Subject 2: The Category of Proof Nets

In the whole subject, we will restrict *ax* nodes in proof nets to have atomic formulas on conclusions (that is pairs X, $X^\perp$). These are called *axiom-expanded* proof nets.

The Category $\mathbb{PN}$

Formulas of multiplicative exponential linear logic with units are given by:

$$A ::= X \mid X^\perp \mid A \otimes A \mid 1 \mid A \parr A \mid \perp \mid !A \mid ?A$$

The notion of proof structure for multiplicative exponential linear logic is extended to the multiplicative units 1 and $\perp$ by considering two new kinds of nodes: 1 and $\perp$ of the same shape as the $?w$ nodes (no premise and one conclusion), with respectively one conclusion with label 1 and one conclusion with label $\perp$. Concerning the correctness criterion, we still consider only the acyclicity constraint on correctness graphs (no difference between the 1 and $\perp$ nodes and the $?w$ nodes for that).

Question 1. Give, for each formula A, a cut-free (axiom-expanded) proof net with conclusions A and $A^\perp$.

Question 2. Define the cut elimination step involving a *cut* node with premisses labelled 1 and $\perp$.

Question 3. Prove that cut elimination of proof nets preserves the fact that axioms have atomic conclusions.

We define objects of $\mathbb{PN}$ to be formulas of multiplicative exponential linear logic with units. Morphisms $\mathbb{PN}(A, B)$ from A to B are cut-free (axiom-expanded) proof nets with two conclusions $A^\perp$ and B.

Question 4. Prove $\mathbb{PN}$ defines a category if we consider the composition $\mathcal{R}_1 ; \mathcal{R}_2$ of a cut-free proof net $\mathcal{R}_1$ (with conclusions $A^\perp$ and B) with a cut-free proof net $\mathcal{R}_2$ (with conclusions $B^\perp$ and C) to be the unique normal from of the proof net obtained by introducing a *cut* node between the conclusions B and $B^\perp$ of $\mathcal{R}_1$ and $\mathcal{R}_2$.

Multiplicative Structure of $\mathbb{PN}$

Question 5. Define an operation $\otimes$ on proof nets which extends the connective $\otimes$ into a bi-functor on $\mathbb{PN}$ (and prove one obtains a bi-functor).

Question 6.

a. For any formulas A and B, define a proof net $s_{A,B}$ with conclusions $A^\perp \parr B^\perp$ and $B \otimes A$.

b. For any formulas A and B, prove $s_{A,B} ; s_{B,A}$ is the identity from $A \otimes B$ to $A \otimes B$.

Question 7.

a. For any formulas A, B and C, define a proof net with conclusions $(A^{\perp} \parr B^{\perp}) \parr C^{\perp}$ and $A \otimes (B \otimes C)$.

b. Prove this family of proof nets defines a natural isomorphism from $(\_ \otimes \_) \otimes \_$ to $\_ \otimes (\_ \otimes \_)$.

Question 8. Prove $(\mathbb{PN}, \otimes, 1)$ has a structure of symmetric monoidal category.

Question 9.

a. For any formulas A and B, define a proof net with conclusions $(A \otimes B^{\perp}) \parr A^{\perp}$ and B.

b. For any formulas A and B, prove $A^{\perp} \parr B$ defines an exponential object of A and B in $\mathbb{PN}$.

We can conclude that $\mathbb{PN}$ is a symmetric monoidal closed category.

Exponential Structure of $\mathbb{PN}$

Question 10. Define an operation ! on proof nets which extends the connective ! into a functor on $\mathbb{PN}$ (and prove one obtains a functor).

Question 11. Extend ! into a co-monad on $\mathbb{PN}$.

Question 12.

a. For any formula A, define a proof net w_A with conclusions $?A^{\perp}$ and 1.

b. For any formula A, define a proof net c_A with conclusions $?A^{\perp}$ and $!A \otimes !A$.

c. Is $(!A, c_A, w_A)$ a co-monoid in $\mathbb{PN}$?

d. Compare $c_A \,;\, s_{!A,!A}$ and c_A (where $s_{A,B}$ is the proof net defined in Question 6).