and D be two objects of $\mathbb{C}$, if $f \in \mathbb{C}(C, A)$, $g \in \mathbb{C}(C, B)$ and $h \in \mathbb{C}(D, C)$ then $h \,; \langle f, g\rangle = \langle h\,; f, h\,; g\rangle$.

PROOF: We have $h;\langle f, g\rangle\,;\pi_A = h;f = \langle h\,; f, h\,; g\rangle\,;\pi_A$ and $h;\langle f, g\rangle\,;\pi_B = h;g = \langle h\,; f, h\,; g\rangle\,;\pi_B$, thus $h\,;\langle f, g\rangle = \langle h\,; f, h\,; g\rangle$ by uniqueness of the pair. □

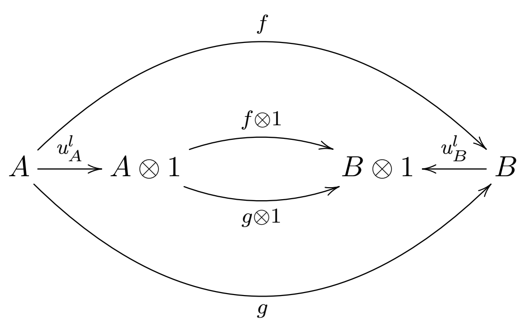
Monoidal Categories

We consider a monoidal category $(\mathbb{C}, \otimes, 1, a, u^l, u^r)$.

Fact 3 (Equality up to $\_ \otimes 1$ and $1 \otimes \_$)
Let A and B be two objects of $\mathbb{C}$ and f and g be two morphisms of $\mathbb{C}$ from A to B, $f \otimes 1 = g \otimes 1 \iff f = g \iff 1 \otimes f = 1 \otimes g$.

PROOF: We have $f = g$ implies both $f \otimes 1 = g \otimes 1$ and $1 \otimes f = 1 \otimes g$.

Now assume $f \otimes 1 = g \otimes 1$, the following diagram commutes:

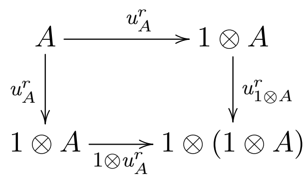


since the two squares commute by naturality of u^l. We conclude $f = g$ because u^l_B is an isomorphism.

Similarly, we obtain the implication $1 \otimes f = 1 \otimes g \Longrightarrow f = g$ by naturality of u^r. □

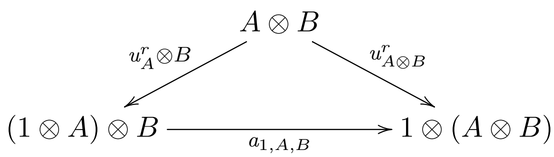
Fact 4 (Unit of Unit)
Let A be an object of $\mathbb{C}$, $u^r_{1\otimes A} = 1 \otimes u^r_A : 1 \otimes A \to 1 \otimes (1 \otimes A)$.

Proof: By naturality of u^r, we have:

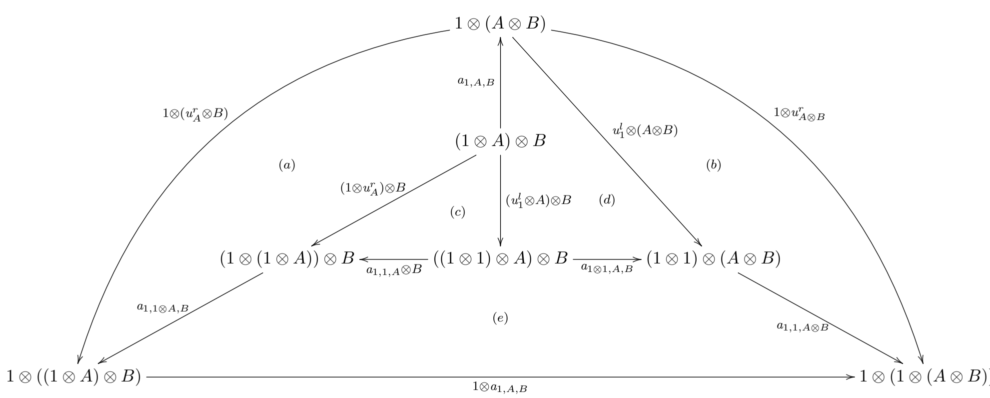


thus, since u^r_A is an isomorphism, $u^r_{1\otimes A} = 1 \otimes u^r_A$. □

Fact 5 (Associativity of Unit)
Let A and B be two objects of $\mathbb{C}$, the following diagram commutes:



Proof: Thanks to Fact 3, it is sufficient to prove the commutation of the following diagram (since a is an isomorphism):



which commutes by:

(a) naturality of a
(b) triangle of monoidal categories
(c) triangle of monoidal categories
(d) naturality of a
(e) pentagon of monoidal categories □

Proofs

Definition 9

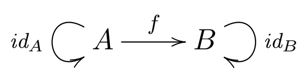
- If $g \,;\, s = h \,;\, s$ then $g = g \,;\, id_A = g \,;\, s \,;\, r = h \,;\, s \,;\, r = h \,;\, id_A = h$.
- If $r \,;\, g = r \,;\, h$ then $g = id_A \,;\, g = s \,;\, r \,;\, g = s \,;\, r \,;\, h = id_A \,;\, h = h$.
- $r \,;\, s \,;\, r \,;\, s = r \,;\, id_A \,;\, s = r \,;\, s$

Property 1

- Let f from A to B be an isomorphism and f^{-1} be its inverse, we have $f \,;\, f^{-1} = id_A$ and $f^{-1} \,;\, f = id_B$.
- There exist $g \in \mathbb{C}(B, A)$ such that $f \,;\, g = id_A$ and $h \in \mathbb{C}(B, A)$ such that $h \,;\, f = id_B$ thus $h = h \,;\, id_A = h \,;\, f \,;\, g = id_B \,;\, g = g$ and we conclude that $g = h$ is an inverse of f.

Comment Page 4

We give a direct proof: let f be an isomorphism from A to B, f^{-1} be its inverse, if g and g' are morphisms from A' to A then $g \,;\, f = g' \,;\, f$ implies $g = g \,;\, id_A = g \,;\, f \,;\, f^{-1} = g' \,;\, f \,;\, f^{-1} = g' \,;\, id_A = g'$. If h and h' are morphisms from B to B' then $f \,;\, h = f \,;\, h'$ implies $h = f^{-1} \,;\, f \,;\, h = f^{-1} \,;\, f \,;\, h' = h'$. In the following category:



with $id_A \,;\, f = f$ and $f \,;\, id_B = id_B$, f is both a monomorphism and an epimorphism but it is not an isomorphism since there is no morphism from B to A.

Example 3

Let A be an object of $\mathbb{C}$, $C_D id_A = id_D = id_{C_D A}$, and if $f \in \mathbb{C}(A, B)$ and $g \in \mathbb{C}(B, C)$ then $C_D(f \,;\, g) = id_D = id_D \,;\, id_D = C_D f \,;\, C_D g$.
A functor F from $\mathbb{C}$ to $\mathbb{T}$ must satisfy $FA = \star$ for any object A of $\mathbb{C}$ since $\star$ is the unique object of $\mathbb{T}$. We must then have $Ff \in \mathbb{T}(\star, \star) = \{id_\star\}$, so $F = C_\star$.

Example 4

We have $I id_A = id_A = id_{IA}$ and $I(f \,;\, g) = f \,;\, g = If \,;\, Ig$.

Example 5

If $\mathbb{C}$ and $\mathbb{D}$ are two (small) categories and F is a functor from $\mathbb{C}$ to $\mathbb{D}$, let A be an object of $\mathbb{C}$, we have $(Id_{\mathbb{C}} \,;\, F)A = F Id_{\mathbb{C}} A = FA = Id_{\mathbb{D}} FA = (F \,;\, Id_{\mathbb{D}})A$ and if $f \in \mathbb{C}(A, B)$ then $(Id_{\mathbb{C}} \,;\, F)f = F Id_{\mathbb{C}} f = Ff = Id_{\mathbb{D}} Ff = (F \,;\, Id_{\mathbb{D}})f$.
If $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{E}$ are three (small) categories, F is a functor from $\mathbb{C}$ to $\mathbb{D}$ and G is a functor from $\mathbb{D}$ to $\mathbb{E}$, let A be an object of $\mathbb{C}$, we have $((F;G);H)A = H(F;G)A = HGFA = (G;H)FA = (F;(G;H))A$ and if $f \in \mathbb{C}(A, B)$ then $((F \,;\, G) \,;\, H)f = H(F \,;\, G)f = HGFf = (G \,;\, H)Ff = (F \,;\, (G \,;\, H))f$.

Example 9

If $(A, B) \in obj(\mathbb{C} \times \mathbb{D})$, $Pid_{(A,B)} = P(id_A, id_B) = id_A = id_{P(A,B)}$.
If $(f, g) \in \mathbb{C} \times \mathbb{D}((A, B), (A', B'))$ and $(f', g') \in \mathbb{C} \times \mathbb{D}((A', B'), (A'', B''))$, $P((f, g) \,;\, (f', g')) = P(f \,;\, f', g \,;\, g') = f \,;\, f' = (P(f, g)) \,;\, (P(f', g'))$.
If $\mathbb{D}$ has at least one morphism between any two objects, let B and B' be two objects of $\mathbb{D}$ and $g \in \mathbb{D}(B, B')$, for any $f \in \mathbb{C}(A, A') = \mathbb{C}(P(A, B), P(A', B'))$, we have $P(f, g) = f$.

Example 10

If A is an object of $\mathbb{C}$, $id_{FA} \in \mathbb{D}(FA, FA)$ is an isomorphism (it is its own inverse).
If $f \in \mathbb{C}(A, B)$, $Ff \,;\, id_{FA} = Ff = id_{FA} \,;\, Ff$.

Definition 19

If $f \in \mathbb{C}(A, B)$, $Ff \,;\, (\alpha \,;^1 \beta)_B = Ff \,;\, \alpha_B \,;\, \beta_B = \alpha_A \,;\, Gf \,;\, \beta_B = \alpha_A \,;\, \beta_A \,;\, Hf = (\alpha \,;^1 \beta)_A \,;\, Hf$.

Definition 20

Since β is a natural transformation from G to G', we have $G\alpha_A \,;\, \beta_{F'A} = \beta_{FA} \,;\, G'\alpha_A$.
If $f \in \mathbb{C}(A, B)$, $(F \,;\, G)f \,;\, (\alpha \,;^0 \beta)_B = GFf \,;\, G\alpha_B \,;\, \beta_{F'B} = G(Ff \,;\, \alpha_B) \,;\, \beta_{F'B} = G(\alpha_A \,;\, F'f) \,;\, \beta_{F'B} = G\alpha_A \,;\, GF'f \,;\, \beta_{F'B} = G\alpha_A \,;\, \beta_{F'A} \,;\, G'F'f = (\alpha \,;^0 \beta)_A \,;\, (F' \,;\, G')f$.

Comment Page 9

For any two objects A and B, we have a product $A \times B$. If $f \in \mathbb{C}(A, B)$ and $f' \in \mathbb{C}(A', B')$, we define $f \times f' = \langle \pi_A \,;\, f, \pi_{A'} \,;\, f' \rangle \in \mathbb{C}(A \times A', B \times B')$.
We have $id_A \times id_{A'} = \langle \pi_A \,;\, id_A, \pi_{A'} \,;\, id_{A'} \rangle = \langle \pi_A, \pi_{A'} \rangle = id_{A \times A'}$ (using Fact 1).
If $f \in \mathbb{C}(A, B)$, $g \in \mathbb{C}(B, C)$, $f' \in \mathbb{C}(A', B')$ and $g' \in \mathbb{C}(B', C')$, we have, using Fact 2, $(f \times f') \,;\, (g \times g') = \langle \pi_A \,;\, f, \pi_{A'} \,;\, f' \rangle \,;\, \langle \pi_B \,;\, g, \pi_{B'} \,;\, g' \rangle = \langle \langle \pi_A \,;\, f, \pi_{A'} \,;\, f' \rangle \,;\, \pi_B \,;\, g, \langle \pi_A \,;\, f, \pi_{A'} \,;\, f' \rangle \,;\, \pi_{B'} \,;\, g' \rangle = \langle \pi_A \,;\, f \,;\, g, \pi_{A'} \,;\, f' \,;\, g' \rangle = (f \,;\, g) \times (f' \,;\, g')$
If $f \in \mathbb{C}(A, B)$, using Fact 2, $f \,;\, \Delta_B = f \,;\, \langle id_B, id_B \rangle = \langle f \,;\, id_B, f \,;\, id_B \rangle = \langle f, f \rangle = \langle id_A \,;\, f, id_A \,;\, f \rangle = \langle \langle id_A, id_A \rangle \,;\, \pi^l_A \,;\, f, \langle id_A, id_A \rangle \,;\, \pi^r_A \,;\, f \rangle = \langle id_A, id_A \rangle \,;\, \langle \pi^l_A \,;\, f, \pi^r_A \,;\, f \rangle = \Delta_A \,;\, (f \times f)$.

Example 12

If $f : C \to A$ and $g : C \to B$, we define:

$$\begin{aligned} \langle f, g \rangle : C &\to A \times B \\ x &\mapsto (f(x), g(x)) \end{aligned}$$

For all $x \in C$, we have $\pi_1 \circ \langle f, g \rangle(x) = f(x)$ and $\pi_2 \circ \langle f, g \rangle(x) = g(x)$. Let $h : C \to A \times B$ be such that any $x \in C$, $\pi_1 \circ h(x) = f(x)$ and $\pi_2 \circ h(x) = g(x)$ then $h(x) = (f(x), g(x)) = \langle f, g \rangle(x)$ that is $h = \langle f, g \rangle$.
For any set C, there is a unique function from C to $\{\star\}$ defined by:

$$\begin{aligned} t_C : C &\to \{\star\} \\ x &\mapsto \star \end{aligned}$$

If $f : A \to C$ and $g : B \to C$, we define:

$$\begin{aligned} [f,g] : A \uplus B &\to C \\ (0,a) &\mapsto f(a) \qquad \text{if } a \in A \\ (1,b) &\mapsto g(b) \qquad \text{if } b \in B \end{aligned}$$

For any $a \in A$, $[f,g] \circ \iota_1(a) = f(a)$ and for any $b \in B$, $[f,g] \circ \iota_2(b) = g(b)$. Let $h : A \uplus B \to C$ be such that for any $a \in A$, $h \circ \iota_1(a) = f(a)$ and for any $b \in B$, $h \circ \iota_2(b) = g(b)$, we have for any $z \in A \uplus B$, $h(z) = [f,g](z)$ that is $h = [f,g]$.
For any set C, there is a unique function from $\emptyset$ to C which is the empty function.

Example 13

If $F : \mathbb{E} \to \mathbb{C}$ and $G : \mathbb{E} \to \mathbb{D}$ are two functors, we define:

$$\begin{aligned} \langle F,G \rangle : \mathbb{E} &\to \mathbb{C} \times \mathbb{D} \\ E &\mapsto (FE, GE) \qquad \text{for objects of } \mathbb{E} \\ f &\mapsto (Ff, Gf) \qquad \text{for morphisms of } \mathbb{E} \end{aligned}$$

For any object E of $\mathbb{E}$, we have $P_{\mathbb{C}}\langle F,G\rangle E = FE$ and $P_{\mathbb{D}}\langle F,G\rangle E = GE$. For any morphism f of $\mathbb{E}$, we have $P_{\mathbb{C}}\langle F,G\rangle f = Ff$ and $P_{\mathbb{D}}\langle F,G\rangle f = Gf$. Let H be a functor from $\mathbb{E}$ to $\mathbb{C} \times \mathbb{D}$ such that $P_{\mathbb{C}}HE = FE$, $P_{\mathbb{D}}HE = GE$, $P_{\mathbb{C}}Hf = Ff$ and $P_{\mathbb{D}}Hf = Gf$ for any object E and any morphism f of $\mathbb{E}$, then $HE = (FE, GE) = \langle F,G\rangle E$ and $Hf = (Ff, Gf) = \langle F,G\rangle f$ that is $H = \langle F,G\rangle$.
Let $\mathbb{E}$ be a category, the unique functor $T_{\mathbb{E}}$ from $\mathbb{E}$ to $\mathbb{T}$ is defined by $T_{\mathbb{E}}E = \star$ for any object E of $\mathbb{E}$ and $T_{\mathbb{E}}f = id_\star$ for any morphism f of $\mathbb{E}$.

Example 14

If $F : \mathbb{C} \to \mathbb{E}$ and $G : \mathbb{D} \to \mathbb{E}$ are two functors, we define:

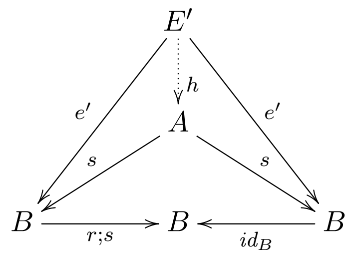
$$\begin{aligned} [F,G] : \mathbb{C} + \mathbb{D} &\to \mathbb{E} \\ (0,C) &\mapsto FC \qquad \text{if } C \in obj(\mathbb{C}) \\ (1,D) &\mapsto GD \qquad \text{if } D \in obj(\mathbb{D}) \\ f &\mapsto Ff \qquad \text{if } f \text{ morphism for } \mathbb{C} \\ g &\mapsto Gg \qquad \text{if } g \text{ morphism for } \mathbb{D} \end{aligned}$$

For any $C \in obj(\mathbb{C})$, $[F,G]I_{\mathbb{C}}C = FC$ and for any $B \in obj(\mathbb{D})$, $[F,G]I_{\mathbb{D}}D = GD$. For any f morphism in $\mathbb{C}$, $[F,G]I_{\mathbb{C}}f = Ff$ and for any g morphism in $\mathbb{D}$, $[F,G]I_{\mathbb{D}}g = Gg$. Let $H : \mathbb{C} + \mathbb{D} \to \mathbb{E}$ be a functor such that for any $C \in obj(\mathbb{C})$, $HI_{\mathbb{C}}C = FC$, for any $B \in obj(\mathbb{D})$, $HI_{\mathbb{D}}D = GD$, for any f morphism in $\mathbb{C}$, $HI_{\mathbb{C}}f = Ff$ and for any g morphism in $\mathbb{D}$, $HI_{\mathbb{D}}g = Gg$, we have for any object A and for any morphism h of $\mathbb{C} + \mathbb{D}$, $HA = [F,G]A$ and $Hh = [F,G]h$, that is $H = [F,G]$.
Let $\mathbb{E}$ be a category, the empty functor is the unique functor from $\perp\!\!\!\perp$ to $\mathbb{E}$.

Definition 26

Let (E,e) be an equalizer of $f \in \mathbb{C}(A,B)$ and $g \in \mathbb{C}(A,B)$, if f' and g' are in $\mathbb{C}(D,E)$ such that $f' ; e = g' ; e$ then $f' ; e ; f = g' ; e ; f = g' ; e ; g$ thus there exists a unique $h \in \mathbb{C}(D,E)$ such that $f' ; e = g' ; e = h ; e$ so that $f' = h = g'$.

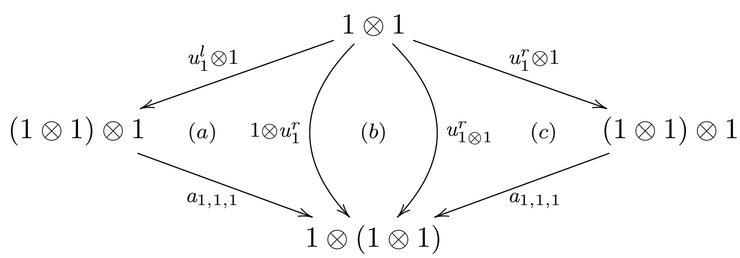
Given a split monomorphism s from A to B coming with its retraction r ($s\,;r = id_A$), we can prove it is the equalizer of $r\,;s$ and id_B:



Indeed, we have $s\,;r\,;s = s = s\,;id_B$, and if $e'\,;r\,;s = e'\,;id_B = e'$ then e' factors through s by means of $h = e'\,;r$. Moreover this h is unique since $h'\,;s = e'$ implies $h' = h'\,;s\,;r = e'\,;r$.

Comment Page 12

The following diagram commutes:



by:

(a) triangle of monoidal categories

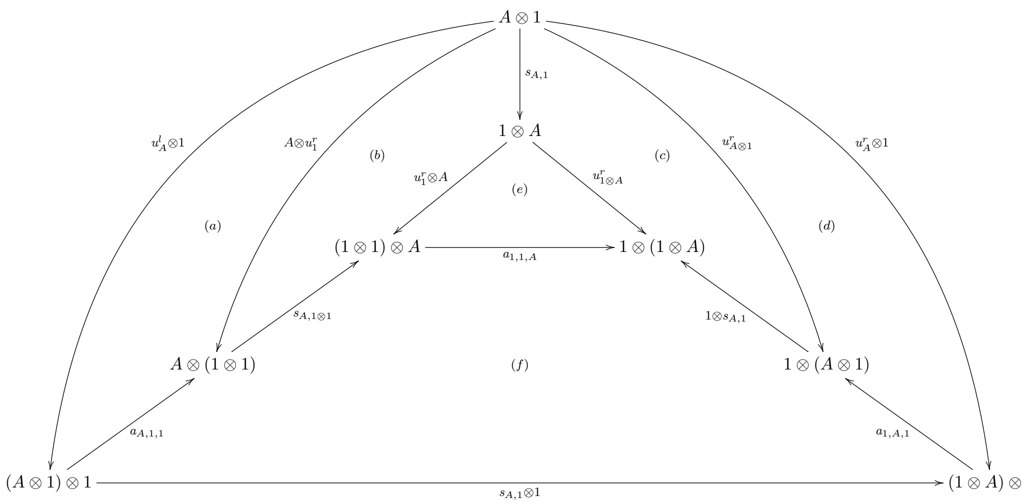
(b) Fact 4

(c) Fact 5

We thus have $u^l_1 \otimes 1 = u^r_1 \otimes 1$ since $a_{1,1,1}$ is an isomorphism, and finally $u^l_1 = u^r_1$ by Fact 3.

Comment Page 13

Thanks to Fact 3, it is sufficient to prove the commutation of the following diagram (since s and a are isomorphisms):



which commutes by:

(a) triangle of monoidal categories

(b) naturality of s

(c) naturality of u^r

(d) Fact 5

(e) Fact 5

(f) hexagon of symmetric monoidal categories

Example 15

$\times$ is a bi-functor from $\mathbb{C}$ and $\mathbb{C}$ to $\mathbb{C}$ (see page 9).
We consider three morphisms $f \in \mathbb{C}(A, A')$, $g \in \mathbb{C}(B, B')$ and $h \in \mathbb{C}(C, C')$. We have:

- using Fact 2 and the definition of the bi-functor $\times$:

$$\begin{aligned}
&(f \times g) \times h \,;\, \langle \pi_{A' \times B'} \,;\, \pi_{A'}, \langle \pi_{A' \times B'} \,;\, \pi_{B'}, \pi_{C'} \rangle \rangle \\
&\qquad = \langle (f \times g) \times h \,;\, \pi_{A' \times B'} \,;\, \pi_{A'}, (f \times g) \times h \,;\, \langle \pi_{A' \times B'} \,;\, \pi_{B'}, \pi_{C'} \rangle \rangle \\
&\qquad = \langle \pi_{A \times B} \,;\, \pi_A \,;\, f, (f \times g) \times h \,;\, \langle \pi_{A' \times B'} \,;\, \pi_{B'}, \pi_{C'} \rangle \rangle \\
&\qquad = \langle \pi_{A \times B} \,;\, \pi_A \,;\, f, \langle (f \times g) \times h \,;\, \pi_{A' \times B'} \,;\, \pi_{B'}, (f \times g) \times h \,;\, \pi_{C'} \rangle \rangle \\
&\qquad = \langle \pi_{A \times B} \,;\, \pi_A \,;\, f, \langle \pi_{A \times B} \,;\, \pi_B \,;\, g, \pi_C \,;\, h \rangle \rangle
\end{aligned}$$

and

$$
\begin{aligned}
&\langle \pi_{A\times B} \,;\, \pi_A, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C\rangle\rangle \,;\, f \times (g \times h)\\
&\qquad = \langle \pi_{A\times B} \,;\, \pi_A, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C\rangle\rangle \,;\, \langle \pi_A \,;\, f, \pi_{B\times C} \,;\, g \times h\rangle\\
&\qquad = \langle\langle \pi_{A\times B} \,;\, \pi_A, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C\rangle\rangle \,;\, \pi_A \,;\, f, \langle \pi_{A\times B} \,;\, \pi_A, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C\rangle\rangle \,;\, \pi_{B\times C} \,;\, g \times h\rangle\\
&\qquad = \langle \pi_{A\times B} \,;\, \pi_A \,;\, f, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C\rangle \,;\, g \times h\rangle\\
&\qquad = \langle \pi_{A\times B} \,;\, \pi_A \,;\, f, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C\rangle \,;\, \langle \pi_B \,;\, g, \pi_C \,;\, h\rangle\rangle\\
&\qquad = \langle \pi_{A\times B} \,;\, \pi_A \,;\, f, \langle\langle \pi_{A\times B} \,;\, \pi_B, \pi_C\rangle \,;\, \pi_B \,;\, g, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C\rangle \,;\, \pi_C \,;\, h\rangle\rangle\\
&\qquad = \langle \pi_{A\times B} \,;\, \pi_A \,;\, f, \langle \pi_{A\times B} \,;\, \pi_B \,;\, g, \pi_C \,;\, h\rangle\rangle
\end{aligned}
$$

Moreover, with Fact 1 and Fact 2:

$$
\begin{aligned}
&\langle \pi_{A\times B} \,;\, \pi_A, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C\rangle\rangle \,;\, \langle\langle \pi_A, \pi_{B\times C} \,;\, \pi_B\rangle, \pi_{B\times C} \,;\, \pi_C\rangle\\
&\qquad = \langle\langle \pi_{A\times B} \,;\, \pi_A, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C\rangle\rangle \,;\, \langle \pi_A, \pi_{B\times C} \,;\, \pi_B\rangle, \langle \pi_{A\times B} \,;\, \pi_A, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C\rangle\rangle \,;\, \pi_{B\times C} \,;\, \pi_C\rangle\\
&\qquad = \langle\langle \pi_{A\times B} \,;\, \pi_A, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C\rangle\rangle \,;\, \langle \pi_A, \pi_{B\times C} \,;\, \pi_B\rangle, \pi_C\rangle\\
&\qquad = \langle\langle\langle \pi_{A\times B} \,;\, \pi_A, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C\rangle\rangle \,;\, \pi_A, \langle \pi_{A\times B} \,;\, \pi_A, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C\rangle\rangle \,;\, \pi_{B\times C} \,;\, \pi_B\rangle, \pi_C\rangle\\
&\qquad = \langle\langle \pi_{A\times B} \,;\, \pi_A, \pi_{A\times B} \,;\, \pi_B\rangle, \pi_C\rangle\\
&\qquad = \langle \pi_{A\times B} \,;\, \langle \pi_A, \pi_B\rangle, \pi_C\rangle\\
&\qquad = \langle \pi_{A\times B}, \pi_C\rangle\\
&\qquad = id_{(A\times B)\times C}
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle\langle \pi_A, \pi_{B\times C} \,;\, \pi_B\rangle, \pi_{B\times C} \,;\, \pi_C\rangle \,;\, \langle \pi_{A\times B} \,;\, \pi_A, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C\rangle\rangle\\
&\qquad = \langle\langle\langle \pi_A, \pi_{B\times C} \,;\, \pi_B\rangle, \pi_{B\times C} \,;\, \pi_C\rangle \,;\, \pi_{A\times B} \,;\, \pi_A, \langle\langle \pi_A, \pi_{B\times C} \,;\, \pi_B\rangle, \pi_{B\times C} \,;\, \pi_C\rangle \,;\, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C\rangle\rangle\\
&\qquad = \langle \pi_A, \langle\langle \pi_A, \pi_{B\times C} \,;\, \pi_B\rangle, \pi_{B\times C} \,;\, \pi_C\rangle \,;\, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C\rangle\rangle\\
&\qquad = \langle \pi_A, \langle\langle\langle \pi_A, \pi_{B\times C} \,;\, \pi_B\rangle, \pi_{B\times C} \,;\, \pi_C\rangle \,;\, \pi_{A\times B} \,;\, \pi_B, \langle\langle \pi_A, \pi_{B\times C} \,;\, \pi_B\rangle, \pi_{B\times C} \,;\, \pi_C\rangle \,;\, \pi_C\rangle\rangle\\
&\qquad = \langle \pi_A, \langle \pi_{B\times C} \,;\, \pi_B, \pi_{B\times C} \,;\, \pi_C\rangle\rangle\\
&\qquad = \langle \pi_A, \pi_{B\times C} \,;\, \langle \pi_B, \pi_C\rangle\rangle\\
&\qquad = \langle \pi_A, \pi_{B\times C}\rangle\\
&\qquad = id_{A\times(B\times C)}
\end{aligned}
$$

- We first prove that $\pi_A \in \mathbb{C}(A \times \top, A)$ is the inverse of $\langle id_A, t_A\rangle \in \mathbb{C}(A, A \times \top)$ using Fact 1 and Fact 2:

$$\langle id_A, t_A\rangle \,;\, \pi_A = id_A$$

and

$$
\begin{aligned}
\pi_A \,;\, \langle id_A, t_A\rangle &= \langle \pi_A \,;\, id_A, \pi_A \,;\, t_A\rangle\\
&= \langle \pi_A, \pi_\top\rangle\\
&= id_{A\times\top}
\end{aligned}
$$

We also have:

$$\begin{aligned}
\langle id_A, t_A \rangle \,;\, f \times id_\top &= \langle id_A, t_A \rangle \,;\, \langle \pi_A \,;\, f, \pi_\top \,;\, id_\top \rangle \\
&= \langle \langle id_A, t_A \rangle \,;\, \pi_A \,;\, f, \langle id_A, t_A \rangle \,;\, \pi_\top \,;\, id_\top \rangle \\
&= \langle f, t_A \rangle \\
&= \langle f \,;\, id_{A'}, f \,;\, t_{A'} \rangle \\
&= f \,;\, \langle id_{A'}, t_{A'} \rangle
\end{aligned}$$

- The results for $\langle t_A, id_A \rangle$ are very similar.
- Using Fact 2:

$$\begin{aligned}
f \times g \,;\, \langle \pi_{B'}, \pi_{A'} \rangle &= \langle f \times g \,;\, \pi_{B'}, f \times g \,;\, \pi_{A'} \rangle \\
&= \langle \pi_B \,;\, g, \pi_A \,;\, f \rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle \pi_B, \pi_A \rangle \,;\, g \times f &= \langle \pi_B, \pi_A \rangle \,;\, \langle \pi_B \,;\, g, \pi_A \,;\, f \rangle \\
&= \langle \langle \pi_B, \pi_A \rangle \,;\, \pi_B \,;\, g, \langle \pi_B, \pi_A \rangle \,;\, \pi_A \,;\, f \rangle \\
&= \langle \pi_B \,;\, g, \pi_A \,;\, f \rangle
\end{aligned}$$

Moreover, with Fact 1 and Fact 2:

$$\begin{aligned}
\langle \pi_B, \pi_A \rangle \,;\, \langle \pi_A, \pi_B \rangle &= \langle \langle \pi_B, \pi_A \rangle \,;\, \pi_A, \langle \pi_B, \pi_A \rangle \,;\, \pi_B \rangle \\
&= \langle \pi_A, \pi_B \rangle \\
&= id_{A \times B}
\end{aligned}$$

We now have to prove to the three additional commutative diagrams of symmetric monoidal categories.

- Pentagon of monoidal categories:

$$\langle \pi_{(A\times B)\times C} \,;\, \pi_{A\times B}, \langle \pi_{(A\times B)\times C} \,;\, \pi_C, \pi_D \rangle \rangle \,;\, \langle \pi_{A\times B} \,;\, \pi_A, \langle \pi_{A\times B} \,;\, \pi_B, \pi_{C\times D} \rangle \rangle$$
$$= \langle \langle \pi_{(A\times B)\times C} \,;\, \pi_{A\times B}, \langle \pi_{(A\times B)\times C} \,;\, \pi_C, \pi_D \rangle \rangle \,;\, \pi_{A\times B} \,;\, \pi_A, \langle \pi_{(A\times B)\times C} \,;\, \pi_{A\times B}, \langle \pi_{(A\times B)\times C} \,;\, \pi_C, \pi_D \rangle \rangle \,;\, \langle \pi_{A\times B} \,;\, \pi$$
$$= \langle \pi_{(A\times B)\times C} \,;\, \pi_{A\times B} \,;\, \pi_A, \langle \langle \pi_{(A\times B)\times C} \,;\, \pi_{A\times B}, \langle \pi_{(A\times B)\times C} \,;\, \pi_C, \pi_D \rangle \rangle \,;\, \pi_{A\times B} \,;\, \pi_B, \langle \pi_{(A\times B)\times C} \,;\, \pi_{A\times B}, \langle \pi_{(A\times B)\times}$$
$$= \langle \pi_{(A\times B)\times C} \,;\, \pi_{A\times B} \,;\, \pi_A, \langle \pi_{(A\times B)\times C} \,;\, \pi_{A\times B} \,;\, \pi_B, \langle \pi_{(A\times B)\times C} \,;\, \pi_C, \pi_D \rangle \rangle \rangle$$

and

$$\langle \pi_{A\times B} \,;\, \pi_A, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C \rangle \rangle \times id_D \,;\, \langle \pi_{A\times(B\times C)} \,;\, \pi_A, \langle \pi_{A\times(B\times C)} \,;\, \pi_{B\times C}, \pi_D \rangle \rangle \,;\, id_A \times \langle \pi_{B\times C} \,;\, \pi_B, \langle \pi_{B\times C} \,;\, \pi_C, \pi_D \rangle \rangle$$
$$= \langle \langle \pi_{A\times B} \,;\, \pi_A, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C \rangle \rangle \times id_D \,;\, \pi_{A\times(B\times C)} \,;\, \pi_A, \langle \pi_{A\times B} \,;\, \pi_A, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C \rangle \rangle \times id_D \,;\, \langle \pi_{A\times(B\times C)} \,;\, \pi_{B\times}$$
$$= \langle \pi_{(A\times B)\times C} \,;\, \pi_{A\times B} \,;\, \pi_A, \langle \langle \pi_{A\times B} \,;\, \pi_A, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C \rangle \rangle \times id_D \,;\, \pi_{A\times(B\times C)} \,;\, \pi_{B\times C}, \langle \pi_{A\times B} \,;\, \pi_A, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C$$
$$= \langle \pi_{(A\times B)\times C} \,;\, \pi_{A\times B} \,;\, \pi_A, \langle \pi_{(A\times B)\times C} \,;\, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C \rangle, \pi_D \rangle \rangle \,;\, \langle \pi_A, \pi_{(B\times C)\times D} \,;\, \langle \pi_{B\times C} \,;\, \pi_B, \langle \pi_{B\times C} \,;\, \pi_C, \pi_D \rangle \rangle \rangle$$
$$= \langle \langle \pi_{(A\times B)\times C} \,;\, \pi_{A\times B} \,;\, \pi_A, \langle \pi_{(A\times B)\times C} \,;\, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C \rangle, \pi_D \rangle \rangle \,;\, \pi_A, \langle \pi_{(A\times B)\times C} \,;\, \pi_{A\times B} \,;\, \pi_A, \langle \pi_{(A\times B)\times C} \,;\, \langle \pi_{A\times}$$
$$= \langle \pi_{(A\times B)\times C} \,;\, \pi_{A\times B} \,;\, \pi_A, \langle \pi_{(A\times B)\times C} \,;\, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C \rangle, \pi_D \rangle \,;\, \langle \pi_{B\times C} \,;\, \pi_B, \langle \pi_{B\times C} \,;\, \pi_C, \pi_D \rangle \rangle \rangle$$
$$= \langle \pi_{(A\times B)\times C} \,;\, \pi_{A\times B} \,;\, \pi_A, \langle \langle \pi_{(A\times B)\times C} \,;\, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C \rangle, \pi_D \rangle \,;\, \pi_{B\times C} \,;\, \pi_B, \langle \pi_{(A\times B)\times C} \,;\, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C \rangle, \pi_D \rangle \,;\, \langle$$
$$= \langle \pi_{(A\times B)\times C} \,;\, \pi_{A\times B} \,;\, \pi_A, \langle \pi_{(A\times B)\times C} \,;\, \pi_{A\times B} \,;\, \pi_B, \langle \langle \pi_{(A\times B)\times C} \,;\, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C \rangle, \pi_D \rangle \,;\, \pi_{B\times C} \,;\, \pi_C, \langle \pi_{(A\times B)\times C} \,;$$
$$= \langle \pi_{(A\times B)\times C} \,;\, \pi_{A\times B} \,;\, \pi_A, \langle \pi_{(A\times B)\times C} \,;\, \pi_{A\times B} \,;\, \pi_B, \langle \pi_{(A\times B)\times C} \,;\, \pi_C, \pi_D \rangle \rangle \rangle$$

- Triangle of monoidal categories:

$$
\begin{aligned}
&\langle id_A, t_A\rangle \times id_B \,;\, \langle \pi_{A\times\top} \,;\, \pi_A, \langle \pi_{A\times\top} \,;\, \pi_\top, \pi_B\rangle\rangle \\
&\quad = \langle \langle id_A, t_A\rangle \times id_B \,;\, \pi_{A\times\top} \,;\, \pi_A, \langle id_A, t_A\rangle \times id_B \,;\, \langle \pi_{A\times\top} \,;\, \pi_\top, \pi_B\rangle\rangle \\
&\quad = \langle \pi_A, \langle \langle id_A, t_A\rangle \times id_B \,;\, \pi_{A\times\top} \,;\, \pi_\top, \langle id_A, t_A\rangle \times id_B \,;\, \pi_B\rangle\rangle \\
&\quad = \langle \pi_A, \langle \pi_A \,;\, t_A, \pi_B\rangle\rangle \\
&\quad = \langle \pi_A, \langle t_{A\times B}, \pi_B\rangle\rangle \\
&\quad = \langle \pi_A, \langle \pi_B \,;\, t_B, \pi_B\rangle\rangle \\
&\quad = \langle \pi_A, \pi_B \,;\, \langle t_B, id_B\rangle\rangle \\
&\quad = id_A \times \langle t_B, id_B\rangle
\end{aligned}
$$

- Hexagon of symmetric monoidal categories:

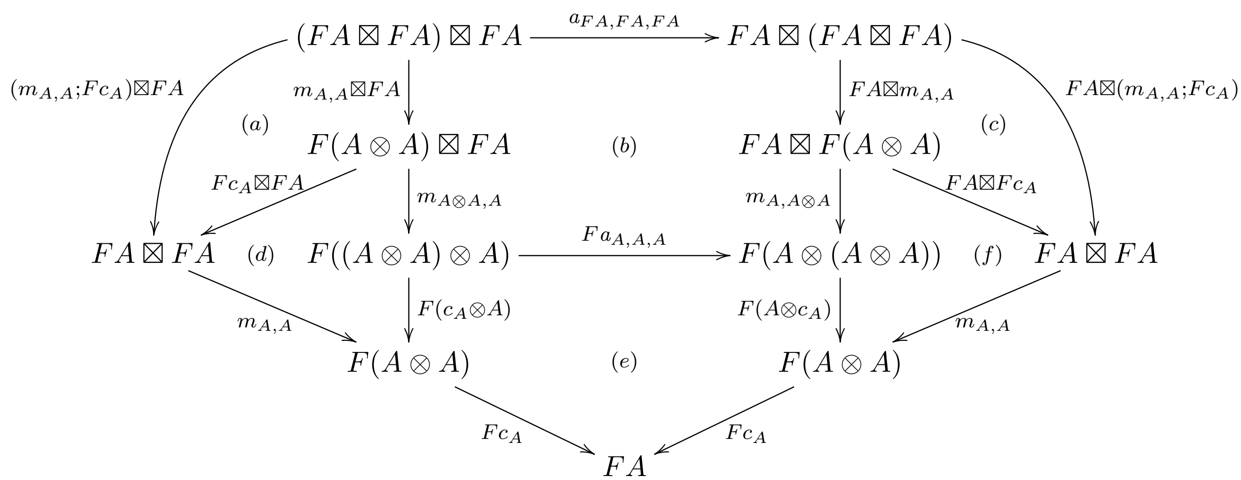
$$
\begin{aligned}
&\langle \pi_{A\times B} \,;\, \pi_A, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C\rangle\rangle \,;\, \langle \pi_{B\times C}, \pi_A\rangle \,;\, \langle \pi_{B\times C} \,;\, \pi_B, \langle \pi_{B\times C} \,;\, \pi_C, \pi_A\rangle\rangle \\
&\quad = \langle \langle \pi_{A\times B} \,;\, \pi_A, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C\rangle\rangle \,;\, \pi_{B\times C}, \langle \pi_{A\times B} \,;\, \pi_A, \langle \pi_{A\times B} \,;\, \pi_B, \pi_C\rangle\rangle \,;\, \pi_A\rangle \,;\, \langle \pi_{B\times C} \,;\, \pi_B, \langle \pi_{B\times C} \,;\, \pi_C, \pi_A\rangle\rangle \\
&\quad = \langle \langle \pi_{A\times B} \,;\, \pi_B, \pi_C\rangle, \pi_{A\times B} \,;\, \pi_A\rangle \,;\, \langle \pi_{B\times C} \,;\, \pi_B, \langle \pi_{B\times C} \,;\, \pi_C, \pi_A\rangle\rangle \\
&\quad = \langle \pi_{A\times B} \,;\, \pi_B, \langle \langle \langle \pi_{A\times B} \,;\, \pi_B, \pi_C\rangle, \pi_{A\times B} \,;\, \pi_A\rangle \,;\, \pi_{B\times C} \,;\, \pi_C, \langle \langle \pi_{A\times B} \,;\, \pi_B, \pi_C\rangle, \pi_{A\times B} \,;\, \pi_A\rangle \,;\, \pi_A\rangle\rangle \\
&\quad = \langle \pi_{A\times B} \,;\, \pi_B, \langle \pi_C, \pi_{A\times B} \,;\, \pi_A\rangle\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle \pi_B, \pi_A\rangle \times id_C \,;\, \langle \pi_{B\times A} \,;\, \pi_B, \langle \pi_{B\times A} \,;\, \pi_A, \pi_C\rangle\rangle \,;\, id_B \times \langle \pi_C, \pi_A\rangle \\
&\quad = \langle \langle \pi_B, \pi_A\rangle \times id_C \,;\, \pi_{B\times A} \,;\, \pi_B, \langle \pi_B, \pi_A\rangle \times id_C \,;\, \langle \pi_{B\times A} \,;\, \pi_A, \pi_C\rangle\rangle \,;\, id_B \times \langle \pi_C, \pi_A\rangle \\
&\quad = \langle \pi_{A\times B} \,;\, \pi_B, \langle \langle \pi_B, \pi_A\rangle \times id_C \,;\, \pi_{B\times A} \,;\, \pi_A, \langle \pi_B, \pi_A\rangle \times id_C \,;\, \pi_C\rangle\rangle \,;\, id_B \times \langle \pi_C, \pi_A\rangle \\
&\quad = \langle \pi_{A\times B} \,;\, \pi_B, \langle \pi_{A\times B} \,;\, \pi_A, \pi_C\rangle\rangle \,;\, \langle \pi_B, \pi_{A\times C} \,;\, \langle \pi_C, \pi_A\rangle\rangle \\
&\quad = \langle \langle \pi_{A\times B} \,;\, \pi_B, \langle \pi_{A\times B} \,;\, \pi_A, \pi_C\rangle\rangle \,;\, \pi_B, \langle \pi_{A\times B} \,;\, \pi_B, \langle \pi_{A\times B} \,;\, \pi_A, \pi_C\rangle\rangle \,;\, \pi_{A\times C} \,;\, \langle \pi_C, \pi_A\rangle\rangle \\
&\quad = \langle \pi_{A\times B} \,;\, \pi_B, \langle \pi_{A\times B} \,;\, \pi_A, \pi_C\rangle \,;\, \langle \pi_C, \pi_A\rangle\rangle \\
&\quad = \langle \pi_{A\times B} \,;\, \pi_B, \langle \langle \pi_{A\times B} \,;\, \pi_A, \pi_C\rangle \,;\, \pi_C, \langle \pi_{A\times B} \,;\, \pi_A, \pi_C\rangle \,;\, \pi_A\rangle\rangle \\
&\quad = \langle \pi_{A\times B} \,;\, \pi_B, \langle \pi_C, \pi_{A\times B} \,;\, \pi_A\rangle\rangle
\end{aligned}
$$

Property 3

The diagram:



commutes by:

(a) functoriality of ⊠

(b) hexagon of monoidal functors

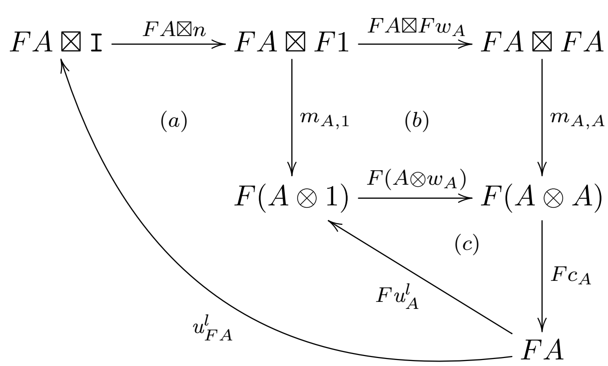
(c) functoriality of ⊠

(d) naturality of m

(e) pentagon of monoids

(f) naturality of m

The diagram:



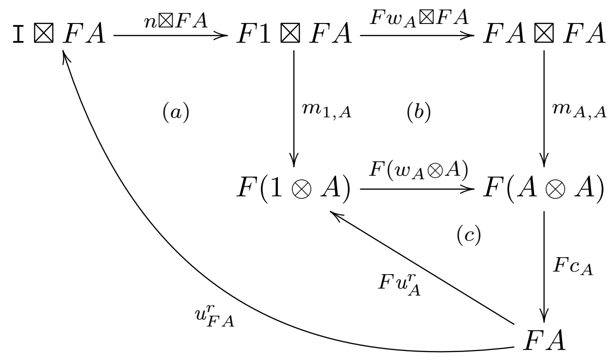
commutes by:

(a) square of monoidal functors

(b) naturality of m

(c) triangle of monoids

The diagram:



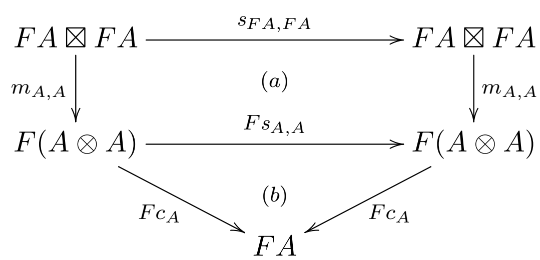
commutes by:

(a) square of monoidal functors

(b) naturality of m

(c) triangle of monoids

In the case of a symmetric monoidal functor and a symmetric monoid, the diagram:



commutes by:

(a) square of symmetric monoidal functors

(b) triangle of symmetric monoids